International Journal of Operations Research

International Journal of Operations Research Vol. 10, No. 3, 134–152 (2013)

A Batch Arrival Unreliable Server Bernoulli Vacation Queue with Two Phases of Service and Delayed Repair

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Received June 2013; *Revised August 2013*; *Accepted September 2013*

Abstract— This paper deals with batch arrival unreliable queue with two phases of service and vacation under Bernoulli vacation schedule, which consist of a breakdown period and a delay period. For this model, we first derive the joint distribution of state of the server and queue size, which is one of the chief objectives of this paper. Secondly, we derive the pgf of the stationary queue size distribution at a departure epoch. Next, we derive the Laplace Stieltjes transform of busy period distribution and waiting time distribution. Finally, we obtain some important performance measures and reliability indices of this model

***Keywords*—** Stationary queue size distribution, random breakdown, repair time, delay time, Bernoulli vacation and Reliability index.

1. Introduction

The study of queueing models with service breakdowns or some other kind of interruptions dates back to 1950s. Among some early papers in this area, we refer the readers to the papers by White and Christie (1958), Heathcote (1959), Keilson (1962), Gaver (1962), Avi-Itzhak and Naor (1963), Thirurengadan (1963) and Mitrany and Avi-Itzhak (1968) for some fundamental works. Sengupta (1990), Ibe and Trivedi (1990), Li *et al.* (1997), Tang (1997), Takin and Sengupta (1998), Madan (2003), Li and Lin (2006), Fiems *et al.* (2008), Krishnamoorthy et al. (2009), and others have studied some queueing systems with interruptions wherein one of the underlying assumptions is that as soon as the service channel fails, it instantaneously undergoes repairs. All these papers assume that the server is immediately repaired upon failure. However, in many real-life situations, it may not be feasible to start the repairs immediately due to the non-availability of the server, in which case the system may also be turned off.

As related works, we should mention the paper of Chao (1995) which dealt with queueing systems in which a catastrophe or disaster removes all the work present in the system. A disaster can be viewed as a general breakdown which causes all the jobs in the system to be lost. For example, if a public telephone breaks down, all the customers in the waiting line leave the telephone booth. This type of model has many applications in day-to-day life. Another typical example is the distributed database system with site failure considered by Towsley and Tripathi (1991).

The classical vacation scheme with Bernoulli service discipline was initiated and developed significantly by Keilson and Servi (1986) and co-workers. Kella (1990) suggested a generalized Bernoulli scheme according to which a single server goes on k consecutive vacations with probability p_k if the queue upon his return is empty. In fact, various aspects of Bernoulli vacation models have been discussed by a number of authors. Recently, there has been considerable attention paid to the study of $M/G/1$ type queueing systems with two phases of service under Bernoulli vacation schedule and different vacation policies, for instance see Choudhury and Madan (2004, 2005), Choudhury and Paul (2006) and Choudhury *et al.* (2007), in which after two successive phases of service, the server may go for a Bernoulli vacation. The motivation for these types of models comes from some computer networks and telecommunication systems, where messages are processed in two stages by a single server. As modern telecommunication systems become more complex and the processing power of microprocessors becomes more expensive, the advantages of more sophisticated scheduling becomes more apparent. The

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need for scheduling the allocation of resources among two or more heterogeneous types of tasks arises also in many other applications. In most of the previous studies, it is assumed that the server is available in the service station on a permanent basis and the service station never fails. However, these assumptions are practically unrealistic. In practice, we often meet cases where service stations fail and are repaired. Similar phenomena always occur in the area of computer communication networks, flexible manufacturing systems, etc. Because the performance of such systems may be heavily affected by the service station breakdown and delay in repair, these systems with a repairable service station are well worth investigating from the queueing theory viewpoint, as well as from the reliability point of view. Hence, Li *et al.* (1997) considered the reliability analysis of a model under Bernoulli vacation schedule with the assumption that the server is subject to breakdowns and repairs.

A wide class of vacation policies for governing the vacation mechanism has also been discussed in the most recent survey by Ke *et al.* (2010). In this context, recently, Choudhury and Deka (2012) investigate an $M / G / 1$ unreliable server Bernoulli vacation queue with two phases of service system. However, in this present paper, our purpose is to generalized such a type of $M / G / 1$ unreliable server queue with batch arrivals for two phases of service system under Bernoulli vacation schedule, where concepts of delay time is also introduced. These types of phenomena usually occur in the area of computer communication networks and flexible manufacturing systems. To this end, the methodology used will be based on the inclusion of supplementary variables.

The remainder of this paper is organized as follows. In section 2, we give a brief description of the mathematical model. Section 3 deals with the derivation of the stationary distribution of the queue size for the server state at a random epoch and at a departure epoch along with some important performance measures. Queue waiting time distribution has been derived in Section 4. Reliability analysis of this model has also been discussed in Section 5, which may lead to remarkable simplification while solving other similar types of queueing problems. Finally, Numerical illustrations are presented in Section 6 to see the effects of some of the system parameters on the system performance.

2. The system:

We consider an $M^x / G / 1$ queueing system, where the number of individual primary customers arrive to the system according to a compound Poisson process with arrival rate λ. The sizes of successive arriving batches are i.i.d random variables $X_1, X_2, \ldots$, distributed with probability mass function $a_n = Prob\left\{X = n\right\}$; $n \geq 1$, probability generating function (*PGF*) $a(z) = E\left[Z^X\right]$, and finite factorial moments $a_{[k]} = E\left[X(X-1)\ldots\ldots\ldots(X-k+1)\right]$.

The server provides to each unit two phases of heterogeneous service in succession, the first phase service (FPS) followed by the second phase service (SPS). The service discipline is assumed to be FCFS. Further, it is assumed that the service time B_i of the i$^{\text{th}}$ phase service follows a general probability law with distribution function (*d.f*) $B_i(x)$, $i = 1, 2$, Laplace Stieltjes Transform (*LST*) $\beta_i^*(\theta) = E\left[e^{-\theta B_i}\right]$, and finite *k*-th moments $\beta_i^{(k)}$, $i = 1, 2$, where sub-index $i = 1$ (respectively $i = 2$) denotes the *FPS* (respectively *SPS*). As soon as the *SPS* of a unit is completed the server may go for a vacation of random length V with probability $p\left(0 \leq p \leq 1\right)$ or may continue to serve the next unit, if any, with probability $q\left(= 1 - p\right)$. Otherwise, it remains in the system. Next, we assume that the vacation time *V* follows a general probability law with d.f. $V\left(y\right)$, LST $\vartheta^*\left(\theta\right)$ and finite moments $\nu^{(k)}$, which is independent of the service time random variables and the arrival process. Further, it is also assumed that if, after returning from a vacation, the server does not find any units in the system, even then it joins the system without taking any further vacations and this policy is termed as single vacation with Bernoulli schedule (BS). While the server is working with any phase of service, it may breakdown at any time and the service channel will fail for a short interval of time (*Breakdown periods*). The breakdowns i.e., *server's life times,* are generated by an exogenous Poisson process with rates α_1 for *FPS* and α_2 for *SPS*, which we may call some sort of *disaster* during *FPS* and *SPS* periods, respectively. As soon as a breakdown occurs, the server is sent for repair during which time it stops providing service to the customers waiting in the queue till the service channel is repaired. The customer being served just before server breakdown waits for repair to start, which we may refer to as *waiting period* of the server. We define this waiting time as *delay time*. The *delay time* D_i of the server for the *i*$^{\text{th}}$ phase of service follows a general law of probability with d.f $D_i(y)$, LST $\gamma_i^*(\theta) = E\left[e^{-\theta D_i}\right]$, and *k*-th finite moments $\gamma_i^{(k)}$; for $i = 1, 2$. The *repair time* (denoted by R_1 for *FPS* and R_2 for *SPS*) distributions of the server for both phases of service are assumed to be arbitrarily distributed with d.f $G_1(y)$

and $G_2(y)$, LST $G_1^*(\theta) = E\left[e^{-\theta R_1}\right]$ and $G_2^*(\theta) = E\left[e^{-\theta R_2}\right]$, and finite *k*-th moments $g_1^{(k)}$ and $g_2^{(k)}$, respectively. Immediately after the broken server is repaired, it is ready to start its remaining service to customers in either phase of service. In this case, the service times are cumulative, i.e., we consider a preemptive-resume policy for service time, which may be referred to as generalized service times. Further we assume that input process, server's lifetime, server's repair time, server's delay time, service time and vacation time random variables are mutually independent of each others.

