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# Dominated parasitic flow loops in networks

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***Abstract*** — The paper introduces the concept 'dominated parasitic flow loops' and demonstrates that these occur naturally in real networks transporting interchangeable commodity. The dominated parasitic flow loops are augmentable broken loops which have a dominating flow in one particular direction of traversing. The dominated parasitic flow loops are associated with transportation losses, congestion and increased pollution of the environment and are highly undesirable in real flow networks.

The paper derives a necessary and sufficient condition for the non-existence of dominated parasitic flow loops in the case of presence of paths with zero and non-zero flow. The necessary and sufficient condition is at the basis of a method for determining the probability of a dominated parasitic flow loop. The results demonstrate that the probability of a dominated parasitic flow loop is very large and increases very quickly with increasing the number of flow paths.

Dominated parasitic flow loops can be drained by augmenting them with flow, which results in an overall decrease of the transportation cost, without affecting the quantity of delivered commodity from sources to destinations. Accordingly, an efficient algorithm for removing dominated parasitic flow loops has been presented and a number of important applications have been identified. The presented algorithm has the potential to save a significant amount of resources to the world economy.

***Keywords*** — parasitic flow loops, flow networks, dominated parasitic flow loops, optimization

## 1. PARASITIC FLOW LOOPS IN NETWORKS

Closed parasitic flow loops in networks are highly undesirable because they affect negatively the quality of service of networks. Parasitic flow loops circulate commodity unnecessarily, increase the cost of transportation in the networks and cause pollution and congestion.

Closed flow loops of matter and energy, which are beneficial for the system, have been studied in ecosystems (Ulanowicz and Kay, 1991). Closed data flow loops in genetic networks have also been recently reported in (Itzhack et al, 2013). Undesirable (parasitic) closed flow loops, in cases where the transported commodity physically travels along a closed contour, have been reported in computer networks, where due to faults in the routing, the packets physically travel along a closed loop (Hengartner et al, 2002; Paxson, 1997). Sharir (1981) considered strongly connected components of a graph, which implies the existence of closed cyclic paths.

Todinov (2013a) established that closed parasitic flow loops could appear even if the transported commodity does not physically travel along a closed contour. This work established for the first time that closed parasitic flow loops occur naturally in real networks dispatching the same type of interchangeable commodity - fuel, electricity, gas, chemicals, particular goods, particular services, etc. by following directed straight-line paths from sources of flow to destinations. The example in Fig.1, taken from (Todinov 2013a) illustrates this phenomenon.

Suppose that the network in Fig.1a transports interchangeable commodity (for example, the same type and quality of petrol, electricity, the same type goods etc.). The throughput capacity of each edge in the network if Fig.1a is 100 units. Assume that the commodity is transported along straight-line paths only: along path (1,2,3,4,5,6), path (7,8,9,4,10,11,12) and path (13,14,10,3,15,16). Despite that the commodity dispatched from each node travels along a straight line to its destination and no commodity physically travels along a closed contour, a directed loop carrying 100 units of flow essentially appears between the intersection nodes 3,4 and 10. Removing 100 units of flow from the directed flow loop (3,4,10,3) reduces the flow along edges (3,4), (4,10) and (10,3) to zero, without diminishing the amount of total flow sent from the source nodes to the destination nodes (Fig.1b).

Even for very few intersecting flow paths, the probability of a directed flow loop is large, which means that the existence of directed parasitic flow loops in real networks is practically inevitable. Thus, for 'n' randomly oriented intersecting straight-line flow paths on a plane, the probability of a parasitic flow loop between some of the intersection

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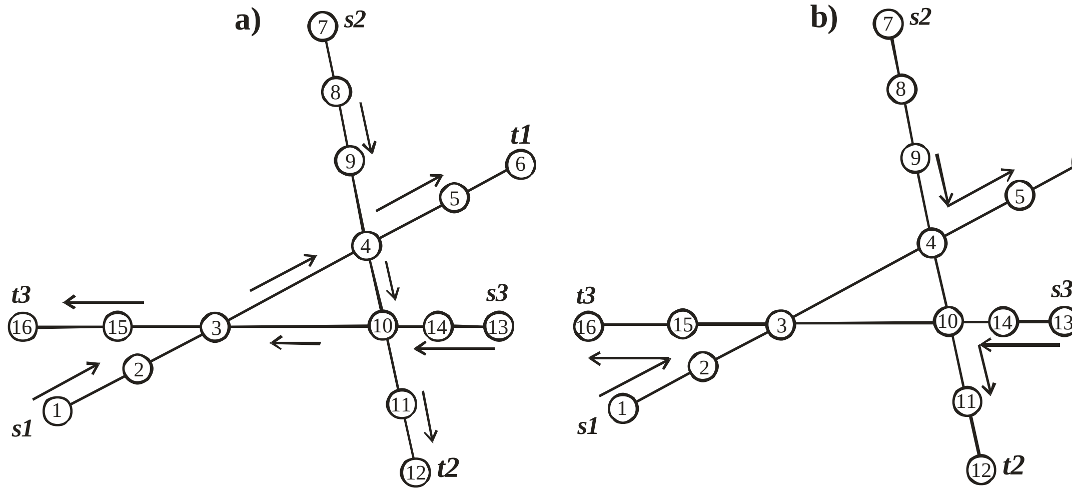


**Figure 1**. a) Closed parasitic flow loops may appear even if the transported commodity does not travel along a closed contour; b) Closed parasitic flow loops can be removed without affecting the throughput flow from sources to destinations (Todinov 2013a).

points is equal to $1 - n / 2^{n-1}$ and approaches very quickly unity with increasing the number of intersecting paths 'n' (Todinov, 2013b).

Despite the large number of published algorithms on optimising the flows in networks (Ahuja et al.,1993; Asano and Asano, 2000; Cormen at al. 2001), the question related to identifying and removing closed parasitic flow loops in flow networks, has evaded the efforts of researchers working in this field. Indeed, despite the years of intensive research on static flow networks, a surprising omission (discussed in detail in Todinov (2013a)) has been made: the algorithms for maximising the throughput flow published since 1956 leave highly undesirable parasitic flow loops in the "optimised" networks.

