Generalized Convex Functions, Nonlinear Programming and Applications

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Abstract

In this talk, the focus is on convex functions and its generalizations. Some important properties of generalized convex functions are given. The role of generalized convex functions in nonlinear programming is shown. Finally, some applications in economics are given.

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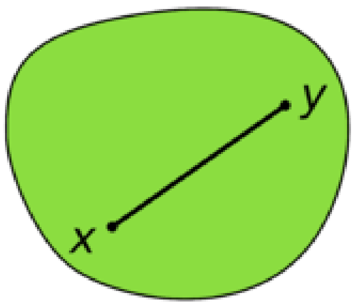
- Elementary notions on convex, quasiconvex and pseudoconvex functions
- Invex functions and some properties
- Optimality and duality involving invex functions
- Vector Variational Like Inequalities
- Applications

Convex Sets

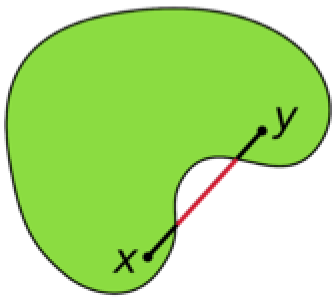
A nonempty set $X \subseteq R^n$ is *convex* if, for any two points in X,

the line segment joining the two points lies entirely within X.

For example:



A convex set



A Nonconvex set

Analytically:

A set $X \subseteq R^n$ is *convex* if, $\forall x_1, x_2 \in X$

$$\lambda x_1 + (1-\lambda) x_2 \in X, \quad \forall \lambda \in [0, 1].$$

Convention:

(i) An empty set is convex.

(ii) A singleton set is also convex.

Definition:

A *convex combination* of finitely many points

$x_i \in R^n$, $i=1,2,\ldots,p,$ is a point x of the form $x=\sum_{i=1}^{p}\lambda_i x_i,$ where

$\sum_{i=1}^{p}\lambda_i=1,\ \ \lambda_i\geq 0, i=1,\ldots,p.$

Some Properties of Convex Sets:

(i) The intersection of an arbitrary family of convex sets is a convex set. (Easy to prove, try yourself)

(ii) A set $X\subseteq R^n$ is convex iff every convex combination of finitely many points of X is in X.

For several other topological properties of convex sets:

Like:

- Closure of a convex set is a convex set.
- Interior of a convex set is a convex set.

For more details see the book:

Generalized Convexity and Optimization,

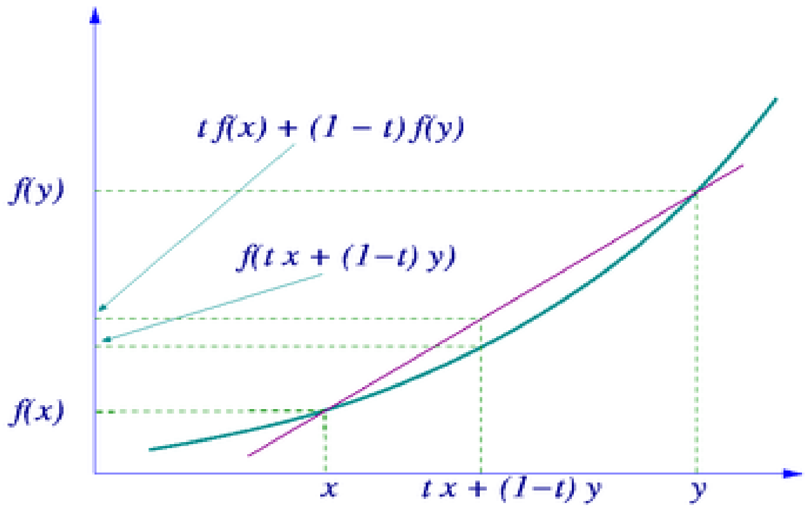
by Alberto Cambini and Laura Martein

Lecture Notes in Economics and Mathematical Systems

No. 616, Springer-Verlag, Berlin, 2009.

Convex Functions

A function f is *convex* provided that the line segment joining any two points of its graph lies on or above the graph.



