International Journal of Operations Research

International Journal of Operations Research Vol. 10, No. 2, 100−108 (2013)

Measuring Inconsistency of Pair-wise Comparison Matrix with Fuzzy Elements

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Received February 2013; Revised April 2013; Accepted May 2013

Abstract — This paper deals with measuring inconsistency and incompatibility of pair-wise comparison matrix with fuzzy elements. Here we deal with some properties of such pair-wise comparisons, particularly, reciprocity consistency and compatibility. Moreover, we show how to measure it defining two new indices. The first index of inconsistency is based on the classical concept of consistency, however, instead of principle eigenvector method used in AHP we apply the logarithmic least squares method. Defining the second index of incompatibility we look for a consistent matrix in the form of the ratio matrix with the maximal membership grade "as close as possible" to the original fuzzy matrix. This leads to solving a nonlinear optimization problem which can be transformed to a sequence of LP ones. We compare properties and application areas of both indices. The first index FI is suitable for non-interactive elements of fuzzy matrices, particularly, when uncertainty of individual elements of the matrix can be reflected/measured. The second index GI is appropriate for interactive elements of fuzzy matrices. By GI a measure of compatibility of fuzzy matrix with the closest consistent matrix is expressed. Illustrating examples and simulations are supplied to characterize the concepts and derived properties.

Keywords — data analysis; decision making; uncertainty; pair-wise comparison; inconsistency; incompatibility

1. INTRODUCTION

Pair-wise comparison is a popular method for solving DM problems of finding the best alternative among more than 2 ones. This method is frequently used when ranking alternatives, evaluating the relative importance of the individual criteria in MCDM problem and/or when evaluating alternatives according to qualitative criteria, e.g. design, taste, likeness, etc.

The core of the method is how to aggregate the results into the final prioritization or ranking. In this paper we deal with some properties of such pair-wise comparisons, particularly with consistency of pair-wise comparisons. Consistency means that if an element i is a-times better than element j and element j is b-times better than element k, then i is a.b-times better than k. If this property is valid for all compared elements i, j and k, we say that the matrix of pair-wise comparisons is consistent. If, at least for one triple of elements, the property is not satisfied, then the matrix is inconsistent. It is important to measure intensity of inconsistency as in some cases the pair-wise comparison matrix can be "close" to a consistent matrix, in the other ones inconsistency can be strong, meaning that there exist numerous inconsistent triples eventually with large differences between corresponding values. For crisp pair-wise comparison matrices there exists numerous inconsistency measurement methods (indices), see e.g. Aguaron (2003), Koczkodaj (1993), Gass (2004), Stein (2007), and others. However, in these papers it was proven that all the consistency indices are linear or nonlinear transformations of the original Saaty's consistency ratio (Saaty, 1991). Moreover, these indices cannot be directly used for measuring consistency of a matrix with fuzzy elements. The earliest work in AHP using fuzzy sets as data was published by van Laarhoven and Pedrycz (1983). They compared fuzzy ratios described by triangular membership functions. The method of logarithmic least squares was used to derive local fuzzy priorities. Later on, using a geometric mean, Buckley *et al.* (1985) determined fuzzy priorities of comparison ratios whose membership functions were assumed trapezoidal. The issue of consistency in AHP using fuzzy sets as elements of the matrix was first tackled by Salo (1996). Departing from the fuzzy arithmetic approach, fuzzy weights using an auxiliary mathematical programming formulation describing relative fuzzy ratios as constraints on the membership values of local priorities were derived. Later on Leung and Cao (2000) proposed a notion of tolerance deviation of fuzzy relative importance that is strongly related to Saaty's consistency ratio.

Instead of principal eigenvector method used in classical AHP, here, we use the logarithmic least squares method, see Van Laarhoven, (1983) and Ramik (2010). The first index – inconsistency index – is based on the logarithmic least squares

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function. Defining the second index – incompatibility index - we look for a consistent crisp matrix in the form of the ratio matrix with the maximal membership grade "as close as possible" to the original fuzzy matrix. This leads to solving a nonlinear optimization problem which can be transformed to a sequence of LP ones. Recently, in the literature we can find papers dealing with applications of pair-wise comparison method where evaluations are performed by fuzzy values, particularly, evaluating regional projects, web pages, e-commerce proposals etc.