The treatment of parasitic flow loops presented in (Todinov 2013a) was limited to closed parasitic flow loops only. In (Todinov 2013b), 'almost-directed flow loops' have been introduced, where the direction of the flows in all but one edge is along a particular traversing direction. The almost-directed flow loops were also shown to be associated with losses. However, no treatment of the probability of an almost-directed flow loops has been presented in (Todinov 2013b) or an algorithm for removing almost-directed flow loops.

The present paper goes significantly beyond the treatment in (Todinov 2013a, 2013b) by showing that the loops do not have to be directed or almost-directed to be associated with losses. The present paper demonstrates that augmentable loops which have only dominating flow along a particular direction of traversal are also associated with transportation losses and congestion, and are highly undesirable in real flow networks.

Consequently, the purpose of this paper is to extend the analyses presented in (Todinov 2013a and 2013b) by:

(i) Introducing and defining the general concept 'dominated parasitic flow loop';

(ii) Demonstrating that similar to the directed and almost-directed parasitic flow loops, the dominated parasitic flow loops are also associated with significant losses and their draining/removal has the potential to save a significant amount of resources to the world economy.

(iii) Suggesting a general algorithm for optimising real networks by removing all directed, almost-directed and dominated parasitic flow loops.

## 2. TRANSPORTATION LOSSES ASSOCIATED WITH DOMINATED PARASITIC Loops

A dominated parasitic flow loop is defined along a closed contour with a specified direction of traversing (Fig.2a). It is a sequence of *n* sections/edges ($n \geq 3$) in which the flow along $n-k$ sections/edges (not necessarily sequential) points against the direction of traversing of the contour and the rest of the *k* sections/edges (the closing edges) are either empty or partially filled with forward flow.

The *n-k* edges with backward flow will be referred to as 'backward edges/sections' while the remaining '*k*' edges will be referred to as 'closing edges/sections'.

For a dominated parasitic flow loop to be present, the cyclic path must be augmentable and the sum of the lengths of the $n-k$ edges with backward flow must be greater than the sum of the lengths of the closing *k* edges:

$$\sum_{i=1}^{n-k} l_i^{(b)} > \sum_{j=1}^{k} l_j^{(c)} \tag{1}$$

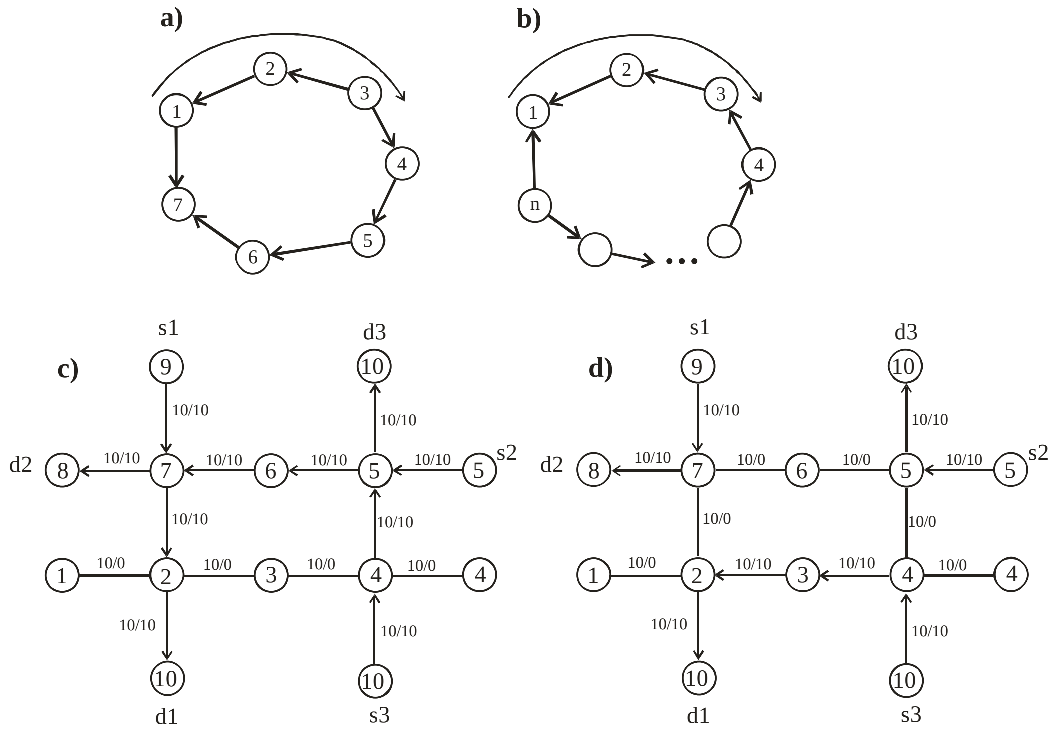


**Figure 2**. a) A dominated parasitic flow loop; b,c) Removal of a dominated parasitic flow loop.

where $l_i^{(b)}$ is the length of the ith backward edge and $l_j^{(c)}$ is the length of the jth closing edge.

Essentially, the sum of the lengths of the edges with backward flows dominates the sum of the lengths of the closing edges. The dominated flow loop can be thought as an augmentable broken flow loop for which some of the flow travels in backward direction and the sum of the lengths of the backward sections dominates the sum of the lengths of the closing sections. To be augmentable, the closing edges must either be empty or partially saturated with forward flow.

Let $\Delta$ be the flow magnitude by which a dominated flow loop can be augmented without violating the capacity constraints on the edges. The dominated parasitic flow loop can be augmented (essentially drained) by decreasing the flow along the *n-k* edges with dominating (backward) flow by $\Delta$ and increasing the flow with the same amount $\Delta$ along the closing *k* edges. The draining operation does not violate the flow conservation at each node and the capacity constraints along the edges and leads to a new feasible flow in the network.