A function f defined on a convex set $X \subseteq R^n$ is said to be *convex* if for every $x, y \in X$

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y), \quad \forall t \in [0, 1].$$

Examples:

$f(x) = x + |x|$ is convex

$f(x) = x^2$ is convex

$f(x) = e^x$ is convex

$f(x) = e^{-x}$ is convex

Theorem (Jensen's Inequality):

A function f is convex on a convex set X iff for every

$x_1,...,x_n \in X$,

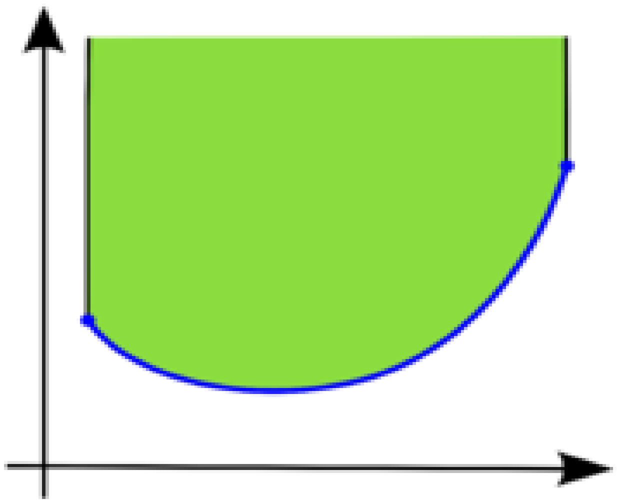
$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f\left(x_i\right),$$

where $\sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \geq 0, \; i = 1,...,n.$

Characterization of Convex Function

Convex functions can be characterized by their epigraph.

Epigraph of a Convex Function:



$$epi\ f = \{(x,z) : x \in X,\ f(x) \le z\}.$$

Theorem: Let f be a function deined on a convex set $X \subseteq R^n$.

Then f is convex if and only if *epi* f is a convex set.

Proof: $(\Rightarrow)$ Suppose f is a convex function on a convex set X .

We have to prove that *epi* f is a convex set.

Let $\left(x_1, z_1\right)$ and $\left(x_2, z_2\right) \in epi\, f$, this implies that $f\left(x_1\right) \le z_1$ and

$f\left(x_2\right) \le z_2$. For every $\lambda \in [0,\ 1]$, we have

$$\lambda\left(x_1, z_1\right) + (1-\lambda)\left(x_2, z_2\right) = \left(\lambda x_1 + (1-\lambda)x_2, \lambda z_1 + (1-\lambda)z_2\right).$$

Note that

$$f\left(\lambda x_1+(1-\lambda)x_2\right)\le \lambda f\left(x_1\right)+(1-\lambda)f\left(x_2\right),$$

(Since f is a convex function)

$$\le \lambda z_1+(1-\lambda)z_2.$$

Thus, $\left(\lambda x_1+(1-\lambda)x_2, \lambda z_1+(1-\lambda)z_2\right)\in epi\ f$,

that is,

$$\lambda\left(x_1, z_1\right)+(1-\lambda)\left(x_2, z_2\right)\in epi\ f\ ,$$

that is, $epi\ f$ is a convex set

$(\Leftarrow)$ Suppose $epi\ f$ is a convex. We have to show that f is convex. Let $x_1, x_2 \in X$, since $\left(x_1, f\left(x_1\right)\right)$ and $\left(x_2, f\left(x_2\right)\right) \in epi\ f$.

We have

$$\lambda\left(x_1, f\left(x_1\right)\right) + (1-\lambda)\left(x_2, f\left(x_2\right)\right) \in epi\ f, \quad \forall\ \lambda \in [0, 1],$$

Since $epi\ f$ is a convex set. That is,

$$\left(\lambda x_1 + (1-\lambda)x_2, \lambda f\left(x_1\right) + (1-\lambda) f\left(x_2\right)\right) \in epi\ f.$$

That is,

$$f\left(\lambda x_1 + (1-\lambda)x_2\right) \leq \lambda f\left(x_1\right) + (1-\lambda) f\left(x_2\right), \text{ by definition of } epi\ f.$$

Thus, f is a convex function on x.□

Definition:

Lower level set of f is:

$$X_{\leq\alpha} = \{x \in X : f(x) \leq \alpha\},\ \alpha \in R.$$

Theorem:

Let f be a convex function defined on a convex set $X \subseteq R^n$.

Then $X_{\leq\alpha}$ is a convex set for every $\alpha \in R$.

Proof:

The result is true if $X_{\le\alpha}$ is an empty set or a singleton set.

Suppose $x_1, x_2 \in X_{\le\alpha}$. By definition of $X_{\le\alpha}$, $f(x_1) \le \alpha$ and $f(x_2) \le \alpha$.

Since f is a convex function, we have

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$$

$$\le \lambda\alpha + (1-\lambda)\alpha, \text{ Since } f(x_1) \le \alpha \text{ and } f(x_2) \le \alpha.$$

$$= \alpha,$$

That is, $\lambda x_1 + (1-\lambda)x_2 \in X_{\le\alpha}$. Thus, $X_{\le\alpha}$ is a convex set.□

Theorem:

Let $f_1,\ldots,f_p$ be convex functions defined on a convex set

$X \subseteq R^n$. Define a new function as

$$f(x) = \sum_{i=1}^{p} \alpha_i f_i(x), \quad \alpha_i \geq 0.$$

Then f is a convex function on X.