This paper is organized as follows. In Section 2 the basic concept of Triangular Fuzzy Positive Reciprocal (TFPR) matrix is introduced. In Section 3, a general concept of inconsistency index FI of TFPR matrix based on the metric function is defined, in Section 4, the fuzzy logarithmic LSQ method is introduced and applied. Section 5, deals with the incompatibility index GI of TFPR matrix. Finally, in Section 6, illustrating examples and simulation experiments are supplied in order to characterize the concepts and derived properties. Finally, in Section 7, concluding remarks of the paper are presented.

2. TRIANGULAR FUZZY POSITIVE RECIPROCAL MATRIX

Triangular fuzzy numbers are suitable for modeling uncertain values by DMs in practice. A *triangular fuzzy number* $\tilde{a}$ can be equivalently expressed by a triple of real numbers, i.e. $\tilde{a} = (a^L; a^M; a^U)$, where a^L is the *Lower number*, a^M is the *Middle number*, and a^U is the *Upper number*, $a^L \leq a^M \leq a^U$. If $a^L = a^M = a^U$, then $\tilde{a}$ is said to be the *crisp number* (non-fuzzy number). Evidently, the set of all crisp numbers is isomorphic to the set of real numbers. If $a^L \neq a^M \neq a^U$, then the *membership function* of is supposed to be continuous, strictly increasing in the interval $[a^L, a^M]$ and strictly decreasing in $[a^M, a^U]$. Moreover, the *membership grade* is equal to zero for $x \notin [a^L, a^U]$ and equal to one for $x = a^M$. As usual, the membership function is assumed to be piece-wise linear, see Figure 1, where the evaluation "moderately more important" is expressed by the triangular fuzzy number on the scale $S = [1/9, 9]$. If $a^L = a^M$ and/or $a^M = a^U$, then the membership function is discontinuous. It is well known that the arithmetic operations +, -, * and / can be extended to fuzzy numbers by the Extension principle, see e.g. Buckley (2001).

The elements of *Triangular Fuzzy Positive Reciprocal (TFPR) matrix* are positive triangular fuzzy numbers $\tilde{a}_{ij} = (a_{ij}^L; a_{ij}^M; a_{ij}^U)$, where $a_{ij}^L > 0$ and

$$\tilde{A} = \begin{bmatrix} (1; & 1; & 1) & (a_{12}^L; a_{12}^M; a_{12}^U) & \cdots & (a_{1n}^L; a_{1n}^M; a_{1n}^U) \\ (\frac{1}{a_{12}^U}; \frac{1}{a_{12}^M}; \frac{1}{a_{12}^L}) & (1; \quad 1; \quad 1) & \cdots & (a_{2n}^L; a_{2n}^M; a_{2n}^U) \\ \vdots & \vdots & \ddots & \vdots \\ (\frac{1}{a_{1n}^U}; \frac{1}{a_{1n}^M}; \frac{1}{a_{1n}^L}) & (\frac{1}{a_{2n}^U}; \frac{1}{a_{2n}^M}; \frac{1}{a_{2n}^L}) & \cdots & (1; \quad 1; \quad 1) \end{bmatrix}. \tag{1}$$

Arithmetic operations with fuzzy numbers are defined as follows, see Buckley (1986) or Ramik (2010). Let $\tilde{a} = (a^L; a^M; a^U)$ and $\tilde{b} = (b^L; b^M; b^U)$, where $a^L > 0$, $b_i^L > 0$, be positive triangular fuzzy numbers. We define the following arithmetic operations:

Addition: $\tilde{a} \tilde{+} \tilde{b} = (a^L + b^L; a^M + b^M; a^U + b^U)$;

Subtraction: $\tilde{a} \tilde{-} \tilde{b} = (a^L - b^U; a^M - b^M; a^U - b^L)$;

Multiplication: $\tilde{a} \tilde{*} \tilde{b} = (a^L * b^L; a^M * b^M; a^U * b^U)$;

Division: $\tilde{a} \tilde{/} \tilde{b} = (a^L / b^U; a^M / b^M; a^U / b^L)$.