Note that the closed parasitic flow loops introduced in Todinov (2013a) and the almost-directed flow loops introduced in Todinov (2013b) are only special cases of a dominated parasitic flow loop. The concept 'dominated parasitic flow loop' is much broader. It includes not only directed parasitic loops and almost-directed parasitic loops (broken at a single edge), but also augmentable loops broken at more than a single edge. Note that for a dominated parasitic flow loop to be present, the sum of the lengths of the edges with backward flow must be greater than the sum of the lengths of the closing edges (see equation (1)). This condition *is not required* for the almost-directed parasitic flow loops introduced in (Todinov, 2013b), which have a single ($k = 1$) closing edge (Fig.2b). Indeed, the length of a polygonal path between two points is always greater than the distance between the two points. Therefore, if the only closing edge is the *n*-th edge, the inequality (1) becomes

$$\sum_{i=1}^{n-1} l_i^{(b)} > l_n^{(c)}, \tag{2}$$

which is automatically fulfilled. As a result, inequality (1) is automatically fulfilled for an almost-directed flow loop. However, for a dominated flow loop with more than a single closing edge, inequality (1) may not be fulfilled.

Dominated parasitic flow loops are associated with significant losses and their removal/draining is highly beneficial. After the draining of a dominated flow loop, either one or more edges with backward flow will become empty or one or more of the *k* closing edges loop will become fully saturated with flow. As a result, the dominated parasitic loop will be removed.

Figure 2c features the dominated parasitic flow loop (2,7,6,5,4,3,2). The backward edges are the edges (2,7),(7,6),(6,5) and (5,4). The closing edges are the empty edges (4,3) and (3,2). The first number of the edge labels denotes the edge

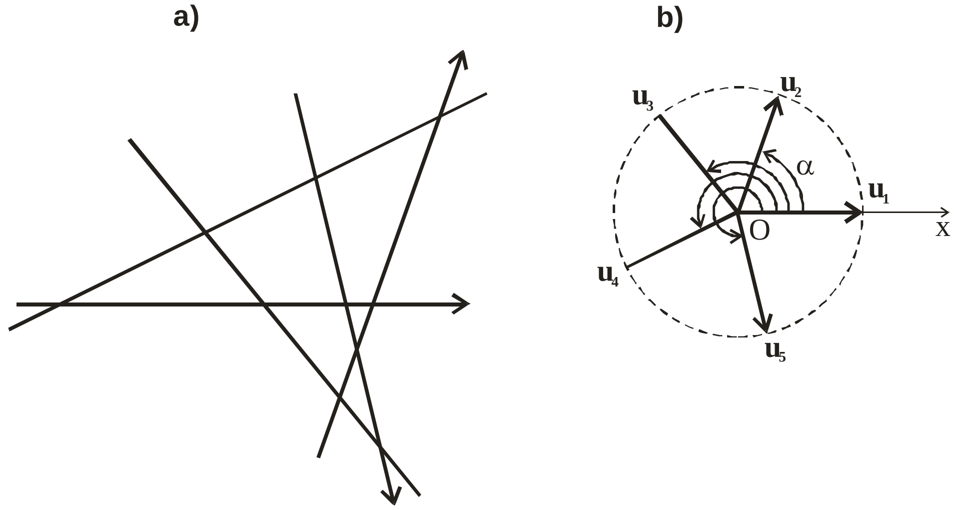


**Figure 3**. a) Randomly oriented intersecting flow paths; b) All direction vectors of the source-destination paths can be translated to start from a common point *O*.

capacity while the second number stands for the actual flow along the edge. The loop can be drained by augmenting it with 10 units of flow in the direction (2,7,6,5,4,3,2). The result is the network in Fig.2d, where no dominated parasitic flow loops exist.

Dominated parasitic flow loops and augmentable broken loops with dominating flow along a particular direction of traversing are equivalent concepts.

If the cost of transportation per unit distance does not vary on the different edges, draining an augmentable broken flow loop with dominating flow always results in a reduction of the transportation cost. In this case, the following theorem can be stated.

**Theorem 1**. *Augmentable broken loops, with dominating flow along a particular direction of traversing, are associated with an increased transportation cost which can be reduced by draining the loops.*

**Proof**. Denote the cost of transportation per unit length by *c*. Consider the *n-k* edgfes with dominating backward flow. Denote the length of these edges by $l_1^{(b)}$ , $l_2^{(b)}$ ,..., $l_{n-k}^{(b)}$ . Denote the lengths of the *k* closing edges by $l_1^{(c)}$ , $l_2^{(c)}$ ,..., $l_k^{(c)}$ . According to the definition of a dominated parasitic flow loop, inequality (1) holds. Because the cost of transportation per unit length is the same, augmenting the loop with the maximum possible (bottleneck) flow $\Delta_{\max}$ will result in a reduction of the transportation cost by $-c\,\Delta_{\max}\sum_{i=1}^{n-k} l_i^{(b)}$ along the backward edges and an increase of the transportation cost by $c\Delta_{\max}\sum_{j=1}^{k} l_j^{(c)}$ along the closing *k* edges. The augmentation (draining) is performed against the direction of the dominating flow along the backward edges. From inequality (1), the inequality

$$c\Delta_{\min}\left(\sum_{j=1}^{k} l_j^{(c)} - \sum_{i=1}^{n-k} l_i^{(b)}\right) < 0$$

is obtained, which means that draining a broken loop with dominating flow along a particular direction of traversing results in a net decrease of the cost of transportation. Consequently, the existence of a dominating flow along augmentable broken loops is associated with increased transportation costs.□

## 3. ESTIMATING THE PROBABILITY OF A DOMINATED PARASITIC FLOW LOOP

The unexpectedly high probability of existence of dominated parasitic flow loops will be demonstrated by considering randomly oriented intersecting straight-line flow paths on a plane (Figure 3a).

All flow paths transport the same type of interchangeable commodity (e.g. petrol) and along each flow path, there is either flow with a particular direction (Fig.3a, the lines with an arrow) or no flow (Fig.3a, the lines without arrows).