Theorem:

Let $f : X \to R$ be a convex function defined on a convex set

$X \subseteq R^n$ and $g : Y \to R$ be a non-decreasing convex function, with

$f(X) \subseteq Y$. Then the composite function $h(x) = g(f(x))$ is a convex

function on X.

Minima of Convex Functions

To know, whether or not a local minimum is also global, is one of the most important questions in optimization. The Presence of convex function answers in positive to this question.

Theorem:

Let $X \subseteq R^n$ be a convex set and let f be a convex function on X. Then

(i) A local minimum point is also global;

(ii) The set X^* of all minimum points is a convex set.

Differentiable Convex Functions

A convex function is continuous in the interior of its domain but not necessarily differentiable.

For example: The convex function $f(x) = |x|$ is continuous on R but it is well known that it is not differentiable at $x = 0$.

Theorem

Let f be differentiable function defined on a nonempty open convex set $X \subseteq R^n$. Then f is convex if and only if $\forall\ u \in X$

$$f(x) - f(u) \geq (x-u)^T \nabla f(u), \quad \forall \quad x \in X.$$

Theorem: Let $X \subseteq R^n$ be a convex set and let f be a differentiable convex function on X. Then, a critical point $u \in X$ of f is a global minimum point.

Proof. Let $u \in X$ be a critical point for f, i.e. $\nabla f(u) = 0$. By convexity of f, one has

$$f(x) - f(u) \geq (x-u)^T \nabla f(u), \quad \forall \quad x \in X.$$

That is, $f(x) - f(u) \geq 0, \quad \forall \quad x \in X.$

Therefore, u is global minimum solution.

In 1949, an Italian Mathematician de Finetti introduced the concept of quasiconvex function as an extension of convex function.

A function f defined on a convex set $X \subseteq R^n$, is said to be quasiconvex on X if

$$f(\lambda x+(1-\lambda)u)\leq \max\{f(x),\ f(u)\},\quad \forall\ x,\ u\in X,\ \forall\ 0\leq\lambda\leq 1.$$

Theorem: Let $X \subseteq R^n$ be a convex set. If f is convex on X, then it is quasiconvex on X.

Proof:

$$f(\lambda x + (1-\lambda)u) \leq \lambda f(x) + (1-\lambda) f(u) \text{ (since } f \text{ is convex)}$$

$$\leq \lambda \max\{f(x),\ f(u)\} + (1-\lambda)\{f(x),\ f(u)\}$$

$$= \max\{f(x),\ f(u)\}. \qquad \square$$

It is clear from the above theorem that the class of quasiconvex functions is larger than the class of convex functions. Unfortunately, some nice properties of convex functions are lost in this new class of functions.

For example: The class of convex functions is closed under addition, where as the sum of quasiconvex functions may not be quasiconvex,

eg: $f(x)=x^3$ and $g(x)=-3x$ both are quasiconvex on R, but the sum $h(x)=x^3-3x$ is not quasiconvex, as $h(-2)=-2$, $h(0)=0$, but $h(-1)=2>\max\{h(-2), h(0)\}$.

One good thing turn out in case of quasiconvex functions:

A quasiconvex function can be characterized by its lower level set.

Invex Functions: Definitions and Properties

Definition. Assume $X \subseteq R^n$ is an open set. The differentiable function $f : X \to R$ is invex if there exists a vector function $\eta : X \times X \to R^n$ such that

$$f(x) - f(y) \geq \eta(x, y)^T \nabla f(y), \quad \forall\ x, y \in X. \qquad (1)$$

It is obvious that the particular case of (differentiable) convex function is obtained from (1) by choosing $\eta(x, y) = x - y$.

Theorem. Let $f : X \to R$ be differentiable. Then f is invex if and only if every stationary point is a global minimizer.

Proof. Necessity: Let f be invex and assume $\bar{x} \in X$ with $f(\bar{x}) = 0$. Then $f(x) - f(\bar{x}) \geq 0,\ \forall\, x \in X$, so $\bar{x}$ is a global minimizer of f over X.

Sufficiency: Assume that every stationary point is a global minimizer.

If $\nabla f(y) = 0$, let $\eta(x, y) = 0$.

If $\nabla f(y) \neq 0$, let $\eta(x, y) = \dfrac{[f(x) - f(y)] \nabla f(y)}{\nabla f(y)^T \nabla f(y)}$. Then f is invex with respect to η.