Particularly: $\dfrac{\tilde{1}}{a} = \left(\dfrac{1}{a^U}; \dfrac{1}{a^M}; \dfrac{1}{a^L} \right)$

It should be noted, that for triangular fuzzy numbers with piece-wise linear membership functions the above formulae for multiplication (and also for division) are not obtained by the Extension principle. They are only approximate ones, e.g. multiplication is a triangular fuzzy number with piece-wise linear membership function, whereas the membership function of the exact operation defined by the Extension principle would be non-linear. For using matrices with triangular fuzzy elements there exist at least following reasons:

- The membership function of triangular fuzzy elements is piece-wise linear, i.e. it is easy to understand.
- Triangular fuzzy numbers can be easily manipulated, e.g. added, multiplied.
- Crisp (non-fuzzy) numbers are special cases of triangular fuzzy numbers.
- The TFRP matrix can be considered by the DM as a model for his/her fuzzy pair-wise preference representations concerning *n* elements (e.g. alternatives). In this model, it is assumed that only $n(n-1)/2$ judgments are needed, the rest is given by reciprocity condition.
- In practice, when interval-valued matrices are employed, the DM often gives ranges narrower than his or her actual perception would authorize, because he/she might be afraid of expressing information which is too imprecise. On the other hand, triangular fuzzy numbers express rich information because the DM provides both the support set of the fuzzy number as the range that the DM believes to surely contain the unknown ratio of relative importance, and the grades of possibility of occurrence (i.e. membership function) within this range.
- Triangular fuzzy numbers are appropriate in group decision making where a^L can be interpreted as the minimum possible value of DMs judgments, a^U is interpreted as the maximum possible value of DMs judgments, and a^M – the geometric mean of the DMs judgments is interpreted as the mean value, or, the most possible value of DMs judgments, see Buckley (1985).

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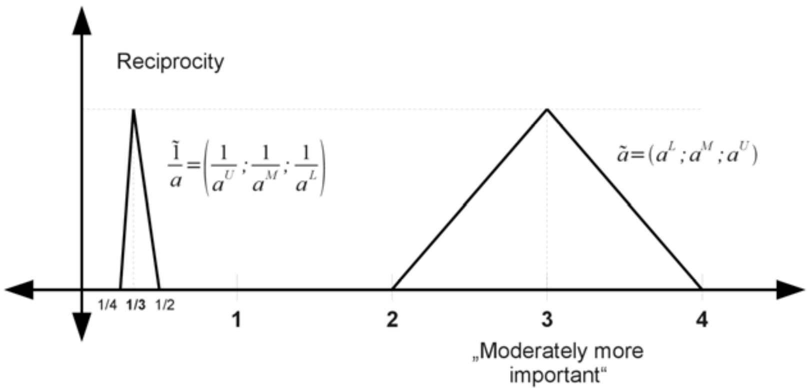

Figure 1. Elements of Triangular Fuzzy Positive Reciprocal (TFRP) matrix

3. INCONSISTENCY INDEX FI OF TFPR MATRIX

Construction of a new measure - inconsistency index of the reciprocal matrix with triangular fuzzy elements is based on the idea of distance of the TFPR matrix to the "ratio" matrix measured by a particular metric function, see Ramik (2010).

Let M be a set of $n\times n$ matrices with triangular fuzzy elements, and let Φ be a real function defined on M×M, i.e. Φ : M×M $\rightarrow$ **R** satisfying the 3 assumptions:

(i) $\Phi(\tilde{A},\tilde{B}) \geq 0$ for all $\tilde{A},\tilde{B} \in$ M.

(ii) If $\Phi(\tilde{A},\tilde{B}) = 0$ then $\tilde{A} = \tilde{B}$.