It is assumed that there are at least three intersecting flow paths; there are no parallel paths and no three paths intersect into a single point. These conditions are natural and common. Indeed, for randomly oriented straight-line flow paths in a plane, it is very unlikely to find two parallel paths or three paths intersecting into a single point. *The likelihood that a dominated*

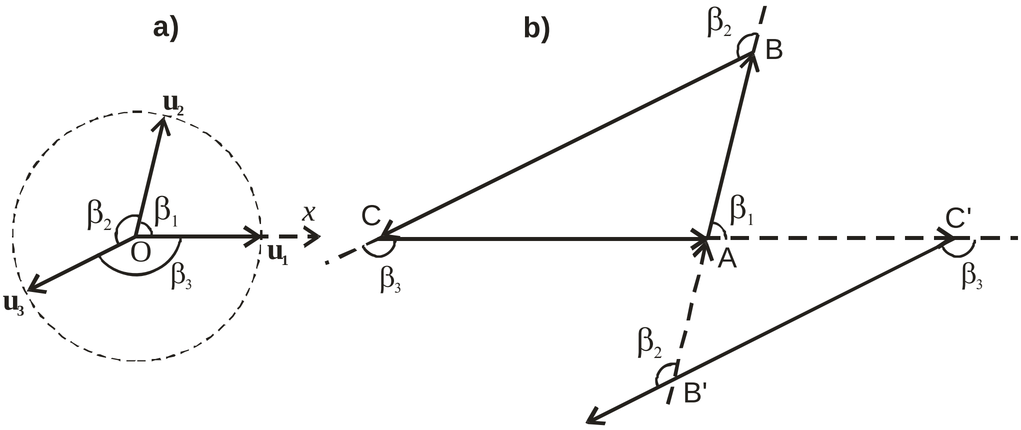


**Figure. 4**. a) Direction vectors of the non-zero flow paths; b) A directed parasitic flow loop formed by the three flow paths;

*parasitic flow loop will be present between the intersection points of the network, given that the orientation of the intersecting flow paths is random, will be termed 'probability of a dominated parasitic flow loop for random flow paths in a plane'.*

The existence of a dominated parasitic flow loop anywhere between the points of intersection implies the existence of a triangular dominated parasitic flow loop. As a result, the existence of a triangular dominated parasitic flow loop is a necessary condition for the existence of a dominated parasitic flow loop. The existence of a triangular dominated parasitic flow loop is also a sufficient condition for the existence of a dominated parasitic flow loop.

Consequently, the probability of a dominated parasitic flow loop for randomly oriented flow paths can be estimated by estimating the probability of a triangular dominated parasitic flow loop. A unit vector can be assigned to each flow path (Fig.3b,4a), pointing in the direction of the flow along the flow path.

The angle $\alpha$ which the unit vector subtends with the horizontal axis (Fig.3b) gives the orientation of the flow path and the direction of the flow along the path, if the path carries non-zero flow. Each flow path is randomly oriented, which means that the angle which the direction vector subtends with the fixed horizontal $x$-axis is uniformly distributed in the interval (0, $2\pi$). In other words, all possible flow path orientations are characterised by the same probability.

The unit vectors assigned to the flow paths will be referred to as 'direction vectors'. They can all be translated at the common origin $O$, as shown in Fig.3b and 4a. Two types of directions will be distinguished: *non-zero flow direction vectors*, corresponding to flow in paths with non-zero flow and *zero flow directions* - corresponding to the orientation of flow paths whose flow is zero. It is assumed that the flow through the non-zero flow paths fills the entire available throughput capacity of the path. For the sake of brevity, the non-zero flow direction vectors will be referred to as *nz-vectors.*

**Theorem 2**. *If it is possible to select three nz-vectors which do not lie in the same half plane, then a directed parasitic flow loop is present.*

**Proof**. Suppose that the three nz-vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, do not lie in a single half-plane (Fig.4a). Then, each of the angles $\beta_1$, $\beta_2$, $\beta_3$ between the vectors, is smaller than $\pi$ and the sum of the angles is exactly equal to $2\pi$:

$$0 < \beta_i < \pi,\ i = 1,2,3;\ \beta_1 + \beta_2 + \beta_3 = 2\pi$$

The intersections of the flow paths form a triangle, the external angles of which correspond to the angles $\beta_1$, $\beta_2$ and $\beta_3$ between the nz-vectors (Fig.4b). Because of the relationship $\beta_1 + \beta_2 + \beta_3 = 2\pi$, the flows along the sides of the triangle determined by the intersections of the flow paths, always form a closed directed flow loop (Fig.4b). □

**Theorem 3**. *If all nz-vectors lie in the same half plane, a directed parasitic flow loop does not exist.*

**Proof**. Suppose that the flow paths with nz-vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, define a triangular directed flow loop (Fig.5a). The sides of the triangular loop can then be presented by $k_1\mathbf{u}_1$, $k_2\mathbf{u}_2$ and $k_3\mathbf{u}_3$, where $k_1$, $k_2$ and $k_3$ are some constants. The sum of the vectors $k_1\mathbf{u}_1$, $k_2\mathbf{u}_2$ and $k_3\mathbf{u}_3$ is zero and as result, the relationship (Todinov 2013a):

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3 = 0 \tag{3}$$

holds. A similar relationship also holds for the projections of the vectors $k_1\mathbf{u}_1$, $k_2\mathbf{u}_2$ and $k_3\mathbf{u}_3$ along any specified axis. Thus, for an axis defined by a direction vector **s,** expanding $(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3)\mathbf{s} = 0$ yields the relationship (Todinov 2013a):

$$m_{1s} + m_{2s} + m_{3s} = 0 \tag{4}$$

**Todinov:** *Dominated parasitic flow loops in networks*
IJOR Vol. 11, No. 1, 001−017 (2014)

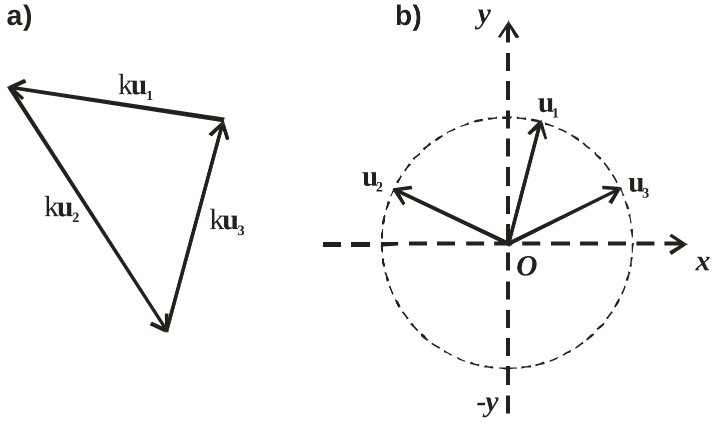


**Figure.5**. a) A closed triangular parasitic flow loop; b) the direction vectors of the flow paths with nonzero flow cannot possibly reside in a single half-plane (Todinov, 2013b).