This is, of course, not the only possible choice of η. Indeed, if $\nabla f(y)=0$, then $\eta(x, y)$ may be chosen arbitrarily, and if $\nabla f(y)\neq 0$, then $\eta(x,y)\in\left\{\frac{[f(x)-f(y)]\nabla f(y)}{\nabla f(y)^T\nabla f(y)}+v:\ v^T\nabla f(y)\leq 0\right\}$, a half-space in R^n.

Example: Take $f(x, y) = x^2 y^2$ on R^2. All stationary points of f given by $(x, 0), (0, y), \; x, , y \in R$ Are global minimum points, so f is invex. On the otherhand, take $a = (0, -4), \; b = (3, -1)$ in R^2,

$$f(a) = 0 < f(b) = 9,$$

but $$(a-b)^T \nabla f(b) = (-3, -3)(6, -18)^T = 36 > 0,$$

which is violating $f(a) \le f(b) \Rightarrow (a-b)^T \nabla f(b) \le 0$

So, f is not quasiconvex.

If f is invex on an open set $X \subseteq R^n$ it is not true that the set $A = \{x \in X, \nabla f(x) = 0\}$ is a convex set (as for convex functions). Let us consider the following.

Example. Let $f(x, y) = y\left(x^2 - 1\right)^2$, defined on the open set $S = \left\{(x, y) \in R^2 : x \in R,\ y > 0\right\}$. The set of all its stationary points coincides with the set of all its minimum points (i.e., f on S). This set is given by $\{(1, y) : y > 0\} \cup \{(-1, y) : y > 0\}$, which is not a convex set in R^2.

We note that the class of functions differentiable on an open set X and all invex with respect to the same $\eta(x, y)$, is closed under addition on any domain contained in X, unlike the classes of quasi-convex and pseudo-convex functions which do not retain this property of convex functions.

However, the class of functions invex on an open set X, but not necessarily with respect to the same $\eta(x, y)$, need not be closed under addition. For instance (see Smart (1990), Mond and Smart (1991a)), consider $f_1 : R \to R$ and $f_2 : R \to R$ defined by $f_1(x) = 1 - e^{-(x+5)^2}$, $f_2(x) = 1 - e^{-(x-5)^2}$. Both f_1 and f_2 are invex, but $f_1 + f_2$ has a stationary point at $\bar{x} = 0$ which is not a global minimizer.

Theorem. Let $f_1, f_2, ..., f_m : X \to R$ all invex on the open set $X \subseteq R^n$, with respect to the same function $\eta(x, y): X \times X \to R^n$. Then:

(i) for each $\alpha \in R,\ \alpha > 0,$ the function $\alpha f_i,\ i = 1, ..., m,$ is invex with respect to the same η;

(ii) the linear combination of $f_1, f_2, ..., f_m,$ with nonnegative coefficients is invex with respect to the same η.

Theorem. Let $f: X \to R$, $g: X \to R$ be invex. A common η, with respect to which both f and g are invex, exists if and only if for all $x, y \in X$ either

(a) $\nabla f(y) \neq -\lambda \nabla g(y)$ for any $\lambda > 0$ or

(b) $\nabla f(y) \neq -\lambda \nabla g(y)$ for some $\lambda > 0$ and $f(x) - f(y) \geq -\lambda[g(x) - g(y)]$.

Theorem. Let $f: X \to R,\ g: X \to R$ be invex. A common η, with respect to which both f and g are invex, exists if and only if $f + \lambda g$ is invex for all $\lambda > 0$.

Theorem

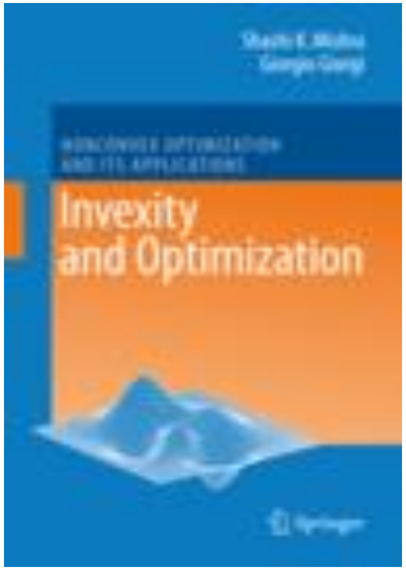
Let $\psi : R \to R$ be a monotone increasing differentiable convex function. If f is invex on X with respect to η, then the composite function $\psi \circ f$ is invex with respect to the same η.