3. The mathematical model:

In this section, we first setup the system state equations for its stationary queue size, by treating *elapsed vacation time*, *elapsed service time*, *elapsed repair time,* and *elapsed delay time* of the server, for both phases of service, as supplementary variables. Then we solve the equations and derive the *PGF*s of the stationary queue size distribution. Assume that the system is in steady-state conditions. Let $N(t)$ be the queue size (including one being served, if any) at time t, $V^0\left(t\right)$ be the elapsed vacation time of the server, and $B_i^0(t)$ be the elapsed service time of the customer for the *i*-th phase of service at time t, with $i = 1,2$ denoting FPS and SPS respectively. In addition, let $R_i^0(t)$ and $D_i^0(t)$ be the *elapsed repair time* and *elapsed delay time* of the server for *i*-th phase of service during which breakdown occurs in the system at time t, where sub-index $i = 1$ (respectively $i = 2$) denotes FPS (respectively SPS). Further, we introduce the following random variable:

$$Y(t) = \begin{cases} 0, & \textit{if the server is idle at time t.} \\ 1, & \textit{if the server is busy with FPS at time t.} \\ 2, & \textit{if the server is busy with SPS at time t.} \\ 3, & \textit{if the server is on vacation at time t.} \\ 4, & \textit{if the server is waiting for repair during FPS at time t.} \\ 5, & \textit{if the server is waiting for repair during SPS at time t.} \\ 6, & \textit{if the server is under repair during FPS at time t.} \\ 7, & \textit{if the server is under repair during SPS at time t.} \end{cases}$$

The supplementary variables $V^0\left(t\right)$, $B_i^0(t)$, $D_i^0(t)$ and $R_i^0(t)$ for $i = 1,2$ are introduced in order to obtain a bivariate Markov process $\left\{N(t), X(t)\right\}$, where $X(t) = 0$ if $Y(t) = 0$, $X(t) = B_1^0(t)$ if $Y(t) = 1$, $X(t) = B_2^0(t)$ if $Y(t) = 2$, $X(t) = V^0(t)$ if $Y(t) = 3$, $X(t) = D_1^0(t)$ if $Y(t) = 4$, $X(t) = D_2^0(t)$ if $Y(t) = 5$, $X(t) = R_1^0(t)$ if $Y(t) = 6$, and $X(t) = R_2^0(t)$ if $Y(t) = 7$. Next we define following limiting probabilities:

$$U_0(t) = P\left\{N(t) = 0, X(t) = 0\right\}$$

$$Q_n(y,t) = P\left\{N(t) = n, X(t) = V^0(t);\ y < V^0(t) \le y + dy\right\};\ y > 0,\ n \ge 0$$

and for $i = 1,2$ and $n \ge 0$

$$P_{i,n}(x,t)dx = P_r\left\{N(t) = n, X(t) = B_i^0(t); x < B_i^0(t) \le x + dx\right\}; x > 0$$

$$S_{i,n}(x,y)dy = P_r\left\{N(t) = n, X(t) = D_i^0(t); y < D_i^0(t) \le y + dy \middle| B_i^0(t) = x\right\}; (x,y) > 0$$

$$R_{i,n}(x,y;t)dy = P_r\left\{N(t) = n, X(t) = R_i^0(t); y < R_i^0(t) \le y + dy \middle| B_i^0(t) = x\right\};\ (x,y) > 0$$

Now, the analysis of the limiting behaviour of this queueing process at a random epoch can be performed with the help of Kolmogorov forward equations provided limiting probabilities

$$U_0 = \lim_{t\to\infty} U_0(t),$$

$$Q_n(y)dy = \lim_{t\to\infty} Q_n(y,t),$$

$$P_{i,n}(x)dx = \lim_{t\to\infty} P_{i,n}(x),$$

$$S_{i,n}(x,y)dy = \lim_{t\to\infty} S_{i,n}(x,y;t)$$

$$R_{i,n}(x,y)dy = \lim_{t\to\infty} R_{i,n}(x,y;t) \quad \textit{for } i = 1,2 \quad \textit{and } n \ge 0$$

exist and positive under the condition that they are independent of the initial state.

Further, it is assumed that

$V(0)=0,\ V(\infty)=1,\ \ B_i(0)=0,\ B_i(\infty)=1,\ D_i(0)=0,\ \ D_i(\infty)=1,\ G_i(0)=0,\ G_i(\infty)=1$ for $i=1,2$, and that $V(y)$ is continuous at $y=0$ for $i=1,2$; $B_i(x)$ is continuous at $x=0$, and $D_i(y)$ and $G_i(y)$ are continuous at $y=0$ for $i=1,2$ respectively, so that

$$v(y)\,dy=\frac{dV(y)}{1-V(y)}$$

$$\mu_i(x)dx=\frac{dB_i(x)}{1-B_i(x)};\ \ \eta_i(y)dy=\frac{dD_i(y)}{1-D_i(y)};\ \zeta_i(y)dy=\frac{dG_i(y)}{1-G_i(y)}\ \text{ for } i=1,2$$

are the first order differential (hazard rate) functions of V, B_i, D_i and G_i respectively for $i=1,2.$

3.1 The steady-state equations

The Kolmogorov forward equations to govern the system under steady-state conditions [e.g. see Cox (1955)] for $i=1,2;$ where sub-index $i=1$ (respectively $i=2$) denotes the *FPS* (respectively *SPS*) can be written as follows:

$$\frac{d}{dx}P_{i,n}(x)+\left[\lambda+\alpha_i+\mu_i(x)\right]P_{i,n}(x)=\lambda\sum_{k=1}^{n}a_kP_{i,n-k}(x)+\int_0^\infty \xi_i(y)R_{i,n}(x,y)dy\ ;\ n\geq 1 \tag{1}$$

$$\frac{d}{dy}Q_n(y)+\left[\lambda+v(y)\right]Q_n(y)=\lambda\left(1-\delta_{0,n}\right)\sum_{k=1}^{n}a_kQ_{n-k}(y)\ ;\quad x>0,\, n\geq 0 \tag{2}$$

$$\frac{d}{dy}S_{i,n}(x,y)+\left[\lambda+\eta_i(y)\right]S_{i,n}(x,y)=\lambda\sum_{k=1}^{n}a_kS_{i,n-k}(x;y)\ ;\ n\geq 1 \tag{3}$$

$$\frac{d}{dy}R_{i,n}(x,y)+\left[\lambda+\xi_i(y)\right]R_{i,n}(x,y)=\lambda\sum_{k=1}^{n}a_kR_{i,n-k}(x;y)\ ;\ n\geq 1 \tag{4}$$

$$\lambda U_0=\int_0^\infty v(y)Q_0(y)dy+q\int_0^\infty \mu_2(x)P_{2,1}(x)dx \tag{5}$$

where $\delta_{m,n}$ denotes Kronecker's function and $P_{i,0}(x)=0$, $Q_{i,0}(x,y)=0$, and $R_{i,0}(x,y)=0$ for $i=1,2$ occurring in equations (1) - (4).

These sets of equations are to be solved under the boundary conditions at $x=0$:

$$P_{1,n}(0)=\lambda a_nU_0+q\int_0^\infty \mu_2(x)P_{2,n+1}(x)dx+\int_0^\infty v(y)Q_n(y)dy;\quad n\geq 1 \tag{6}$$

$$P_{2,n}(0)=\int_0^\infty \mu_1(x)P_{1,n}(x)dx\ ;n\geq 1 \tag{7}$$

and at $y=0$:

$$Q_n(0)=p\int_0^\infty \mu_2(x)P_{2,n+1}(x)\,dx\ ;n\geq 0 \tag{8}$$

also at $y=0$ for $i=1,2$ and fixed values of x

$$S_{i,n}(x,0)=\alpha_iP_{i,n}(x)\,;x>0,n\geq 1 \tag{9}$$

$$R_{i,n}(x,0)=\int_0^\infty \eta_i(y)S_{i,n}(x;y)dy\,;x>0,n\geq 0 \tag{10}$$

These equations are to be solved under the following normalizing condition

$$U_0+\sum_{n=0}^{\infty}\left[\int_0^\infty Q_n(y)\,dy+\sum_{i=1}^{2}\left\{\int_0^\infty P_{i,n}(x)dx+\int_0^\infty\int_0^\infty S_{i,n}(x,y)dxdy+\int_0^\infty\int_0^\infty R_{i,n}(x,y)dxdy\right\}\right]=1\,. \tag{11}$$

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3.2 The model solution

To solve the system of equations (1) - (10), let us introduce the following *PGF*s for $\left|z\right| < 1$ and $i = 1,2$:

$$S_i(x,y;z) = \sum_{n=1}^{\infty} z^n S_{i,n}(x;y) \;; \qquad S_i(x,0;z) = \sum_{n=1}^{\infty} z^n S_{i,n}(x;0)$$

$$R_i(x,y;z) = \sum_{n=1}^{\infty} z^n R_{i,n}(x;y) \;; \qquad R_i(x,0;z) = \sum_{n=1}^{\infty} z^n R_{i,n}(x;0)$$

$$Q(y,z) = \sum_{n=1}^{\infty} z^n Q_n(y) \;; \qquad Q(0,z) = \sum_{n=1}^{\infty} z^n Q_n(0)$$

$$P_i(x,z) = \sum_{n=1}^{\infty} z^n P_{i,n}(x) \;; \qquad P_i(0,z) = \sum_{n=1}^{\infty} z^n P_{i,n}(0)$$

and $a\left(z\right) = \sum_{n=1}^{\infty} z^n a_n$

Let $b(z) = \lambda\left(1 - a(z)\right)$.Then, proceeding in the usual manner with equations (2) - (4), we get a set of differential equations of Lagrangian type whose solutions are given by:

$$Q(y;z) = Q(0;z)[1 - V(y)]\exp\{-b(z)y\}\,;(x,y) > 0 \tag{12}$$

$$S_i(x,y;z) = S_i(x,0;z)[1 - D_i(y)]\exp\{-b(z)y\}\,;(x,y) > 0 \text{ for } i = 1,2 \tag{13}$$

$$R_i(x,y;z) = R_i(x,0;z)\left[1 - G_i(y)\right]\exp\left\{-b(z)y\right\} \tag{14}$$

where $S_i(x,0;z)$ and $R_i(x,0;z)$ for $i = 1,2$ can be obtained from equations (9) and (10), which after simplification yield

$$S_i(x,0;z) = \alpha_i P_i(x;z) \tag{15}$$

$$R_i(x,0;z) = \alpha_i P_i(x,z)\gamma_i^*\left(b(z)\right) \tag{16}$$

Now solving the differential equations (1), we get

$$P_i(x;z) = P_i(0;z)[1 - B_i(x)]\exp\left\{-A_i(z)x\right\},\, x > 0 \text{ for } i = 1,2; \tag{17}$$

where $A_i(z) = b(z) + \alpha_i\left(1 - \gamma_i^*\left(b(z)\right)G_i^*\left(b\left(z\right)\right)\right)$ for $i = 1,2$.