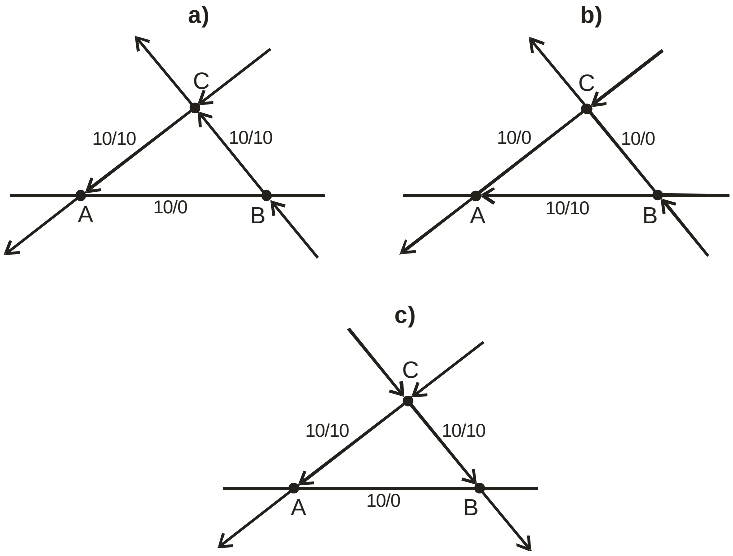


**Figure 6**. a) Intersecting two non-zero flow paths with a zero flow path yields a dominated parasitic flow loop b) The drained dominated parasitic flow loop from a); c) Intersection of two non-zero flow paths with a zero flow path does not always result in a dominated parasitic flow loop.

for the projections $m_{1s}=k_1\mathbf{u}_1\mathbf{s}$, $m_{2s}=k_2\mathbf{u}_2\mathbf{s}$ and $m_{3s}=k_3\mathbf{u}_3\mathbf{s}$. Consequently, equation (4) is a necessary condition for the existence of a directed triangular loop.

Suppose that there is a single half plane where all nz-vectors reside (Fig.5b) and a directed parasitic flow loop exists. The existence of a directed parasitic plow loop implies the existence of a triangular directed parasitic flow loop. The nz-vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ of the paths forming the directed triangular loop will also reside in the same half-plane. Without loss of generality, suppose that the axis *x* and the half-axis (O,y) define this half-plane (Fig.5b). The sum of the projections of the nz-vectors along the y-axis will now violate the necessary condition (4), because there will be a non-zero sum of projections along the axis (O,y) ($m_{1y}, + m_{2y} + m_{3y} > 0$). We arrived at a contradiction. Therefore, if all nz-vectors lie in the same half-plane, a directed parasitic flow loop does not exist. □

Here, it needs to be pointed out immediately, that although three non-zero flow paths may not form a parasitic flow loop, two non-zero flow paths intersected by a zero flow path may form a dominated parasitic flow loop.

For example, in Fig.6a, the non-zero flow paths BC and CA, each with capacity 10 units flow per unit time and each carrying 10 units of flow per unit time, have been intersected by the zero flow path AB with capacity 10 units of flow per unit time and carrying no flow. (The first number on the edge labels stands for the capacity of the edge and the second number stands for the actual flow along the edge). The cyclic path ACBA is augmentable. This cyclic path is essentially a dominated parasitic flow loop from which 10 units of flow can be drained. The result is the network in Fig.6b.

The non-zero flow paths in Fig.6c have also been intersected with a zero flow path. In this case however, no dominated parasitic flow loop is present.

**Lemma 1.** *If a single path with zero flow and more than one non-zero flow paths are present, a dominated flow loop is absent if all random nz-vectors corresponding to the non-zero flow paths reside in a single half-plane defined by the direction of the zero flow path.*

**Lemma 2.** *If a single path with zero flow and more than one non-zero flow paths are present, a dominated parasitic flow loop is present if the non-zero flow paths do not reside in a single half-plane defined by the direction of the zero flow path.*

These lemmas (see Fig.7a) follow from Theorem 3 and will not be proved here. The probability that all *n* random nz-vectors will reside in one of the two half-planes defined by the direction of the zero flow path is $1/2^{n-1}$. Consequently, for *n* non-zero flow paths and a single zero flow path, the probability of existence of a dominated flow loop is $p = 1 - 1/2^{n-1}$.

**Theorem 4**. *If more than one zero flow paths and more than one non-zero flow paths are present, dominated parasitic flow loops are absent if and only if all nz-vectors reside in one of the sectors defined by the directions of the zero flow paths.*

In Fig.7b, the four sectors defined by the directions of the two zero flow paths are $\beta_1$, $\beta_2$, $\beta_3$ and $\beta_4$. No dominated parasitic flow loop is present because all nz-vectors, defining the directions of the flows in the nz-paths, reside in sector $\beta_3$. In Fig.7c, however, the nz-vectors do not reside in a single sector defined by the direction of the single zero flow path $z_1$. As a result, a dominated parasitic flow loop will be present.

**Proof of Theorem 4**. Given that a dominated parasitic flow loop is absent, suppose the contrary – that not all nz-vectors reside in a single sector defined by the directions of the zero flow paths. In this case, there will be two nz-vectors, each residing in a different sector (Fig.7d, the vectors $u_1$ and $u_2$ residing in sectors $\beta_1$ and $\beta_2$). Because the sectors $\beta_1$ and $\beta_2$ do not overlap, there is a zero flow path whose direction $z_0$ defines two half-planes each of which contains one of the two sectors. As a result, the two nz-vectors residing in these sectors will reside in different half planes, with respect to the direction $z_0$ of the zero flow path (Fig.7d).

In this case, according to Lemma 2, a dominated parasitic flow loop will be present, which contradicts our assumption that a dominated flow loop is absent. Consequently, the nz-vectors cannot reside in different sectors; they all must reside in a single sector.