Proof

By the fact that $\psi(x+h) \geq \psi(x) + \psi'(x)h, \quad \forall\ x, h \in R,$ we get

$$\psi(f(x)) \geq \psi(f(y)) + \nabla f(y)\eta(x, y) \geq \psi(f(y)) + \psi'(f(y))\nabla f(y)\eta(x, y)$$

$$= \psi(f(y)) + \nabla(f \circ \psi)(y)\eta(x, y).$$



Invexity and Optimization

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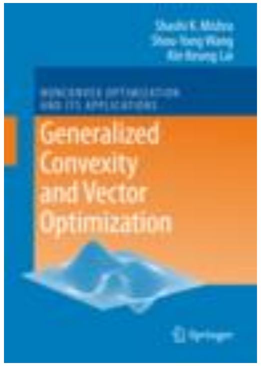

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Invexity in necessary and sufficient optimality conditions

Let us consider the following nonlinear programming problem:

$$(\mathrm{P}) \quad \begin{cases} \text{minimize} \quad f(x) \\ x \in K \\ K = \{x : x \in C,\ g(x) \leq 0\}, \end{cases}$$

where $f : C \to R$ and $g : C \to R^m$ are (Frechet) differentiable on the open set $C \subseteq R^n$.

Note:

If we have a problem with equality constraints, of the type $h(x)=0,\ h:C\to R^{p}$, we could re-write these constraints as $h(x)\leq 0, -h(x)\leq 0$.

It is well known that under certain regularity assumptions on the vector function g (“constraint qualifications”) the Karush-Kuhn-Tucker conditions are necessary for optimality in (P), that is, if x^* is a solution of (P) or even if it is a point of local minimum of f on K, then there exists a vector $\lambda^* \in R^m$ such that

$$\nabla f\left(x^*\right) + \lambda^{*T} \nabla g\left(x^*\right) = 0 \qquad (1)$$

$$\lambda^{*T} g\left(x^*\right) = 0 \qquad (2)$$

$$\lambda^{*T} \geq 0. \qquad (3)$$

Karush-Kuhn-Tucker type Sufficient Optimality Conditions

If $\left(x^*, \lambda^*\right)$, with $x^* \in K$, $\lambda^* \in R^m$, satisfies (1)-(3), then x^* is optimal for (P), provided one of the following assumptions is imposed:

1. (Kuhn-Tucker (1951)) $f(x)$ convex and $g_i(x)$ convex, $i = 1, \ldots, m$.
2. (Mangasarian (1965, 1969)) $f(x)$ pseudoconvex and $g_i(x)$ quasiconvex, with $i \in I = \left\{ i : g_i\left(x^*\right) = 0 \right\}$ the set of the active or effective constraints at $x^* \in K$.

3. (Mond (1983) $f(x)$ pseudoconvex and $\lambda^{*T} g(x)$quasiconvex.

4. (Mond (1983)) $f(x)+\lambda^{*T} g(x)$ pseudoconvex.

Hanson (1981) observed that the (generalized) convexity requirements appearing in the (1)-(4) above can be further weakened as in the related proofs of the sufficiency for problem (P) there is no explicit dependence on the linear term $(x-u)$, appearing in the definition of differentiable convex, pseudoconvex and quasiconvex functions.

That is, if $x^* \in K$ and $\left(x^*, \lambda^*\right)$ satisfies (1)-(3), then x^* solves (P) if any one of the following conditions is satisfied:

(1) $f(x)$ and every $g_i(x)$, $i \in I$, are invex with respect to the same η.

(2) $f(x)$ is pseudoinvex and every $g_i(x)$, $i \in I$, is quasiinvex with respect to the same η.

(3) $f(x)$ is pseudoinvex and $\lambda^{*T} g(x)$ is quasiinvex with respect to the same η.

(4) The Lagrangian function $f(x)+\lambda^{*T} g(x)$ is pseudoinvex with respect to an arbitrary η.

We give only the proof for (1) (see, Hanson (1981)):

For any $x \in C$ satisfying $g(x) \leq 0$, we have

$$f(x) - f\left(x^*\right) \geq \eta\left(x, x^*\right)^T \nabla f\left(x^*\right)$$

$$= -\eta\left(x, x^*\right)^T \nabla\left(\lambda^{*T} g(x)\right)$$

$$\geq -\lambda^{*T}\left(g(x) - g\left(x^*\right)\right)$$

$$= -\lambda^{*T} g(x)$$
$$\geq 0.$$

So x^* is minimal.

Jeyakumar (1985) gives the following result that weakens the sufficient optimality conditions for problem (P) by means of $\rho-$invex functions:

Theorem

Let $x^* \in K$ and let $\left(x^*, \lambda^*\right)$ satisfy (1)-(3); let $f(x)$ be ρ_0-pseudoinvex at x^* and let every $g_i(x)$, $i \in I$, be ρ_i-quasiinvex at x^*, with respect to the same functions η and θ. Let $\rho_0 + \sum_{i\in I} \lambda_i^* \rho_i \geq 0$. Then x^* solves (P).