(iii) $\Phi(\tilde{A},\tilde{B}) + \Phi(\tilde{B},\tilde{C}) \geq \Phi(\tilde{A},\tilde{C})$ for all $\tilde{A},\tilde{B},\tilde{C} \in$ M.

Then Φ is called the *metric function on* M.

Let $\tilde{X} = \{\frac{\tilde{x}_i}{\tilde{x}_j}\}$, *i,j* = 1, 2,...,*n*, be a *ratio matrix* with positive triangular fuzzy numbers $\tilde{x}_k = (x_k^L; x_k^M; x_k^U)$. The vector of positive triangular fuzzy numbers $\tilde{x} = (\tilde{x}_1,\ldots,\tilde{x}_n)$ is called the *vector of fuzzy weights* if

$$\sum_{k=1}^{n} x_k^M = 1 . \tag{2}$$

Given $\tilde{A} = \{\tilde{a}_{ij}\}$ - $n\times n$ reciprocal matrix with triangular fuzzy elements (n>2), where the support $supp(\tilde{a}_{ij}) \subset S$, $S = [1/\sigma, \sigma]$, $\sigma > 1$, $\tilde{a}_{ij} = (a_{ij}^L; a_{ij}^M; a_{ij}^U)$, i,j = 1, 2,...,n, and let Φ and Ψ be metric functions on M.

The new inconsistency index $FI(\tilde{A})$ of $\tilde{A} = \{\tilde{a}_{ij}\}$ is designed in two steps (Ramik, 2010):

Step 1: Solve the following optimization problem:

$$\Phi(\tilde{A}, \tilde{X}) \to \min; \tag{3}$$

subject to

$$\sum_{k=1}^{n} x_k^M = 1 ,\; x_k^U \geq x_k^M \geq x_k^L \geq 0 ,\; k = 1, 2,...,n. \tag{4}$$

Step 2: Set the *inconsistency index* I_n of $\tilde{A}$ as

$$I_n(\tilde{A}) = \inf\{ \Psi(\tilde{A}, \tilde{W});\; \tilde{W} \text{ - optimal solution of (3), (4)}\}. \tag{5}$$

Remark. In Ramik (2010), Ramik and Korviny defined Φ and Ψ as follows:

$$\Phi(\tilde{A}, \tilde{X}) = \sum_{i,j} \max\left\{\left(\log\frac{x_i^L}{x_j^L} - \log a_{ij}^L\right)^2, \left(\log\frac{x_i^M}{x_j^M} - \log a_{ij}^M\right)^2, \left(\log\frac{x_i^U}{x_j^U} - \log a_{ij}^U\right)^2\right\}, \tag{6}$$

$$\Psi(\tilde{A}, \tilde{W}) = \gamma.\max_{i,j}\left\{\max\{|\frac{w_i^L}{w_j^U} - a_{ij}^L|, |\frac{w_i^M}{w_j^M} - a_{ij}^M|, |\frac{w_i^U}{w_j^L} - a_{ij}^U|\}\right\}, \tag{7}$$

where γ is a normalizing constant. Here, in Step 1 and Step 2, in contrast to Ramik (2010), we apply metric function Φ according to (6) and we set $\Psi = \Phi$.

Here, *inconsistency index* of TFPR matrix $\tilde{A}$ is defined by (5) as follows:

$$FI(\tilde{A}) = \sum_{i,j} \max\left\{\left(\log\frac{x_i^L}{x_j^L} - \log a_{ij}^L\right)^2, \left(\log\frac{x_i^M}{x_j^M} - \log a_{ij}^M\right)^2, \left(\log\frac{x_i^U}{x_j^U} - \log a_{ij}^U\right)^2\right\}. \tag{8}$$

This definition was recommended also in Brunelli (2011). A TFRP matrix is said to be *F-consistent* if = 0. Clearly, this new index (8) is identical to the former inconsistency index, for crisp matrices all three parts within the inner parentheses in (8) coincide.

4. FUZZY LOGARITHMIC LSQ METHOD

Instead of principle eigenvector method used in classical AHP, here, we use the logarithmic least squares method modified for fuzzy values. The advantage of the logarithmic least squares method over the classical least squares method is the symmetry of values from the scale [1, 9] and the reciprocal values from [1/9, 1].