Suppose now that all nz-vectors reside in a single sector. We will show that, in this case, no dominated parasitic flow loop exists.

Indeed, the fact that all nz-vectors reside in a single sector implies that they all reside in a single half-plane. The existence of a dominated flow loop implies the existence of a dominated triangular flow loop. The dominated triangular parasitic flow loop cannot contain only nz-paths, because all nz-vectors reside in a single half-plane. According to Theorem 3, no directed parasitic flow loop exists in this case. The only possibility that remains is the triangular dominated flow loop to contain two nz-paths and a single zero flow path. However, this is impossible because with respect to any selected zero flow path, both of the nz-vectors will reside in the same half-plane formed by the direction of the zero flow path. According to Lemma 1, the zero flow path and the two nz-paths cannot form a dominated triangular flow loop. Consequently, no dominated parasitic flow loop exists if all nz-vectors reside in a single sector. The theorem has been proved. □

According to Lemma2, in the flow path configurations presented in Fig.8a and Fig.8b, a dominated parasitic flow loop exists. This is because the nz-vectors do not reside in one of the two sectors(half-planes) defined the direction of the zero flow path. In addition, the sum of the lengths of the two edges with non-zero flow, both pointing against the specified direction of traversing the loop, are always greater than the length of the edge with zero flow.

In Fig.8c and 8d, both nz-vectors reside solely in one of the two sectors defined by the direction of the zero flow path and, according to Lemma 1, no dominated parasitic flow loop exists.

Dominated parasitic flow loops do not exist in the case where two zero flow paths are intersected with a single non-zero flow path. In this case, condition (1) for the existence of a dominated parasitic flow loop is violated because the sum of the lengths of the edges with zero flow (the closing edges) is always greater than the length of the edge with non-zero flow (the backward edge).

### 3.1 Simulation algorithm and results

The orientation of the nz-vectors characterising the non-zero flow paths is determined simply by generating random numbers, uniformly distributed in the interval $(0, 2\pi)$. The orientation of the zero flow paths is determined by generating random numbers distributed in the interval $(0, \pi)$.

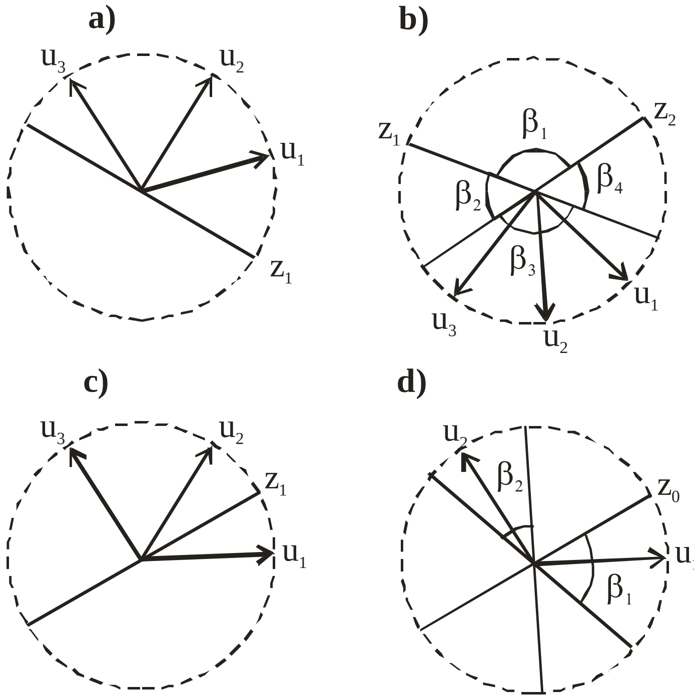


**Figure 7**. a) and b) A dominated parasitic flow loop does not exists; c) A dominated triangular parasitic flow loop exists; d) Two nz-vectors in two different sectors.

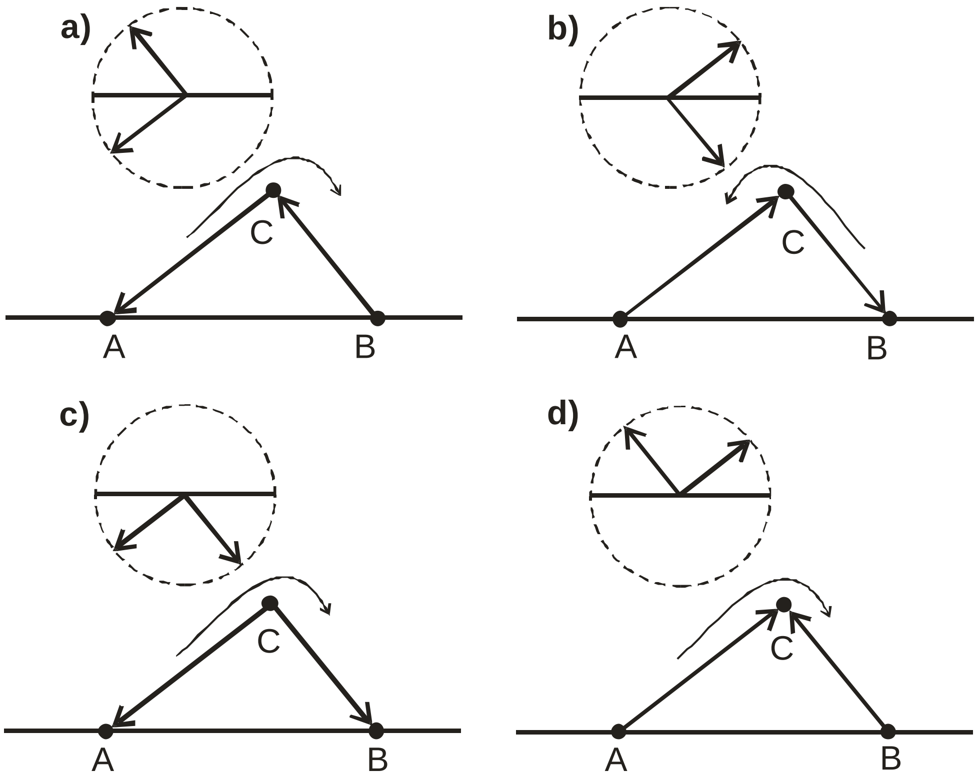


**Figure 8**. a,b) Flow configurations which: a,b) result in a dominated flow loop; c,d) do not result in a dominated flow loop.