Duality

Hanson (1981) demonstrated that invexity of f and $g_i, i=1,\ldots,m,$ with respect to a common η was also sufficient for weak and strong duality to hold between the primal problem (P) and its Wolfe dual (Wolfe (1961)):

(WD) $\quad \underset{u,\lambda}{\text{Maximize}} \; f(u)+\lambda^T g(u)$

subject to $\quad \nabla f(u)+\nabla\left(\lambda^T g(u)\right)=0$

$\lambda \geq 0.$

Theorem (Weak duality):

Let x be feasible for (P) and (u, λ) be feasible for (WD) and let f and $g_i, i=1,...,m$, are all invex with respect to a common η.

Then, we have

$$f(x) \geq f(u) + \lambda^T g(u).$$

Theorem (Strong Duality):

Under the conditions of a suitable constraint qualification for (P), if x^0 is minimal in the primal problem (P), then $\left(x^0, \lambda^0\right)$ is maximal in the dual problem (WD), where λ^0 is given by the Kuhn-Tucker conditions and f and $g_i, i=1,\ldots,m$, are all invex with respect to a common η. Moreover, the extremal values are equal for the two problems.

Proof. Let (u, λ) be a feasible for (WD). Then

$$\left(f\left(x^0\right) + \lambda^{0T} g\left(x^0\right)\right) - \left(f(u) + \lambda^T g(u)\right)$$

$$= f\left(x^0\right) - f(u) - \lambda^T g(u)$$

$$\geq \eta\left(x^0, u\right)^T \nabla f(u) - \lambda^T g(u)$$

$$= -\eta\left(x^0, u\right)^T \lambda^T \nabla g(u) - \lambda^T g(u)$$

$$\geq -\lambda^T g\left(x^0\right) \geq 0.$$

So $\left(x^0, \lambda^0\right)$ is maximal in the dual problem, and since $\lambda^{0T} g\left(x^0\right) = 0$, the extreme of the two problems are equal.

Definition. Let $f: C \to R$ be invex with respect to some function $\eta: C \times C \to R^n$; f is said to be *strictly invex at* $\bar{x}$ if

$$f(x) - f(\bar{x}) > \eta(x, \bar{x})^T \nabla f(\bar{x}), \ \forall\, x \in C, x \neq \bar{x}.$$

Let $f: C \to R$ be pseudoinvex with respect to some function $\eta: C \times C \to R^n$; f is said to be *strictly pseudoinvex at* $\bar{x}$ if

$$\eta(x, \bar{x})^T \nabla f(\bar{x}) \geq 0 \Rightarrow f(x) > f(\bar{x}), \ \forall\, x \in C, x \neq \bar{x}.$$

Theorem (Strict Converse Duality):

Assume f and g_i, $i=1,...,m$, are invex with respect to a common kernel function η. Let x^* be optimal for (P) and $(\bar{x}, \bar{\lambda})$ be optimal for (WD). If a constraint qualification is satisfied for (P) and f is strictly invex for (P) at $\bar{x}$, then $x^* = \bar{x}$.

The original version of the Mond-Weir dual to (P) is defined as follows:

(MWD) Maximize $f(u)$

subject to $\nabla f(u) + \lambda^T \nabla g(u) = 0$

$$\lambda^T g(u) \geq 0, \quad \lambda \geq 0.$$

General Mond-Weir dual is obtained by partitioning the set $M=\{1,\ldots,m\}$ into $r+1$ subsets $I_0, I_1, \ldots, I_r$, $(r \le m-1)$, such that

$$I_\alpha \cap I_\beta = \phi,\ \alpha \ne \beta, \text{ and}$$

$$\bigcup_{\alpha=0}^{r} I_\alpha = M.$$

The General Mond-Weir dual problem is now (Mond and Weir (1981)):

(GMWD) Maximize $f(u)+\sum_{i\in I_0}\lambda_i g_i(u)$ (4)

subject to $\nabla f(u)+\nabla\left(\lambda^T g(u)\right)=0$ (5)

$\lambda \geq 0$ (6)

$\sum_{i\in I_\alpha}\lambda_i g_i(u)\geq 0,\ \alpha=1,\ldots,r.$ (7)

Remark:

We remark that if $I_0 = M$, $r = 1$, and $I_1 = \phi$, then (GMWD) reduces to the Wolfe dual. If $I_0 = \phi$, $r = 1$, and $I_1 = M$, then (GMWD) yields the Mond-Weir dual (MWD).