Let be a TFPR matrix. Now, solve the optimization problem:

$$\sum_{i,j} \max\left\{\left(\log\frac{x_i^L}{x_j^L} - \log a_{ij}^L\right)^2, \left(\log\frac{x_i^M}{x_j^M} - \log a_{ij}^M\right)^2, \left(\log\frac{x_i^U}{x_j^U} - \log a_{ij}^U\right)^2\right\} \to \min; \tag{9}$$

subject to

$$x_k^U \geq x_k^M \geq x_k^L \geq 0 ,\; k = 1,2,...,n. \tag{10}$$

Theorem 1. Let $w_k^S = \left(\prod_{j=1}^{n} a_{kj}^S\right)^{1/n}$, $S\in\{L,M,U\}$ and k=1,2,…,n, Then $\tilde{w}_k = (w_k^L; w_k^M; w_k^U)$, k=1,2,…,n, is an optimal solution of problem (9), (10).

The proof is easy, see Ramik (2010).

Notice, that optimal solution of problem (9), (10): , k =1,2,...,n, can be used for ranking the alternatives and/or criteria in MCDM problem.

The following properties of inconsistency index $FI(\tilde{A})$ of TFPR matrix $\tilde{A}$ follow directly from Theorem 1:

(i) $0 \le FI(\tilde{A})$.

(ii) If $FI(\tilde{A}) = 0$, i.e. $\tilde{A}$ is F-consistent, then $\tilde{A}$ is crisp (i.e. nonfuzzy).

Now, we will show that if fuzziness of the elements of a TFPR matrix increases then the inconsistency index *FI* of this matrix increases, too.

Let $\tilde{A} = \{\tilde{a}_{ij}\} = \{(a_{ij}^L; a_{ij}^M; a_{ij}^U)\}$ be a FPR matrix and let $\delta \ge 1$ and let $\tilde{A}_\delta = \{(a_{ij}^L \frac{1}{\delta}; a_{ij}^M; a_{ij}^U \delta)\}$ be called *δ-fuzzification of matrix* $\tilde{A} = \{\tilde{a}_{ij}\}$.

Now, we show that if fuzziness of the elements of the matrix $\tilde{A}_\delta$ increases then the inconsistency index $FI(\tilde{A}_\delta)$ increases, too. In order to show this property, we need the following result.

Proposition 1. Let $\delta \ge 1$ and $w_i = w_i^L = w_i^M = w_i^U$, i = 1,2,...,n, be an optimal solution of (9), (10) with respect to crisp matrix $A = \{a_{ij}\}$, $FI(A) = \sum_{i,j=1}^{n} \left(\log \frac{w_i}{w_j} - \log a_{ij} \right)^2$. Then the inconsistency index *FI* of δ-fuzzification of matrix $A = \{a_{ij}\}$ satisfies the following formula:

$$FI(\tilde{A}_\delta) = 2\sum_{i,j=1}^{n} \left(|\log \frac{w_i}{w_j} - \log a_{ij}| + \log \delta \right)^2 \tag{11}$$

The proof of formula (11) follows directly from Theorem 1, as ,for i,j = 1, 2,...,n, $i \ne j$, and , for i=1,2,...,n. Now, it follows easily from (11) that inconsistency index $FI(\tilde{A}_\delta)$ increases with increasing $\delta \ge 1$ as $FI(\tilde{A}_\delta) = 2\sum_{i,j=1}^{n} \left(|\log \frac{w_i}{w_j} - \log a_{ij}| + \log \delta \right)^2$ $\log\delta$ increases with increasing δ.