Utilizing (15) - (17) in (13) we get for $i = 1,2$

$$S_i(x,y;z) = \alpha_i P_i(0;z)[1 - B_i(x)]\exp\left\{-A_i\left(z\right)x\right\}\times\left[1 - D_i\left(y\right)\right]\exp\left\{-b(z)y\right\} \tag{18}$$

Utilizing (16) - (17) in (14) we get for $i = 1,2$

$$R_i(x,y:z) = \alpha_i P_i(0;z)[1 - B_i(x)]\exp\left\{-A_i(z)x\right\}\left[1 - G_i\left(y\right)\right]\exp\left\{-b(z)y\right\}\gamma_i^*\left(b(z)\right) \tag{19}$$

Multiplying equation (3.6) by z^n and then taking summation over all possible values of $n \geq 1$, we get on simplification

$$zP_1(0,z) = qP_2(0;z)\beta_2^*\left(A_2(z)\right) + zQ(0;z)\vartheta^*\left(b(z)\right) - zU_0 b(z)\,. \tag{20}$$

Similarly from equations (7) and (8), we have

$$P_2(0,z) = P_1(0;z)\beta_1^*\left(A_1\left(z\right)\right). \tag{21}$$

$$zQ(0,z) = pP_2(0;z)\beta_2^*\left(A_2\left(z\right)\right) \tag{22}$$

Utilizing (21) and (22) in (20) and simplifying we get

$$P_1(0,z) = \frac{zU_0 b(z)}{\left[q + p\vartheta^*\left(b\left(z\right)\right)\right]\beta_1^*\left(A_1\left(z\right)\right)\beta_2^*\left(A_2\left(z\right)\right) - z} \tag{23}$$

Letting $z \to 1$ in (23), we obtain by L'Hospital's rule

$$P_1(0,1) = \frac{\lambda a_{[1]} U_0}{(1 - \rho_H)}\;, \tag{24}$$

Where $\rho_H = \rho_1\left\{1+\alpha_1\left(g_1^{(1)}+\gamma_1^{(1)}\right)\right\}+\rho_2\left\{1+\alpha_2\left(g_2^{(1)}+\gamma_2^{(1)}\right)\right\}+p\lambda a_{[1]}\vartheta^{(1)}$ is the utilizing factor of the system and $\rho_i = \lambda a_{[1]}\beta_i^{(1)}$ for $i=1,2$.

This gives for i=1, 2

$$P_i(x,1)=\frac{\lambda a_{[1]}U_0\left[1-B_i(x)\right]}{(1-\rho_H)} \tag{25}$$

$$S_i(x,y,1)=\frac{\alpha_i\lambda a_{[1]}U_0\left[1-B_i(x)\right]\left[1-D_i(y)\right]}{(1-\rho_H)} \tag{26}$$

$$R_i(x,y,1)=\frac{\alpha_i\lambda a_{[1]}U_0\left[1-B_i(x)\right]\left[1-G_i(y)\right]}{(1-\rho_H)} \tag{27}$$

and $$Q(y,1)=\frac{p\lambda U_0 a_{[1]}\left[1-V(y)\right]}{(1-\rho_H)} \tag{28}$$

Now utilizing normalizing condition (11), we get

$$U_0=(1-\rho_H); \tag{29}$$

Note that equation (25) represents steady-state probability that the server is idle but available in the system. Also, from equation (25), we have $\rho_H<1$, which is the necessary and sufficient condition under which steady-state solution exists. Thus, we summarize our results in the following **Theorem 3.1.**

Theorem 3.1:- Under the stability condition $\rho_H<1$, the joint distribution of the state of the server and the queue size has the following partial *PGFs*

$$P_1(x;z)=\frac{zb(z)(1-\rho_H)[1-B_1(x)]\exp\left\{-\left(A_1(z)\right)x\right\}}{\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z}, \tag{30}$$

$$P_2(x;z)=\frac{zb(z)(1-\rho_H)\beta_1^*\left(A_1(z)\right)[1-B_2(x)]\exp\left\{-\left(A_2(z)\right)x\right\}}{\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z} \tag{31}$$

$$Q(y;z)=\frac{pb(z)(1-\rho_H)\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2\left(z\right)\right)[1-V(y)]\exp\left\{-b\left(z\right)y\right\}}{\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z} \tag{32}$$

$$R_1(x,y;z)=\frac{\alpha_1 zb(z)(1-\rho_H)[1-B_1(x)]\exp\left\{-\left(A_1(z)\right)x\right\}\gamma_1^*(b(z))[1-G_1(y)]\exp\left\{-\left(b(z)\right)y\right\}}{\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z} \tag{33}$$

$$R_2(x,y;z)=\frac{\alpha_2 zb(z)(1-\rho_H)[1-B_2(x)]\beta_1^*(A_1(z))\exp\left\{-\left(A_2(z)\right)x\right\}\times[1-G_2(y)]\gamma_2^*(b(z))\exp\left\{-\left(b(z)\right)y\right\}}{\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z} \tag{34}$$

$$S_1(x,y;z)=\frac{\alpha_1 zb(z)(1-\rho_H)[1-B_1(x)]\exp\left\{-\left(A_1(z)\right)x\right\}\times[1-D_1(y)]\exp\left\{-\left(b(z)\right)y\right\}}{\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z} \tag{35}$$

$$S_2(x,y;z)=\frac{\alpha_2 zb(z)(1-\rho_H)\beta_1^*\left(A_1(z)\right)[1-B_2(x)]\exp\left\{-\left(A_2(z)\right)x\right\}\times[1-D_2(y)]\exp\left\{-\left(b(z)\right)y\right\}}{\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z}; \tag{36}$$

where $A_i(z)=b(z)+\alpha_i\left(1-G_i^*\left(b(z)\right)\gamma_i^*\left(b(z)\right)\right)$ for $i=1,2$ and $b(z)=\lambda(1-a(z))$.

Remark 3.1 It is important to note here that such types of joint distributions are important to obtain the distribution of each state of the server in more comprehensive manner, which helps us to obtain marginal distributions of the server's states as well as stationary queue size distribution at a departure epoch.

Theorem 3.2:- Under the stability condition $\rho_H < 1$, the marginal *PGFs* of the server's state queue size distribution are given by

$$P_1(z) = \frac{\left(1-\rho_H\right) z b\left(z\right)\left[1-\beta_1^*\left(A_1(z)\right)\right]}{A_1(z)\left[\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z\right]} \tag{37}$$

$$P_2(z) = \frac{\left(1-\rho_H\right) z b\left(z\right)\beta_1^*\left(A_1\left(z\right)\right)\left[1-\beta_2^*\left(A_2(z)\right)\right]}{A_2(z)\left[\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z\right]} \tag{38}$$

$$Q(z) = \frac{p\left(1-\rho_H\right)\beta_1^*\left(A_1\left(z\right)\right)\beta_2^*\left(A_2(z)\right)\left[1-\vartheta^*\left(b(z)\right)\right]}{\left[\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z\right]} \tag{39}$$

$$R_1(z) = \frac{\alpha_1\left(1-\rho_H\right) z\left[1-G_1^*(b(z))\right]\left[1-\beta_1^*\left(A_1(z)\right)\right]}{A_1(z)\left[\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z\right]} \tag{40}$$

$$R_2(z) = \frac{\alpha_2\left(1-\rho_H\right) z\beta_1^*\left(A_1(z)\right)\gamma_2^*(b(z))\left[1-\beta_2^*\left(A_2(z)\right)\right]\left[1-G_2^*(b(z))\right]}{A_2(z)\left[\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z\right]} \tag{41}$$

$$S_1(z) = \frac{\alpha_1\left(1-\rho_H\right) z\left[1-\gamma_1^*(b(z))\right]\left[1-\beta_1^*\left(A_1(z)\right)\right]}{A_1(z)\left[\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z\right]} \tag{42}$$

And $$S_2(z) = \frac{\alpha_2\left(1-\rho_H\right) z\beta_1^*\left(A_1(z)\right)\left[1-\beta_2^*\left(A_2(z)\right)\right]\left[1-\lambda_2^*(b(z)\right]}{A_2(z)\left[\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)-z\right]} \tag{43}$$

Proof: - Integrating (30), (31) and (32) with respect to x and y respectively, then

$$\int_0^\infty e^{-sx}\left(1-B_i(x)\right)dx = \frac{\left[1-\beta_i^*(s)\right]}{s} \text{ for } i=1,2$$

and $$\int_0^\infty e^{-sx}\left(1-V(y)\right)dy = \frac{\left[1-\vartheta^*(s)\right]}{s};$$

we get formulae (37) - (39).
Similarly, integrating equations (33) - (36) with respect to y, we get for $i = 1,2$

$$R_i(x,z) = \int_0^\infty R_i(x,y;z)dy = \alpha_i\left[b\left(z\right)\right]^{-1}\gamma_i^*\left(b\left(z\right)\right)\left[1-G_i^*\left(b\left(z\right)\right)\right]P_i(0;z)\left[1-B_i(x)\right]\exp\left\{-A_i\left(z\right)x\right\} \tag{44}$$

and $$S_i(x,z) = \int_0^\infty S_i(x,y;z)dy = \alpha_i\left[b\left(z\right)\right]^{-1}\left[1-\gamma_i^*\left(b\left(z\right)\right)\right]P_i\left(0;z\right)\left[1-B_i(x)\right]\exp\left\{-\left(A_i(z)x\right)\right\} \tag{45}$$

Further integrating (44) and (45) with respect to x, we claimed in formulae (40) - (43).