Theorem (Weak duality):

If $f+\sum_{i\in I_0}\lambda_i g_i$ is pseudoinvex with respect to some $\eta: C\times C\to R^n$ and $\sum_{i\in I_\alpha}\lambda_i g_i$ is quasiinvex with respect to the same $\eta: C\times C\to R^n$, $\alpha=1,\ldots,r$, for any $\lambda\in R_+^m$, then $\inf(\text{P})\geq\sup(\text{GMWD})$.

Theorem (Strong duality):

Let x^* be optimal for (P), and assume the invexity assumptions of Weak duality theorem are satisfied. Assume also that a suitable constraint qualification is satisfied for (P). Then there exists $\lambda^* \in R^m$ such that $\left(x^*, \lambda^*\right)$ is optimal for (MWD), and the objective values are equal.

f is pseudolinear if and only if there exists a positive functional $p(x, y) \in R$ such that

$$f(x) = f(y) + [p(x,y)(x-y)]^T \nabla f(y).$$

Definition. A differentiable functions f defined on an open set $X \subseteq R^n$ is called $\eta-$*pseudolinear* if f and $-f$ are pseudo-invex with respect to the same η.

Theorem. Let f be a differentiable function defined on an open set $X \subseteq R^n$ and K be an invex subset of X such that $\eta : K \times K \to R^n$ satisfies Condition C. Suppose that f is $\eta-$ pseudolinear on K. The for all $x, y \in K$, $\eta(x,y)^T \nabla f(y) = 0$ if and only if $f(x) = f(y)$.

Definition. The function $\eta : K \times K \to R^n$ defined on the invex set $K \subseteq R^n$ satisfies Condition C (Mohan and Neogy (1995)), if for evey $x,\, y \in K$:

$$\eta\left(y, y+\eta(x,y)\right) = -\lambda \eta(x,y) \text{ and}$$

$$\eta\left(x, y+\eta(x,y)\right) = \left(1-\lambda\right)\eta(x,y) \text{ for all } \lambda \in \left[0,1\right].$$

Theorem. Let f be a differentiable function defined on an open set $X \subseteq R^n$ and K an invex subset of X with respect to η. Then f is $\eta-$pseudolinear on K if and only if there exists a function p defined on $K \times K$ such that $p(x, y) > 0$ and

$$f(x) = f(y) + \left(p(x, y)\eta(x, y)\right)^T \nabla f(y), \quad \forall \ x, y \in K.$$

A vector variational-like inequality problem (VVLIP), is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $F(\bar{x})\eta(y,\bar{x}) \leq 0$.

A weak vector variational-like inequality problem (WVVLIP), is to find a point $\bar{x} \in X$, such that there exists no $y \in X$, such that $F(\bar{x})\eta(y,\bar{x}) < 0$.

Example 3.1. Consider the problem

$$\text{(VOP)} \quad V - \min f(x)$$
$$\text{subject to } x \in [-1, 0]$$

where $f(x) = (x, x^2)^t$.

It is clear that every $x \in [-1, 0]$ is an efficient solution.

Let $x = 0$, then $\exists y = -1$, such that

$$\nabla f(x)(y - x) = \left(f_1'(x)(y - x), f_2'(x)(y - x)\right)$$
$$= (-1, 0)^t \leqslant (0, 0)^t.$$

Thus $x = 0$ is not a solution to (VVIP).

Let $f : R^n \to R^p$, the vector optimization problem (VOP)

is to find the *efficient points* for

(VOP) $V - \min f(x)$

subject to $x \in X$.

Theorem. Let $f: X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f, f$ is invex with respect to η and $\bar{x}$ solves the generalized vector variational-like inequality problem (VVLIP) with respect to the same η, then $\bar{x}$ is an efficient point to the vector optimization problem (VOP).

Theorem. Let $f: X \subset R^n \to R^p$ be differentiable function on X. If $F = \nabla f, -f$ is strictly-invex with respect to η. If $\bar{x}$ is a weakly efficient solution to the vector optimization problem (VOP) then $\bar{x}$ also solves the generalized vector variational-like inequality problem (VVLIP).

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THE LE CHATELIER PRINCIPLE IN INVEX PROGRAMMING

In 1884 the French chemist formulated a very nice principle regarding the interaction of parameters and variables. We can do no better than cite this principle the way it has been stated in the Eichhorn and Oettli [7] paper. We quote ([7], page 711):

If a system is in stable equilibrium and one of the conditions is changed, then the equilibrium will shift in such a way as to tend to annul the applied change in the conditions

Consider a maximization problem:

(P) Max $f(x)$

$$X \equiv \left\{ x \in X^0 : h(x) \leqq b \right\},$$

where $X^0 \subset R^n$ is an open convex set; $f : R^n \to R$ is a incave scalar function; $h : R^n \to R^m$ is invex vector function; $b \in R^m$.