5. RESULTS AND ANALYSIS

Here we look for a (crisp) consistent matrix in the form of the ratio matrix with the maximal membership grade i.e. "as close as possible" to the original fuzzy matrix. We define an incompatibility index *GI* based on Ohnishi (2008). Later on, we will show that *GI* has different behavior when comparing to *FI*. Again, let $\tilde{A} = \{\tilde{a}_{ij}\} = \{(a_{ij}^L; a_{ij}^M; a_{ij}^U)\}$ be a TFPR matrix, where μ_{ij} is a membership function of $\tilde{a}_{ij}$. Let

$$G(w_1, \dots, w_n) = \min \{\mu_{ij}\left(\tfrac{w_i}{w_j}\right) | 1 \le i, j \le n\}. \tag{12}$$

Then $GI(\tilde{A})$, called the *incompatibility index of* , is defined as

$$GI(\tilde{A}) = 1 - \max \left\{ G(w_1, \dots, w_n) | \sum_{j=1}^{n} w_j = 1, w_i \ge 0, i = 1, 2, \dots, n \right\}. \tag{13}$$

Theorem 2. Let be a TFPR matrix.
Then $GI(\tilde{A}) = 1 - x_0^*$, where is an optimal solution of the following optimization problem:

$$x_0 \to \max; \tag{14}$$

subject to

$$a_{ij}^L + x_0(a_{ij}^M - a_{ij}^L) \le \frac{x_i}{x_j} \le a_{ij}^U - x_0(a_{ij}^U - a_{ij}^M), \tag{15}$$

$$0 \le x_0 \le 1, \quad x_k \ge 0, \quad 1 \le i, j, k \le n\,. \tag{16}$$

The proof of Theorem 2 is easy and it follows directly from (12) and (13).

Optimization problem (14) – (16) is nonlinear in variables $x_0, x_1, \ldots, x_n$, however, it can be transformed into a sequence of LP problems which can be solved by "dichotomy" method, see Ohnishi (2008).

The following two properties of incompatibility index $GI(\tilde{A})$ of TFPR matrix $\tilde{A}$ follow directly from (12) – (16):

(i) $0 \le GI(\tilde{A}) \le 1$.

(ii) If $GI(\tilde{A}) = 0$, then $FI(\tilde{A}) = 0$, i.e. $\tilde{A}$ is F-consistent.

(iii) There exists a TFPR non-crisp matrix $\tilde{A}^*$ with $GI(\tilde{A}^*) = 0$.

Now, we will prove the following property: Incompatibility index $GI(\tilde{A})$ could decrease when fuzziness of the elements of the matrix would increase.

Let be a TFPR matrix, and let $\tilde{A}_\delta = \{(a^L_{ij}\frac{1}{\delta}; a^M_{ij}; a^U_{ij}\delta)\}$ be δ-fuzzification of matrix $\tilde{A} = \{\tilde{a}_{ij}\}$, $\delta \ge 1$. Notice that .

Let optimization problem (P_δ) be as follows:

(P_δ)

subject to

$$x_0 \to \max; \tag{17}$$

$$a^L_{ij}\frac{1}{\delta} + x_0(a^M_{ij} - a^L_{ij}\frac{1}{\delta}) \le \frac{x_i}{x_j} \le a^U_{ij}\delta - x_0(a^U_{ij}\delta - a^M_{ij}), \tag{18}$$

$$\sum_{j=1}^{n} x_j = 1\,, \tag{19}$$

$$0 \le x_0 \le 1, \quad x_k \ge 0, \quad 1 \le i, j, k \le n \tag{20}$$

Proposition 2. Let $0 < \delta_1 < \delta_2$ and let $x = (x_0, x_1, \ldots, x_n)$ be a vector of feasible solution of (P_δ), i.e. x_k satisfies (17)-(20) with $\delta = \delta_1$. Then is a feasible solution of (P_{δ_2}), i.e. x_k satisfies (17)–(20) with $\delta = \delta_2$.