Theorem 3.3:- Let ψ_j be the stationary distribution of the number of customers in the queue at a random epoch, then its corresponding *PGF*, i.e. $\psi(z) = \sum_{j=0}^{\infty} z^j \psi_j$ is given by

$$\psi(z) = \frac{\left(1-\rho_H\right)\left(1-z\right)\left[q+p\vartheta^*\left(b(z)\right)\right]\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)}{\left[\left\{q+p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left\{A_1(z)\right\}\beta_2^*\left(A_2(z)\right)-z\right]}. \tag{46}$$

Proof:- The result follows directly from *Theorem 3.2* with the help of *PGFs* $Q(z)$, , $P_i(z)$, $R_i(z)$ and $S_i(z)$ for $i = 1,2$, since the distribution of the number of customers in the queue has *PGF*

$$\psi(z) = \sum_{i=1}^{2}\left\{P_i(z) + R_i(z) + S_i(z)\right\} + zQ\left(z\right).$$

By direct calculation we obtain (46).

Remark 3.2:- Now if we put $D_i = 0;\ i = 1,2$ (i.e. there is no delay in the system) or equivalently $\gamma_i^*\left(\theta\right) = 1$ and $a\left(z\right) = z$ (i.e. for single unit arrival case) , then $\rho_H = \rho_1\left\{1 + \alpha_1 g_1\right\} + \rho_2\left\{1 + \alpha_2 g_2\right\} + p\lambda\vartheta^{(1)}$,

$\rho_i = \lambda\beta_i^{(1)}$ and therefore our expression (46) is reduces to

$$\psi(z) = \frac{\left(1-\rho_H\right)\left(1-z\right)\left[q + p\vartheta^*\left(b(z)\right)\right]\beta_1^*\left(b(z) + \alpha_1\left(1 - G_1^*\left(b\left(z\right)\right)\right)\right)\beta_2^*\left(b(z) + \alpha_2\left(1 - G_2^*\left(b\left(z\right)\right)\right)\right)}{\left[\left\{q + p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(b(z) + \alpha_1\left(1 - G_1^*\left(b\left(z\right)\right)\right)\right)\beta_2^*\left(b(z) + \alpha_2\left(1 - G_2^*\left(b\left(z\right)\right)\right)\right) - z\right]};$$

which is consistent with expression (52) of Choudhury and Deka (2012). Also, we note that for $\alpha_i = 0$ for $i = 1,2$ our expression (53) is consistent with the expression (20) of Choudhury and Madan (2004).

Corollary 3.1 If the system is in steady state conditions, then

(i) The probability that the system is idle is,
$P_1 = 1 - \rho_1\left\{1 + \alpha_1\left(g_1^{(1)} + \gamma_1^{(1)}\right)\right\} - \rho_2\left\{1 + \alpha_2\left(g_2^{(1)} + \gamma_2^{(1)}\right)\right\} - p\lambda a_{[1]}\vartheta^{(1)}$

(ii) The probability that the server is busy with FPS is $P_{B_1} = \rho_1$

(iii) The probability that the server is busy with SPS is $P_{B_2} = \rho_2$

(iv) The probability that the server is on vacation, $P_\vartheta = p\lambda a_{[1]}\vartheta^{(1)}$

(v) The probability that the server is waiting for repair during FPS is $P_{w_1} = \alpha_1\rho_1\gamma_1^{(1)}$

(vi) The probability that the server is waiting for repair during SPS is $P_{w_2} = \alpha_2\rho_2\gamma_2^{(1)}$

(vii) The probability that the server is under repair during FPS is $P_{R_1} = \alpha_1\rho_1 g_1^{(1)}$

(viii) The probability that the server is under repair during SPS is $P_{R_2} = \alpha_2\rho_2 g_2^{(1)}$

Proof:- Noting that $P_V = \lim_{z\to1} Q(z)$, $P_{B_i} = \lim_{z\to1} P_i(z)$, $P_{R_i} = \lim_{z\to1} R_i(z)$ **,** $P_{S_i} = \lim_{z\to1} S_i(z)$ for $i = 1,2$, and $P_I = 1 - \sum_{i=1}^{2}\left\{P_{B_i} + P_{R_i} + P_{S_i}\right\} - P_V$, the stated formulae follow by direct calculation.

Finally, the derivation of the stationary queue size distribution at a departure epoch of this model is given in the proof of Theorem 3.4.

Theorem 3.4:- Under the steady-state condition, the PGF of the stationary queue size at a departure epoch of this model is given by,

$$\pi(z) = \frac{(1-\rho_H)\left(1 - a(z)\right)\left\{q + p\vartheta^*(b(z))\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2\left(z\right)\right)}{a_{[1]}\left[\left\{q + p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right) - z\right]} \tag{47}$$

Proof:- Following the argument of PASTA [see Wolf (1982)], we state that a departing customer will see 'j' customer in the queue just after a departure if and only if there were 'j' customer in the queue SPS or a vacation just before the departure. Now denoting $\left\{\pi_j : j \geq 0\right\}$ as the probability that there are j units in the queue at a departure epoch, then for $j \geq 0$ we may write

$$\pi_j = K_0\left[q\int_0^\infty \mu_2(x)P_{2,j+1}(x)dx + \int_0^\infty \vartheta(y)Q_j(y)dy\right] \tag{48}$$

Where K_0 is the normalizing constant.

Now multiplying both sides of equation (48) by z^j and then taking summation over $j \in Z^+$ and utilizing equations (21) and (23), we get on simplification

$$\pi(z) = \frac{K_0 U_0 b(z)\left\{q + p\vartheta^*(b(z))\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right)}{\left[\left\{q + p\vartheta^*\left(b(z)\right)\right\}\beta_1^*\left(A_1(z)\right)\beta_2^*\left(A_2(z)\right) - z\right]} \tag{49}$$

Utilizing normalizing condition $\pi(1) = 1$, we get

$$K_0 = \frac{1-\rho_H}{\lambda U_0 a_{[1]}} \tag{50}$$

Hence formula (46) follows by inserting (49) in (48).

Next, the relationship between the stationary queue size distributions at a random epoch and at a departure epoch is stated in *Corollary 3.2*.

Corollary 3.2:- Under the steady-state condition, the relationship between the *PGF*s of the queue size distributions at a random epoch and at a departure epoch of an $M^x / G / 1$ unreliable server queue with two phases of service and Bernoulli vacation schedule under *N*-policy is given by

$$\pi(z) = \frac{\left[1 - a(z)\right]}{a_{[1]}(1-z)}\psi(z) = H_t(z)\psi(z); \tag{51}$$

where $H_t(z)$ is the *PGF* of the number of customers placed before an arbitrary test customer (tagged customer) in a batch in which the tagged customer arrives. This number is given as the backward recurrence time in the discrete time renewal process where renewal points are generated by the arrival size random variable [e.g., see Takagi (1991), p. 46], i.e.,

$$H_t(z) = \frac{\left[1-a(z)\right]}{a'(1)(1-z)} \quad \text{and} \quad a_{[1]} = a'(1).$$

Remark 3.3:- From *Corollary 3.2*, it is observed that queue size distribution at the departure epoch of an $M^x / G / 1$ unreliable server queue with two phases of service and Bernoulli vacation schedule is the convolution of the distributions of two independent random variables. The first one is the number of units placed before a tagged customer in a batch in which the tagged customer arrives. This is due to the randomness property of the arriving batch size. The interpretation of the other random variables is the stationary distribution an $M^x / G / 1$ unreliable server queue with two phases of service and Bernoulli vacation schedule.

Remark 3.4:- Now setting $z = 0$ in equation (51), we get

$$\pi(0) = P_r \text{ \{No customer is waiting in the system at the departure epoch\}}$$
$$= \frac{(1-\rho_H)}{a_{[1]}} = \pi_0$$

Thus the relationship between U_0 and π_0 is given by

$$\pi_0 a_{[1]} = U_0.$$

This exhibits an interesting phenomenon. It states that a random observer is more likely to find the system empty than a departing customer leaving the system.

Next the mean queue size and mean busy and cycle periods of this model is given in the corollary 3.3 and corollary 3.4 respectively.