Because the existence of a dominated flow loop between the points of intersection of the flow paths implies the existence of a triangular dominated flow loop, determining the probability of a dominated parasitic flow loop can be reduced to determining the probability of a triangular dominated flow loop. If no zero flow paths are present, determining the

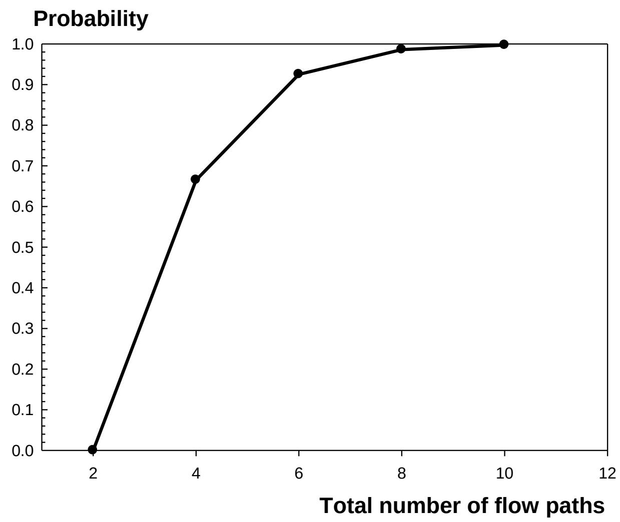


**Figure 9**. Probability of a dominated parasitic flow loop as a function of the total number of flow pats. The number of non-zero flow paths is equal to the number of zero flow paths.

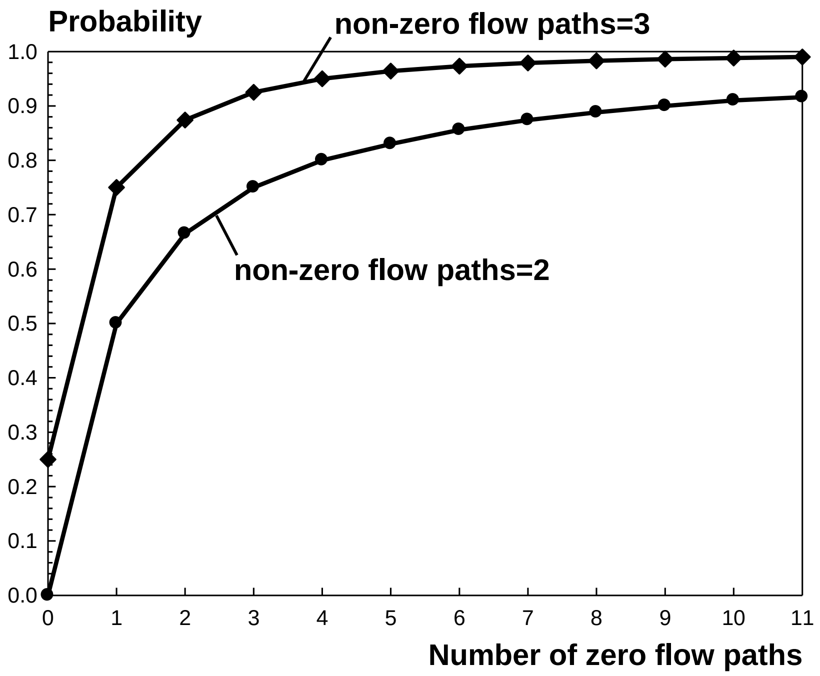


**Figure 10**. Probability of a dominated parasitic flow loop as a function of the number of randomly oriented zero flow paths, for n=2 and n=3 non-zero flow paths.

probability that no parasitic flow loop is present reduces to determining the probability that all non-zero vectors lie in a single half-plane. This is equivalent to determining the existence of an angle greater than $\pi$ between two adjacent nz-vectors. According to Todinov (2013a) this probability is $n / 2^{n-1}$, where n is the number of non-zero flow paths. Subtracting this probability from one, gives the probability $(1 - n / 2^{n-1})$ that there will be a parasitic flow loop.

If zero flow paths are also present, determining the probability that no dominated parasitic flow loop is present reduces to determining the probability that all nz-vectors will reside in a single sector formed by the directions of the zero flow paths. This check can be performed easily, if the random orientation angles of the zero flow paths and the nz-vectors are sorted (separately) in ascending order.

The algorithm is given in Appendix A. For randomly oriented non-zero flow paths and zero flow paths, such as in Fig.3a, the probability of a dominated parasitic flow loop for a different total number of flow paths has been plotted in Fig.9. The number of zero flow paths is equal to the number of non-zero flow paths.

With increasing the total number of flow paths, the probability of a dominated flow loop approaches unity very quickly.

In Fig.10, the probability of a dominated flow loop has been plotted as a function of the number of zero flow paths, for a mixture of zero flow paths and non-zero flow paths, where the number of non-zero flow paths is n=2 and n=3, respectively.

Again, with increasing the number of randomly oriented zero flow paths, the probability of a dominated parasitic flow loop increases very quickly. For three intersecting non-zero flow paths and two zero flow paths, the probability of a dominated parasitic flow loop is about 87%!

These results show that the existence of dominated flow loops on a set of intersecting zero and non-zero flow paths is practically inevitable.

## 4. COST FACTOR AND CYCLIC PATHS WITH A NEGATIVE COST FACTOR

### 4.1 Cost factor of an augmentable path and cancelling dominated parasitic flow loops from networks

Consider a network with undirected edges. A *path* in a flow network is a unique sequence of edges between two nodes. Suppose that the flow through a particular path can be increased by $\Delta$ without violating the capacity constraints on the edges and the flow conservation law at the nodes along the path. Such a path will be referred to as *augmentable path*.

Consider an edge *i* whose flow direction is opposite to the direction of traversing of the augmentable path. According to an earlier convention, such edges are referred to as 'backward edges'. Similarly, edges which are empty or whose flow direction is along the direction of traversing the augmentable path are referred to as 'closing edges'.