Proof: To prove the proposition it is sufficient to show that for all $i,j = 1,2,\ldots,n$

$$a^L_{ij}\frac{1}{\delta_2} + x_0(a^M_{ij} - a^L_{ij}\frac{1}{\delta_2}) \le a^L_{ij}\frac{1}{\delta_1} + x_0(a^M_{ij} - a^L_{ij}\frac{1}{\delta_1}), \tag{21}$$

and

$$a^U_{ij}\delta_1 + x_0(a^U_{ij}\delta_1 - a^M_{ij}) \le a^U_{ij}\delta_2 + x_0(a^U_{ij}\delta_2 - a^M_{ij})\,. \tag{22}$$

Evidently, (21) is equivalent to the following inequality:

$$0 \le a^L_{ij}\left(\frac{1}{\delta_1} - \frac{1}{\delta_2}\right)(1 - x_0)\,. \tag{23}$$

Similarly, (22) is equivalent to

$$0 \le a^U_{ij}\left(\delta_2 - \delta_1\right)(1 - x_0)\,. \tag{24}$$

As $0 < \delta_1 < \delta_2$, $0 \le x_0 \le 1$, $0 \le a^L_{ij}$, and $0 \le a^U_{ij}$, inequations (23), (24) are satisfied for all $i,j = 1,2,\ldots,n$, q.e.d.

From Proposition 2 it is clear that if x_0 is the maximum value of the objective function of and is the maximum value of the objective function of , and $0 \le \delta_1 \le \delta_2$, then $x_0 \le x^*_0$. Moreover, $GI(\tilde{A}_{\delta_1}) = 1 - x_0$ and $GI(\tilde{A}_{\delta_2}) = 1 - x^*_0$, then . Consequently, is non-increasing function of $\delta \ge 1$. Comparing this result to that of Proposition 1, inconsistency index $FI(\tilde{A}_\delta)$ increases with increasing $\delta \ge 1$.

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6. EXAMPLES AND SIMULATIONS

In this section we present some illustrating examples showing that the new inconsistency/incompatibility indices are convenient tools not only for measuring consistency/compatibility of pair-wise comparison matrices with fuzzy elements, but also for measuring consistency/compatibility of crisp pair-wise comparison matrices. The last example is a simulation case study comparing the both presented indices.

Example 1.

Consider the following TFPR (crisp) matrix:

$$\tilde{A} = \begin{bmatrix} (1;1;1) & (2;2;2) & (6;6;6) \\ (\frac{1}{2};\frac{1}{2};\frac{1}{2}) & (1;1;1) & (3;3;3) \\ (\frac{1}{6};\frac{1}{6};\frac{1}{6}) & (\frac{1}{3};\frac{1}{3};\frac{1}{3}) & (1;1;1) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ \frac{1}{2} & 1 & 3 \\ \frac{1}{6} & \frac{1}{3} & 1 \end{bmatrix}$$

Let $S = [^1/_9, 9]$ be the scale. Then by Theorem 1 we calculate: $\tilde{w}_1 = (w_1^L; w_1^M; w_1^U)$ =(2.289; 2.289; 2.289), $\tilde{w}_2 = (w_2^L; w_2^M; w_2^U)$ =(1.145; 1.145; 1.145), $\tilde{w}_3 = (w_3^L; w_3^M; w_3^U)$ =(0.382; 0.382; 0.382). Consequently, $FI(\tilde{A}) = 0$, then $\tilde{A}$ is F-consistent .

Example 2.

Consider the following TFPR (fuzzy) matrix:

$$\tilde{B} = \begin{bmatrix} (1;1;1) & (1;3;4) & (4;5;6) \\ (\frac{1}{4};\frac{1}{3};1) & (1;1;1) & (3;4;5) \\ (\frac{1}{6};\frac{1}{5};\frac{1}{3}) & (\frac{1}{5};\frac{1}{4};\frac{1}{3}) & (1;1;1) \end{bmatrix}.$$

The scale is $S = [^1/_9, 9]$ and by Theorem 1 we calculate: $\tilde{w}_1 = (w_1^L; w_1^M; w_1^U)$ =(1.587; 2.466; 2.884), $\tilde{w}_2 = (w_2^L; w_2^M; w_2^U)$ =(0.909; 1.101; 1.710), $GI(\tilde{B}) = 0.511$.