Corollary 3.3 Under the stability conditions, the mean number of customers in the system (i.e. mean queue length) L_S is given by

$$L_S = \rho_H + \frac{\left(\lambda a_{[1]}\right)^2 \left[\beta_1^{(2)}\left\{1+\alpha_1\left(g_1^{(1)}+\gamma_1^{(1)}\right)\right\}^2 + \beta_2^{(2)}\left\{1+\alpha_2\left(g_2^{(1)}+\gamma_2^{(1)}\right)\right\}^2 + p\vartheta^{(2)}\right]}{2\left(1-\rho_H\right)} +$$

$$\frac{\left(\lambda a_{[1]}\right)^2\left[\alpha_1\beta_1^{(1)}\left(g_1^{(2)}+\gamma_1^{(2)}+2g_1^{(1)}\gamma_1^{(1)}\right)+\alpha_2\beta_2^{(1)}\left(g_2^{(2)}+\gamma_2^{(2)}+2g_2^{(1)}\gamma_2^{(1)}\right)\right]}{2(1-\rho_H)} +$$

$$\frac{\left(\lambda a_{[1]}\right)^2\left[p\vartheta^{(1)}\left\{\beta_1^{(1)}\left(1+\alpha_1\left(g_1^{(1)}+\gamma_1^{(1)}\right)\right)+\beta_2^{(1)}\left(1+\alpha_2\left(g_2^{(1)}+\gamma_2^{(1)}\right)\right)\right\}\right]}{(1-\rho_H)} +$$

$$\frac{\left(\lambda a_{[1]}\right)^2\left[\beta_1^{(1)}\beta_2^{(1)}\left\{1+\alpha_1\left(g_1^{(1)}+\gamma_1^{(1)}\right)\right\}\left\{1+\alpha_2\left(g_2^{(1)}+\gamma_2^{(1)}\right)\right\}\right]}{(1-\rho_H)} + \frac{a_{[2]}}{2a_{[1]}(1-\rho_H)} \tag{52}$$

Proof:- The result follows directly by differentiating (47) with respect to z and then taking limit $z \to 1$ by using the L'Hospital's rule.

Corollary 3.4:-Let T_b and T_c be the length of a busy period and length of a busy cycle respectfully, then under the steady state condition, we have

$$E(T_b) = \left(\frac{\rho_H}{1-\rho_H}\right)\frac{1}{\lambda a_{[1]}} \tag{53}$$

$$E(T_c) = \left(\frac{1}{\lambda a_{[1]}(1-\rho_H)}\right) \tag{54}$$

Proof :- The result follows directly by applying the argument of alternating renewal theory . We now define T_0 as length of an idle period. Since arrival process is Poisson, therefore T_B and T_0 generate an alternating renewal Process and hence we may write

$$E\left(T_b\right) = \frac{P_r\left[T_b\right]}{1-P_r\left[T_b\right]}E\left(T_0\right); \tag{55}$$

where $P_r\left[T_b\right]$ is the probability that long fraction of time system remains busy and this is equivalent to ρ_H . Again it is well known to us that for batch arrival (compound) Poisson input queue we have $E\left(T_0\right)=\left[\lambda a_{[1]}\right]^{-1}$ . .Now inserting $E\left(T_0\right)$ in the above relationship (55), we get the necessary result (53). Further, utilizing the relationship $E\left(T_C\right) = E\left(T_0\right) + E\left(T_b\right)$ we get (54).

4. Waiting time distribution:

To obtain the waiting time distribution in the queue, we first derive the waiting time of the first customer in an arriving batch, W_1 (say) and use $W_1^*(\theta)$ to denote *LST* of W_1.

Now if we identify a batch with a single customer, then its service time is just the modified service time of customers constituting the batch. In this case, the batch will have as its batch size X (z) =z. The mean arrival rate will λ and LST of the modified service time of the batch will replace

$$U^*(\theta) = \left(q + pV^*(\theta)\right)T_1^*(\theta)T_2^*(\theta), \quad where \quad T_i^*(\theta) = \beta_i^*\left(A_i(\theta)\right) = \beta_i^*\left\{\theta + \alpha_1\left(1 - G_1^*(\theta)\gamma_i^*(\theta)\right)\right\} \text{ by } X\left(U^*(\theta)\right)$$

Using the transformation and the results by Chaudhry and Templeton (1983) (see Chapter 3), from equation (46) we have,

$$\pi(z) = \frac{\left(1-\rho_H\right)\lambda(1-z)X\left[U^*(b(z))\right]}{\lambda\left[X\left(U^*(b(z))\right)-z\right]} \tag{56}$$

If the waiting time of each batch is independent of the part of arrival process following the arrival time of the batches left behind a departing batch are those that arrive during the time it spends in the queue and in service, it follows that

$$\pi(z) = W_1^*\left(\lambda - \lambda z\right) X\left[U^*\left(\lambda - \lambda z\right)\right] \tag{57}$$

Now putting $\theta = \lambda - \lambda z$ in (57) and utilizing (56) in (57), we get finally,

$$W_1^*(\theta) = \frac{\left(1-\rho_H\right)\theta}{\left[\theta - \lambda + \lambda X\left[U^*(\theta)\right]\right]} \tag{58}$$

Now, let W be the waiting time of an arbitrary customer in a batch and denote by $W^*(\theta)$ the LST of W. If $j \geq 1$ is the position of the customer within arrival batch, then

$$W = W_1 + \sum_{i=1}^{j-1} U_i' \qquad ; \qquad j \geq 1 \tag{59}$$

Where U_i' denotes the difference between modified service time and inter arrival time of the i customer in the batch.

If χ_j is the probability of an arbitrary customer being the j[th] position of an arriving batch, then applying the results of Chaudhry & Templeton (1983), we may write,

$$\Pr\left[\sum_{i=1}^{j-1} G_i' \leq t\right] = \sum_{j=1}^{\infty} \chi_j G(t)^{(j-1)*} \quad ;$$

where $U(t) = \Pr\left[U_i' \leq t\right]$ *and* $\chi_j = \left(1 - \sum_{i=1}^{j-1} a_i\right) \Big/ E(X)$

Consequently, taking LST of (59), we get on simplification

$$\begin{aligned} W^*(\theta) &= E\left[e^{-\theta w}\right] \\ &= E\left[e^{-\theta w_1}\right] E\left[e^{-\theta \sum_{i=1}^{j-1} U_i'}\right] \\ &= \frac{W_1^*(\theta)}{a_{[1]}} \frac{\left[1 - X\left(U^*(\theta)\right)\right]}{\left[1 - U^*(\theta)\right]} \end{aligned}$$

and therefore LST of the waiting time distribution in the queue for this model is given by

$$W^*(\theta) = \frac{\theta\left[1 - X\left(U^*(\theta)\right)\right]}{a_{[1]}\left[\theta - \lambda + \lambda X\left(U^*(\theta)\right)\right]\left[1 - U^*(\theta)\right]} \tag{60}$$

5. Reliability Analysis:-

Our final goal is to derive some reliability indices of this model. First of all we will discuss two reliability indices of the system viz. - the system availability and failure frequency under the steady state conditions. Suppose that the system is initially empty. Let $A_v(t)$ be the point wise availability of the server at time 't' that is, the probability that the server is either serving a customer or the server is available if the server is free and up during an idle period, such that the steady state availability of the server will be $A_v = \lim_{t \to \infty} A_v(t)$

Theorem 5.1:- The steady state availability of the server is given by,

$$A_v = 1 - \rho_1 \alpha_1 \left(g_1^{(1)} + \gamma_1^{(1)}\right) - \rho_2 \alpha_2 \left(g_2^{(1)} + \gamma_2^{(1)}\right) - \lambda p a_{[1]} \vartheta^{(1)} \tag{61}$$

Proof:- The result follows directly from theorem (3.2) by considering the following equation

$$A_v = U_0 + \sum_{i=1}^{2} \int_0^{\infty} P_i(x,1)dx = U_0 + \lim_{z \to 1}\left[P_1(z) + P_2(z)\right]$$

By using (29), (37) and (38), we get (61).

Theorem 5.2:- The steady state failure frequency of the server is given by,

$$M_f = \alpha_1\rho_1 + \alpha_2\rho_2 \tag{62}$$

Proof: - The result follows directly from equation (25) by utilizing the argument of Li et. al.(1997).

$$M_f = \alpha_1\int_0^\infty P_1(x,1)dx + \alpha_2\int_0^\infty P_2(x,1)dx$$

Now since $\int_0^\infty \left[1-B_i(x)\right]dx = \int_0^\infty x dB_i(x) = \beta_i^{(1)}$; for i=1, 2; therefore from equation (25) we have (62)

Next we denote by τ the time to the first failure of the server, and then the reliability function of the server is $R(t) = P(\tau > t)$

Theorem 5.3:- The Laplace transform of $R(t)$ is given by

$$R^*(\theta) = U_0^*(\theta) + \frac{\left[\left(\theta+\alpha_2\right)\left(1-\beta_1^*\left(\theta+\alpha_1\right)\right)+\left(\theta+\alpha_1\right)\beta_1^*\left(\theta+\alpha_1\right)\left(1-\beta_2^*\left(\theta+\alpha_2\right)\right)\right]\left(1-\theta U_0^*\left(\theta\right)\right)}{\left(\theta+\alpha_1\right)\left(\theta+\alpha_2\right)\left[1-\left\{q+p\vartheta^*\left(\theta\right)\right\}\beta_1^*\left(\theta+\alpha_1\right)\beta_2^*\left(\theta+\alpha_2\right)\right]}$$
$$+\frac{p\beta_1^*\left(\theta+\alpha_1\right)\beta_2^*\left(\theta+\alpha_2\right)\left(1-\vartheta^*\left(\theta\right)\right)\left(1-\theta U_0^*\left(\theta\right)\right)}{\theta\left[1-\left\{q+p\vartheta^*\left(\theta\right)\right\}\beta_1^*\left(\theta+\alpha_1\right)\beta_2^*\left(\theta+\alpha_2\right)\right]}; \tag{63}$$

where $U_0^*\left(\theta\right) = \dfrac{1}{\lambda+\theta-\lambda\omega_0\left(\theta\right)}$

and $\omega_0\left(\theta\right)$ is the unique root of the equation

$$z = \left\{q+p\vartheta^*\left(\theta+b\left(z\right)\right)\right\}\beta_1^*\left(\theta+\alpha_1+b\left(z\right)\right)\beta_2^*\left(\theta+\alpha_2+b\left(z\right)\right)$$

inside $\left|z\right| = 1,\ \mathrm{Re}(\theta) > 0$ and $b(z) = \lambda(1-a\left(z\right))$.