Let x be a vector of production levels, $f(x)$ the firm's objective function and b a vector of available resources and the function $h(x)$ the vector of resource use. In this set up the variations in b are analyzed by the Le Chatelier Principle.

Kuhn-Tucker necessary optimality conditions

If x^* is an optimal solution for (P) then there exists a $v^* \in R^m$,

$v_i^* \geqq 0$, such that

$$\nabla f\left(x^*\right) - \sum_{i=1}^{m} v_i^* \nabla h_i\left(x^*\right) = 0,$$

$$v_i^*\left(b_i - h_i\left(x^*\right)\right) = 0, \ i = 1, \ldots, m,$$

$$h_i\left(x^*\right) \leqq b_i, \ i = 1, \ldots, m.$$

Theorem (Kuhn-Tucker Sufficient Optimality Conditions):

Let x^* be a feasible solution for the maximization problem. Suppose that f is incave and each h_i for $i=1,\dots,m$, is invex with respect to the same η at x^* and there exists $v^* \in R^m$, $v_i^* \geqq 0$, such that

$$\nabla f\left(x^*\right) - \sum_{i=1}^{m} v_i^* \nabla h_i\left(x^*\right) = 0,$$

$$v_i^*\left(b_i - h_i\left(x^*\right)\right) = 0, \; i=1,\dots,m,$$

$$h_i\left(x^*\right) \leqq b_i, \; i=1,\dots,m.$$

Then x^* is an optimal solution for (P).

Theorem (Kuhn-Tucker Sufficient Optimality Conditions):

Let x^* be a feasible solution for the maximization problem. Suppose that f is pseudo-incave and each h_i for $i = 1, \ldots, m$, is quasi-invex with respect to the same η at x^* and there exists $v^* \in R^m$, $v_i^* \geqq 0$, such that necessary conditions given in above theorem hold. Then x^* is an optimal solution for (P).

Now recall the Lagrangian function for (P):

$$L(x,v) \equiv f(x) + \sum_{i=1}^{m} v_i \left[b_i - h_i(x) \right].$$

Using one of the above Theorems of sufficiency, one can prove that: x^* solves (P)-assuming the Slater regularity condition $h\left(x^0\right) < b$, for some $x^0 \in X^0$ if and only if there exists $v^* \in R^m, v^* \geqq 0$ such that $\left(x^*, v^*\right)$ is a saddle point of the Lagrangian $L(x,v) \equiv f(x) + \sum_{i=1}^{m} v_i \left[b_i - h_i(x) \right]$.

Note: $\left(x^*, v^*\right)$ is a saddle point of the Lagrangian

$$L(x, v) \equiv f(x) + \sum_{i=1}^{m} v_i \left[b_i - h_i(x)\right],$$

That is,

$$f(x) + v^*(b - h(x)) \leqq f\left(x^*\right) + v^*\left(b - h\left(x^*\right)\right) \leqq f\left(x^*\right) + v\left(b - h\left(x^*\right)\right)$$

for all $x \in X^0, v \geqq 0$.

Clearly, the second inequality holds if and only if

$$v^*\left(b - h\left(x^*\right)\right) = 0.$$

We consider a problem $(\overline{\mathrm{P}})$, where $\bar{b}$ has been substituted for b,

so that

$(\overline{\mathrm{P}})$ Max $f(x)$

$$\bar{X} \equiv \left\{ x \in X^0 : h(x) \leqq \bar{b} \right\}.$$

Let $(\bar{x}, \bar{v})$ be a saddle point of $(\overline{\mathrm{P}})$ so that

$$f(x) + \bar{v}\left(\bar{b} - h(x)\right) \leqq f(\bar{x}) + \bar{v}\left(\bar{b} - h(\bar{x})\right) \leqq f(\bar{x}) + v\left(\bar{b} - h(\bar{x})\right)$$

for all $x \in X^0, v \geqq 0$.

Again, the second inequality holds if and only if

$$\bar{v}\left(\bar{b} - h(\bar{x})\right) = 0.$$

Theorem:

Let f be incave (pseudo-incave)and h invex (quasi-invex). If there is a saddle point $\left(x^*, v^*\right)$ for (P) and a saddle point $(\bar{x}, \bar{v})$ for $(\bar{\mathrm{P}})$, then $\Delta v \Delta b \leqq 0$.

Here, $\Delta b = \bar{b} - b$, $\Delta v = \bar{v} - v^*$.

Remark:

The above Theorem is a generalized version of the Le Chatelier Principle as it now applies to a wider class of problems. The Principle I of Leblanc and Moeseke (1976) is a particular case of the above Theorem by setting $\eta(x, y) = x - y$.

Thank You