Example 3. Consider the following TFPR (fuzzy) matrix

$$\tilde{A}_\delta = \begin{bmatrix} (1;1;1) & (\frac{2}{\delta}; 2; 2\delta) & (\frac{a_{13}}{\delta}; a_{13}; a_{13}\delta) \\ (\frac{1}{2\delta};\frac{1}{2};\frac{\delta}{2}) & (1;1;1) & (\frac{3}{\delta}; 3; 3\delta) \\ (\frac{1}{a_{13}\delta};\frac{1}{a_{13}};\frac{\delta}{a_{13}}) & (\frac{1}{3\delta};\frac{1}{3};\frac{\delta}{3}) & (1;1;1) \end{bmatrix},$$

where $\delta \in [1,9]$.

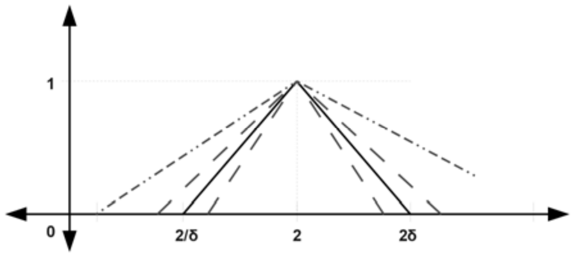

Figure 2. δ -fuzzification of element "2"

Comparison of *FI* and *GI* index for TFPR matrix $\tilde{A}_\delta$:

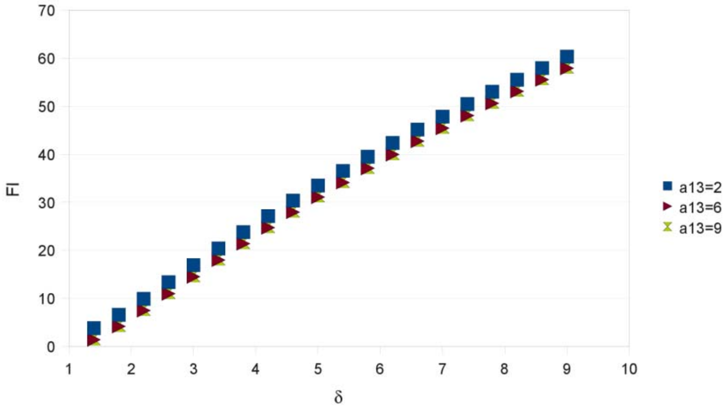

Figure 3. $FI(\delta)$

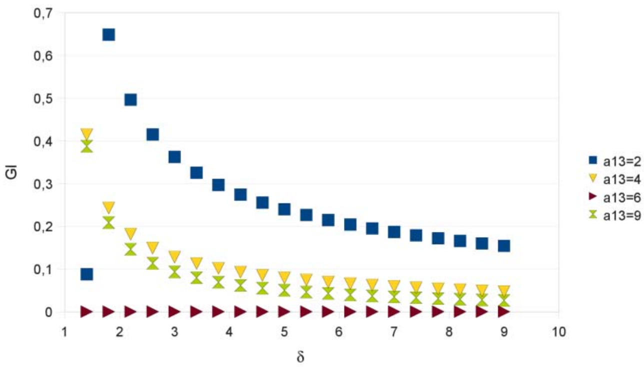

Figure 4. $GI(\delta)$

It is clear that the results depicted in Figure 3 and Figure 4 confirm the conclusions of Propositions 1 and 2.

7. CONCLUSION

When comparing *FI* and *GI* we can give the following recommendation:

- *FI* is suitable for non-interactive elements of fuzzy matrices;
- *GI* is appropriate also for interactive elements of fuzzy matrices;
- by *FI* uncertainty of individual elements of the matrix can be reflected/measured;
- by *GI* a measure of compatibility of fuzzy matrix with the closest consistent matrix is expressed.

Consistency itself is a necessary condition for a better understanding of relations between elements in MCDM. In MCDM every criterion may have its own inconsistency/incompatibility index. Then inconsistency/incompatibility of the whole problem is given as maximum of the individual inconsistencies/incompatibilities.

ACKNOWLEDGMENT

This research has been supported by GACR Project No. 402090405.

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