Proof: - In order to find the reliability function of the server, we assume that the failure state of the server be the absorbing states, then we obtain a new system. In this new system, we use the same notations as in the previous sections, and then we can write the following set of Kolmogorov forward equations:

$$\frac{d}{dt}U(t) + \lambda U_0(t) = \int_0^\infty \vartheta(y)Q_0(y,t)dy + q\int_0^\infty \mu_2(x)P_{2,1}(x;t)dx \tag{64}$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}\right)Q_n(y,t) + \left[\lambda+\vartheta(y)\right]Q_n(y,t) = \lambda\left(1-\delta_{0,n}\right)\sum_{k=1}^n a_k Q_{n-k}(y,t), \quad n \geq 0 \tag{65}$$

And for i=1,2 and $n \geq 1$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)P_{i,n}(x,t) + \left[\lambda+\alpha_i+\mu_i(x)\right]P_{i,n}(x,t) = \lambda\sum_{k=1}^n a_k P_{i,n-k}(x,t), \quad n \geq 1 \tag{66}$$

These equations are to be solved with initial condition $U_0(0) = 1$ subject to the boundary conditions at x=0:

$$P_{1,n}(0,t) = \lambda a_n U_0(t) + q\int_0^\infty \mu_2(x)P_{2,n+1}(x;t)dx + \int_0^\infty \vartheta(y)Q_n(y,t)dy, \quad n \geq 1 \tag{67}$$

$$P_{2,n}(0,t) = \int_0^\infty \mu_1(x)P_{1,n}(x,t)dx, \quad n \geq 1 \tag{68}$$

and $Q_n(0,t) = p\int_0^\infty \mu_2(x)P_{2,n+1}(x,t)dx, \quad n \geq 0$ (69)

We now introduce following Laplace transform of generating functions for $\left|z\right| \leq 1$;

$P_i^*(x,\theta;z) = \sum_{n=1}^\infty z^n P_{i,n}^*(x;\theta)$, $\quad P_i^*(0,\theta;z) = \sum_{n=1}^\infty z^n P_{i,n}^*(0;\theta)$ for i=1,2

for i=1,2

and $U_0^*(\theta) = \int_0^\infty e^{-\theta t} dU_0(t)$

Now performing Laplace transform with respect to these equations (64) and (65) we get

$$(\theta+\lambda)U_0^*(\theta) - 1 = \int_0^\infty \vartheta(y) Q_0^*(y;\theta)dy + q\int_0^\infty \mu_2(x) P_{2,1}^*(x;\theta)dx \tag{70}$$

$$\left(\theta+\lambda+\vartheta(y)\right)Q_n^*(y;\theta) + \frac{\partial}{\partial y}Q_n^*(y;\theta) = \lambda\left(1-\delta_{0,n}\right)\sum_{k=1}^{n} a_k Q_{n-k}^*(y;\theta) \qquad n \geq 0 \tag{71}$$

Similarly from equations (66)-(69) we have for i=1, 2

$$\left(\theta+\lambda+\alpha_i+\mu_i(x)\right)P_{i,n}^*(x;\theta) + \frac{\partial}{\partial \theta}P_{i,n}^*(x;\theta) = \lambda\sum_{k=1}^{n} a_k P_{i,n-k}^*(x;\theta) \qquad n \geq 1 \tag{72}$$

$$P_{1,n}^*(0;\theta) = \lambda a_n U_0^*(\theta) + q\int_0^\infty \mu_2(x) P_{2,n+1}(x;\theta)dx + \int_0^\infty \vartheta(y) Q_n^*(y;\theta)dy \;\; n \geq 1 \tag{73}$$

$$P_{2,n}^*(0;\theta) = \int_0^\infty \mu_1(x) P_{1,n}^*(x;\theta)dx \qquad n \geq 1 \tag{74}$$

and $$Q_n^*(0;\theta) = p\int_0^\infty \mu_2(x) P_{2,n+1}^*(x;\theta)dx \qquad n \geq 0 \tag{75}$$

Now multiplying equation (71) by z^n and then taking summation over all possible values of $n \geq 0$, we get a set of differential equation of Lagrangian type whose solution is given by,

$$Q^*(y,\theta;z) = Q^*(0,\theta;z)\exp\left\{-\left(b(z)+\theta\right)y\right\}\left[1-V(y)\right] \tag{76}$$

Where $b(z) = \lambda\left(1-a(z)\right)$ is as defined in section 3.

Again from equation (75) we have

$$zQ^*(0,\theta;z) = pP_2^*(0,\theta;z)\beta_2^*\left(\theta+\alpha_2+b(z)\right) \tag{77}$$

Now similarly multiplying equation (72) by z^n and then taking summation over all positive values of $n \geq 0$, we get a set of similar type of differential equation of type whose solution is given by,

$$P_i^*(x,\theta;z) = P_i^*(0,\theta;z)\exp\left\{-\left(b(z)+\alpha_i+\theta\right)x\right\}\left[1-B_i(x)\right] \qquad \text{for } i = 1,2 \tag{78}$$

Again from equations (73) and (74) we have

$$zP_1^*(0,\theta;z) + z\left\{b(z)+\theta\right\}U_0^*(\theta) = z + qP_2^*(0,\theta;z)\beta_2^*\left\{b(z)+\theta+\alpha_2\right\} + zQ^*(0,\theta;z)\vartheta^*\left(\theta+b(z)\right) \tag{79}$$

$$P_2^*(0,\theta;z) = P_1^*(0,\theta;z)\beta_1^*\left(\theta+\alpha_1+b(z)\right) \tag{80}$$

Similarly from equation (79) after utilizing (77) and (80)

$$P_1^*(0,\theta;z) = \frac{z\left[\left(b(z)+\theta\right)U_0^*(\theta)-1\right]}{\left[\left\{q+p\vartheta^*\left(\theta+b(z)\right)\right\}\beta_1^*\left(\theta+\alpha_1+b(z)\right)\beta_2^*\left(\theta+\alpha_2+b(z)\right)-z\right]} \tag{81}$$

Further from equation (78) for i=1 we obtain

$$P_1^*(\theta,z) = \frac{z\left[\left(b(z)+\theta\right)U_0^*(\theta)-1\right]\left[1-\beta_1^*\left(b(z)+\alpha_1+\theta\right)\right]}{\left[\left\{q+p\vartheta^*\left(\theta+b(z)\right)\right\}\beta_1^*\left(\theta+\alpha_1+b(z)\right)\beta_2^*\left(\theta+\alpha_2+b(z)\right)-z\right]\left(b(z)+\alpha_1+\theta\right)} \tag{82}$$

Similarly from equation (78) for i=2 we obtain

$$P_2^*(\theta,z) = \frac{z\left[\left(b(z)+\theta\right)U_0^*(\theta)-1\right]\beta_1^*\left(\theta+\alpha_1+b(z)\right)\left[1-\beta_2^*(b(z)+\alpha_2+\theta)\right]}{\left[\left\{q+p\vartheta^*\left(\theta+b(z)\right)\right\}\beta_1^*\left(\theta+\alpha_1+b(z)\right)\beta_2^*\left(\theta+\alpha_2+b(z)\right)-z\right]\left(b(z)+\alpha_2+\theta\right)} \tag{83}$$

Finally from equation (5.16)

$$Q^*(\theta;z) = \frac{p\left[\left(b(z)+\theta\right)U_0^*(\theta)-\right]\beta_1^*\left(\theta+\alpha_1+b(z)\right)\beta_2^*\left(\theta+\alpha_2+b(z)\right)\left[1-\vartheta^*\left(b(z)+\theta\right)\right]}{\left[\left\{q+p\vartheta^*\left(\theta+b(z)\right)\right\}\beta_1^*\left(\theta+\alpha_1+b(z)\right)\beta_2^*\left(\theta+\alpha_2+b(z)\right)-z\right]\left(b(z)+\theta\right)} \tag{84}$$

Now consider the coefficient

$$f(z) = \left[\left\{ q + p\vartheta^* \left(\theta + b(z) \right) \right\} \beta_1^* \left(\theta + \alpha_1 + b(z) \right) \beta_2^* \left(\theta + \alpha_2 + b(z) \right) - z \right]$$

from which it can be shown that the function $f(z)$ is convex. Hence by Rouche's theorem $f(z)$ has only exactly one root $w_0(\theta)$ inside the unit circle $|z| = 1$ for Re(z). Therefore we have

$$U_0^*(\theta) = \frac{1}{s + \lambda - \lambda w_0(\theta)}$$

where $w_0(\theta)$ is the unique root of the equation

$$z = \left\{ q + p\vartheta^* \left(\theta + b(z) \right) \right\} \beta_1^* \left(\theta + \alpha_1 + b(z) \right) \beta_2^* \left(\theta + \alpha_2 + b(z) \right)$$

Hence from equations (81), (82) and 83), we have,

$$R^*(\theta) = U_0^*(\theta) + P_1^*(\theta) + P_2^*(\theta) + Q^*(\theta)$$
$$= U_0^*(\theta) + P_1^*(\theta,1) + P_2^*(\theta,1) + Q^*(\theta,1);$$

From which we get the required expression (63)

6. Numerical Illustration

This section contains some numerical illustrative examples. Note that for the sake of computational convenience; let us assume that the distributions of the service times, the delay times, the repair times and the vacation times are exponential. The distribution of the sizes of successive arriving batches is assumed to be geometric with parameter a, so that $a_{[1]} = \frac{1}{a}$ and $a_{[2]} = \frac{(2-a)}{a^2}$.

6.1 Optimal design

The optimal design of a queueing system is to determine the optimal system parameters using some cost functions. To illustrate, let c_h be the holding cost per unit time for each customer present in the system, c_0 be the cost per unit time for keeping the server on and in operation, c_s be the setup cost per busy cycle, and c_a be the startup cost per unit time for the preparatory work of the server before staring the service. These unit costs can be combined with the performance measures obtained above to write the system total expected cost per unit of time as

$$TC = c_h L_s + c_0 \frac{E(T_b)}{E(T_c)} + c_s \frac{1}{E(T_c)} + c_a \frac{E(T_0)}{E(T_c)}$$
$$= c_h L_s + c_0 \rho_H + c_s \lambda a_{[1]} \left(1 - \rho_H\right) + c_a \left(1 - \rho_H\right)$$

Here $E(T_0) = \frac{1}{\lambda a_{[1]}}$ is the expected length of an idle period, while $E(T_b)$ and $E(T_c)$ are the expected length of a busy period and a busy cycle, respectively, and are given by (52) and (63) respectively. To show how any parameter can affect the system performance, let us assume that the various distributions involved are exponential.

For the following values of the parameters: $p = 0.5, \lambda = 0.325, c_h = 5, c_0 = 100, c_S = 1000, c_a = 100,$ $\alpha_1 = 0.5,\ \alpha_2 = 0.5, a = 0.5,\ \beta_1^{(1)} = 0.5,\ \beta_2^{(1)} = 0.4,\ g_1^{(1)} = \beta_1^{(1)} / 5,\ g_2^{(1)} = \beta_2^{(1)} / 5,\ \gamma_1^{(1)} = g_1^{(1)} / 2,$ $\gamma_2^{(1)} = g_2^{(1)} / 2,\ \nu^{(1)} = 0.45$, we find a total expected cost per unit of time is $TC = 294.769$.

The effect of the system parameters on the optimal policy can be easily undertaken numerically. For example,

Tables 1-5 show the effect on the mean queue size L_S and on the optimal cost TC of the parameters of interest to us in this paper, namely, the breakdown rates α_1 and α_2, the probability of the Bernoulli vacation p, the parameter of the geometric distribution a and the arrival rate λ, respectively. All the other parameters are kept unchanged.

Again in Table 6 we observe that for higher values of λ, the rate of increase in L_S is faster than the lower values of λ for various values of p for both the cases, as it should be.

Finally in Table 7 and in Table 8, the numerical results are summarized in which the steady-state server availability A_V and failure frequency M_f are calculated with the given data. Clearly, high value of α_i (i=1, 2) results in low server availability and high failure frequency.

Table 1. Effect of the FPS breakdown rate α_1 on the optimal cost

α_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
L_S	8.43809	8.62019	8.80974	9.00719	9.21306	9.4279	9.6523	9.88692	10.1325
TC	303.569	301.311	299.09	296.908	294.769	292.674	290.628	288.632	286.691

Table 2. Effect of the SPS breakdown rate α_2 on the optimal cost

α_2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
L_S	8.70725	8.82873	8.95344	9.08151	9.21306	9.34826	9.48725	9.63019	9.77725
TC	300.352	298.931	297.527	296.139	294.769	293.417	292.084	290.771	289.478

Table 3. Effect of the Bernoulli vacation on the L_S and optimal cost

p	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
L_S	5.654225	6.53064	7.24865	8.12283	9.21306	10.6144	12.4871	15.1245	19.1269
TC	354.399	338.394	322.972	308.33	294.769	282.763	273.114	267.289	268.288

Table 4. Effect of the arriving batch sizes on the L_S and optimal cost

a	0.45	0.5	0.55	0.6	0.65	0.7	0.75	0.8	0.85	0.9
L_S	16.6146	9.21306	6.30395	4.74601	3.77372	3.10832	2.62388	2.25518	1.965	1.73059
TC	286.411	294.769	308.134	317.274	322.244	324.064	323.654	321.707	318.719	315.043

Table 5. Effect of the arrival rate on the L_S and optimal cost

λ	.125	0.15	0.175	0.2	0.225	0.25	0.275	0.3	0.325	0.35
L_S	2.5199	2.82761	3.19444	3.64043	4.1958	4.90842	5.85884	7.19396	9.21306	12.6345
TC	288.443	307.353	320.626	328.362	330.713	327.917	320.378	308.83	294.769	281.788

Table 6. Effect of the arrival rate and the Bernoulli vacation on the mean queue size

λ	p								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	2.11775	2.15181	2.18644	2.22168	2.25753	2.29403	2.33119	2.36903	2.4076
0.125	2.32	2.346682	2.41755	2.4681	2.5199	2.57303	2.62754	2.6835	2.74098
0.15	2.54965	2.61582	2.68412	2.75467	2.82761	2.90309	2.98129	3.06238	3.14656
0.175	2.81308	2.90247	2.99562	3.09283	3.19444	3.30079	3.41231	3.52944	3.65269
0.2	3.11883	3.23875	3.36521	3.49886	3.64043	3.79076	3.95084	4.12176	4.30482
0.225	3.47858	3.63954	3.81176	3.99663	4.1958	4.41123	4.64523	4.90059	5.18068
0.25	3.90874	4.12629	4.36327	4.62274	4.90842	5.22493	5.578	5.97493	6.42505
0.275	4.43313	4.73116	5.06326	5.43624	5.85884	6.34247	6.90235	7.55918	8.34197
0.3	5.08766	5.5046	5.9831	6.53897	7.19396	7.97884	8.93854	10.1414	11.6969
0.325	5.92915	6.53064	7.24865	8.12283	9.21306	10.6144	12.4871	15.1245	19.1269

Table 7. Effect of α_1 on reliability indices

α_1	A_V	M_f
0.1	0.833275	0.1625
0.2	0.8284	0.195
0.3	0.823525	0.2275
0.4	0.81865	0.26
0.5	0.813775	0.2925
0.6	0.8089	0.325
0.7	0.804025	0.3575
0.8	0.79915	0.39
0.9	0.794275	0.4225

Table 8. Effect of α_2 on reliability indices

α_2	A_V	M_f
0.1	0.826255	0.1885
0.2	0.823135	0.2145
0.3	0.820015	0.2405
0.4	0.816895	0.2665
0.5	0.813775	0.2925
0.6	0.810655	0.3185
0.7	0.807535	0.3445
0.8	0.804415	0.3705
0.9	0.801295	0.3965

The graphs below show the effect of some of the system parameters on the total expected cost per unit of time and on mean queue size in the system. To investigate the effect of server failures on the total expected cost per unit of time, we give different values to mean failure rates and record the corresponding value of the system total expected cost and mean queue size. Figure 1 and Figure 3 below show that TC decreases as mean failure rates increases. From Figure 2 and Figure 4, we observe that L_S increases as α_i $(i = 1,2)$ increases. When breakdown rates increase, the server is unable to provide service for the customers, which leads to the expected number of customers in the system becoming larger and the completion period longer.

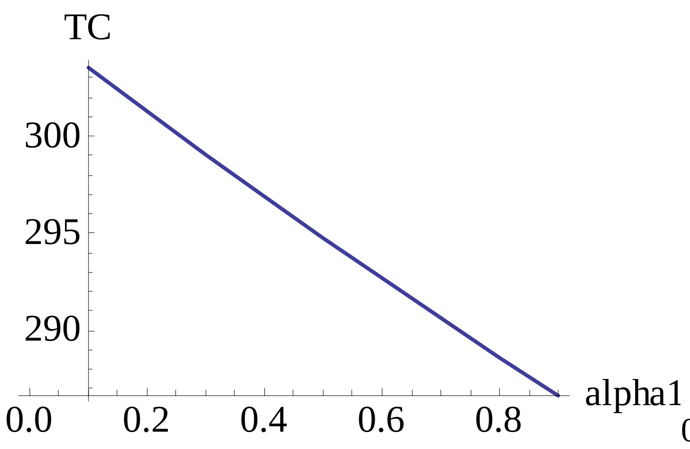

Figure 1

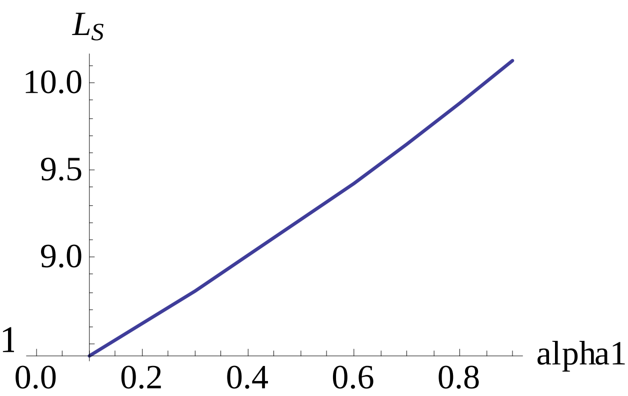

Figure 2

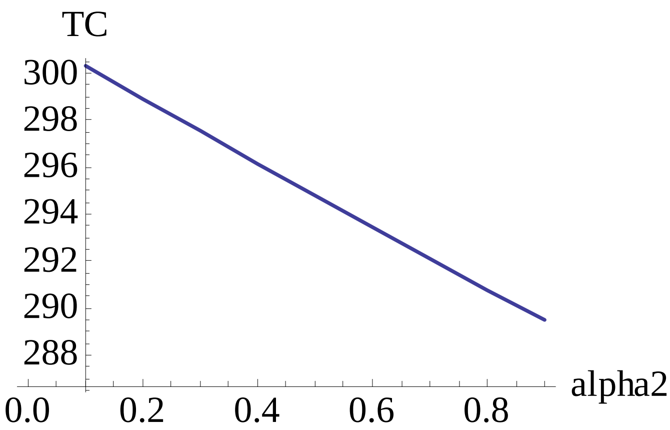

Figure 3

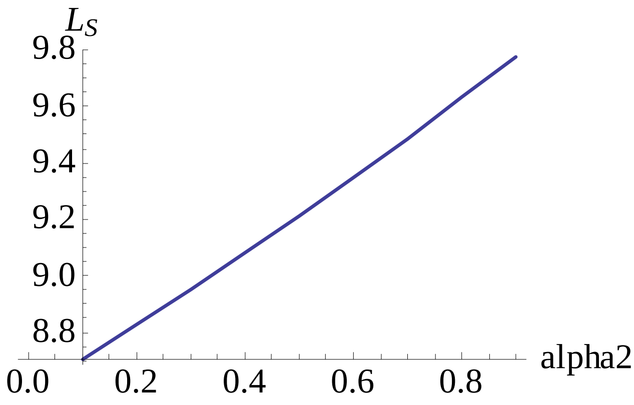

Figure 4

Suppose now we are interested in the effect of the Bernoulli vacation schedule on the system performance. Keeping the values of the system parameters unchanged, we vary the probability of a vacation from 0 to 0.9 and again record the corresponding values of the system total expected cost and mean queue size in the system. Figures 5 and 6 below depict the variations of the cost and mean queue size respectively with the probability of a vacation. We see that TC first decreases as p increases and then becomes stably as p becomes large. Figure 6 reports that mean queue size increases as p increases. As expected, a larger p implies that the number of customers and the completion period becomes larger, due to ongoing preventative maintenance having a higher probability.

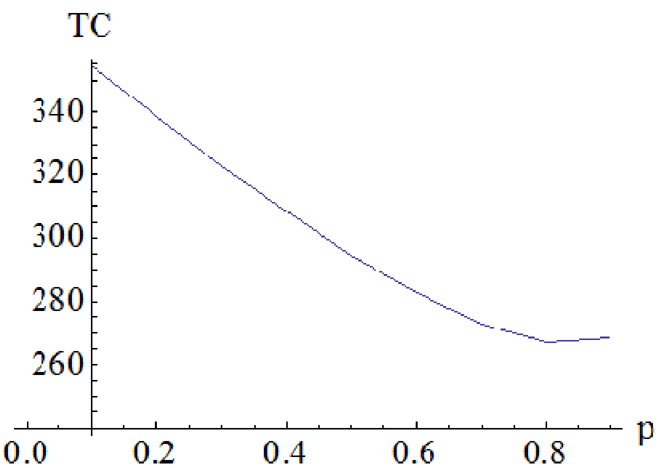

Figure 5

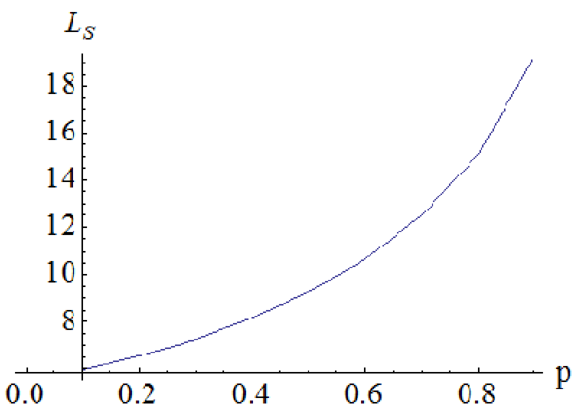

Figure 6

Again we want to see the effect of arriving batch sizes on the system performance. Using the same values of the system parameters as above, we vary the values of the arriving batch sizes from 0.45 to 0.9 and record the corresponding values of the system total expected cost and mean queue size in the system. Figures 7 and 8 below depict the variations of the cost and mean queue size respectively with the variation of arriving batch sizes. From Figure 7 one sees that TC first increases $(a \le 0.7, \rho_H \le 0.550875)$ and then decreases $\left(a > 0.7, \rho_H > 0.550875\right)$ with increasing a. We find that the maximum cost is $TC = 324.064$, and it is obtained when a is 0.7; on the other hand L_S decreases as batch size a increases.

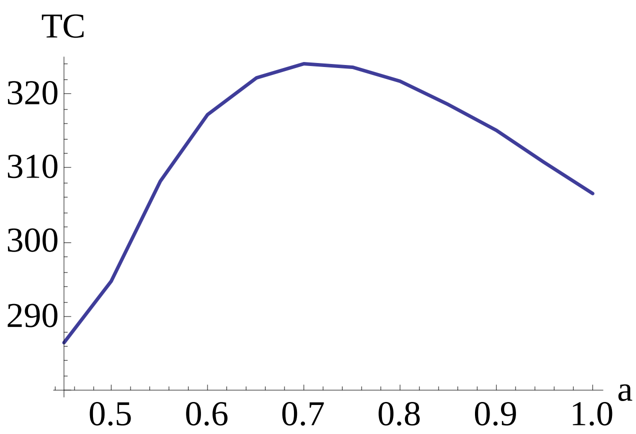

Figure 7

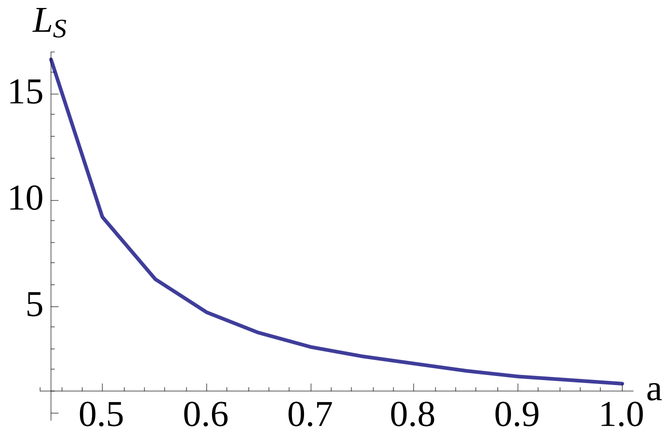

Figure 8

Finally, we want to see the effect of arrival rate when all the data are kept unchanged. Figure 9 and Figure 10 show the variations of the system total cost and mean queue size in the system respectively when the arrival rate varies from 0.125 to 0.35. Figure 9 reveals that TC first increases ($\lambda \leq 0.225$) and then decreases$\left(\lambda > 0.225\right)$. We find that the maximum cost is $TC = 330.713,$, and it is obtained when $\lambda = 0.225$ and in Figure 10 it appears that L_s increases as λ increases.

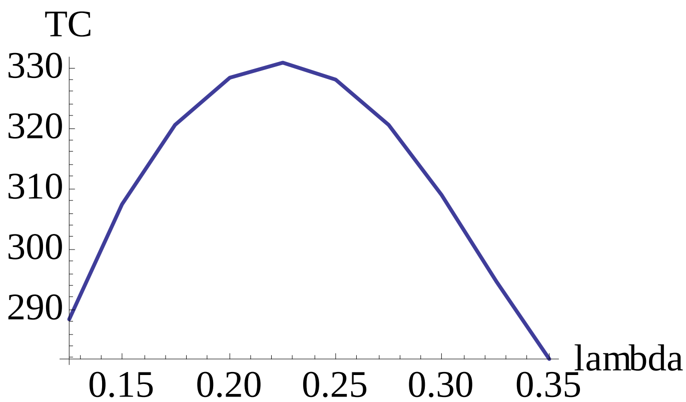

Figure 9

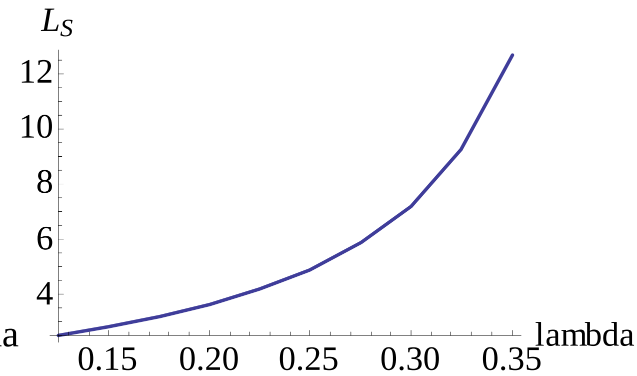

Figure 10

ACKNOWLEDGEMENT

The authors thank the referees for their comments and suggestions, which brings the paper in current form.

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