Suppose that the cost of transportation per unit length is $c$ units. Augmenting the path with $\Delta$ means that the flow along each backward edge is decreased and the flow along each forward edge is increased.

Because, during the path augmentation, the flow along the closing edges is increased, the transportation cost along the closing edges will increase. For the *i*th closing edge, the increase of the cost of transportation $\Delta C_i$ is equal to

$$\Delta C_i = \Delta \times c \times l_i^{(c)} \tag{5}$$

where $l_i^{(c)}$ is the length of the *i*th closing edge.

The increase of the cost of transportation for a path composed of $M_c$ closing edges only, is given by

$$\Delta C_c \approx \Delta \times c \times \sum_{i=1}^{M_c} l_i^{(c)} \tag{6}$$

Suppose that a path containing both closing edges and backward edges is augmented with flow with magnitude $\Delta$.

After the path augmentation, along any backward edge, the flow $\Delta$ is no longer transported but is prevented from being transported, because the flow along backward edges has been *decreased* by $\Delta$. Therefore, the increase in the transportation cost $\Delta C_j$, along a backward edge with index '*j*', has a negative sign:

$$\Delta C_j = -\Delta \times c \times l_j^{(b)} \tag{7}$$

For $M_b$ backward edges, the total increase of the cost of transportation can be approximated by

$$\Delta C_b \approx -\Delta \times c \times \sum_{j=1}^{M_b} l_j^{(b)} \tag{8}$$

where $l_j^{(b)}$ is the length of the *j*th backward edge.

Consequently, the total increase of the cost of transportation $\Delta C$ along a path including both backward and closing edges can be approximated by

$$\Delta C = \Delta C_f + \Delta C_b \approx \Delta \times c \left( \sum_{i=1}^{M_c} l_i^{(c)} - \sum_{j=1}^{M_b} l_j^{(b)} \right) \tag{9}$$

The quantity

$$\gamma = \sum_{i=1}^{M_c} l_i^{(c)} - \sum_{j=1}^{M_b} l_j^{(b)} \tag{10}$$

will be referred to as a '*cost factor of a path*'. As it will be shown later, this concept is of fundamental importance to the optimal solution.

A positive sign of the cost factor $\gamma$ means that if the path is augmented with flow, the transportation cost will increase. A negative sign of the cost factor $\gamma$ means that the transportation cost will decrease following the path augmentation. The larger the magnitude of the cost factor $\gamma$, the larger is the amount of the transportation cost increase/decrease following the path augmentation.

A dominated parasitic flow loop is always characterised by a negative cost factor. The augmentation of a dominated parasitic flow loop results in a decrease of the transportation cost. At the same time, the amount of flow transported from sources to destinations is not affected. Effectively, augmenting a parasitic flow loop in a direction opposite to the direction of its dominant (backward) flow, is a process of draining the flow loop. An indication of the presence of a dominated flow loop is the presence of an augmentable cyclic path with a negative cost factor, anywhere in the network.

Now suppose that there is a set of sources with specified generation and a set of consumers with specified consumption. The generated and consumed commodity is exchangeable, which means that any given consumer could be supplied from any given source. The following theorem then holds.

**Theorem 5.** *A necessary and sufficient condition for a minimum transportation cost is the non-existence of a cyclic path with a negative cost factor.*

**Proof**. Proving that the non-existence of augmentable cyclic paths with a negative cost factor is a necessary condition for a minimum transportation cost is straightforward. Suppose that for a given total throughput flow $f^*$ from sources to consumers, the transportation cost is the smallest possible. If there exists an augmentable cyclic path with a negative cost factor, this path could be augmented, which will result in different feasible edge flows, with the same throughput flow $f^*$ from sources to consumers but with a smaller transportation cost. However, this is impossible because, by assumption, the transportation cost is the smallest possible. We arrive at a contradiction.

Suppose now, that in a network characterised by edge flows $f_1(i,j)$ and throughput flow $f^*$ from sources to consumers, there is no augmentable cyclic path with a negative cost factor. Then, the transportation cost $C_1$, associated with edge flows $f_1(i,j)$ is the smallest possible. Indeed, suppose that edge flows $f_2(i,j)$ exist, associated the same throughput flow $f^*$ and a smaller transportation cost $C_2 < C_1$.

To prove this part of the theorem, it is necessary to use a theorem, stated by Ahuja et al. (1993). Here, this theorem will be stated as Lemma 3, to reflect the circumstance that it will be used to prove Theorem 5.

**Lemma 3**. *If there are two different feasible edge flows $f_1(i,j)$ and $f_2(i,j)$, resulting the same throughput flow, the flow $f_2(i,j)$ can be presented as a sum of the flow $f_1(i,j)$ and the augmented flows along a set of augmentable cyclic paths.*

According to Lemma 3, the set of feasible edge flows $f_2(i,j)$ resulting in throughput flow $f^*$, can be obtained from the set of feasible edge flows $f_1(i,j)$, resulting in the same throughput flow $f^*$, by a sum of augmentations along cyclic paths only. Without loss of generality, suppose that the edge flows $f_2(i,j)$ have been obtained from edge flows $f_1(i,j)$, after adding the augmented flows along *k* cyclic paths. The expected transportation cost $C_2$ is then given by

$$C_2 = C_1 + \Delta_1 \times c \times \gamma_1 + \Delta_2 \times c \times \gamma_2 + ... + \Delta_k \times c \times \gamma_k \quad (11)$$

where $\gamma_i$ is the cost factor of the *i*th cyclic path and $\Delta_i$ is the flow with which the *i*th cyclic path is augmented. Because of the assumption $C_2 < C_1$ (the edge flows $f_2(i,j)$ are associated with smaller losses than edge flows $f_1(i,j)$), the inequality

$$\Delta_1 \times c \times \gamma_1 + \Delta_2 \times c \times \gamma_2 + ... + \Delta_k \times c \times \gamma_k < 0 \quad (12)$$

must necessarily hold. This inequality however is impossible because, according to our assumption, there is no augmentable cyclic path with a negative cost factor. This completes the proof. □

Consequently, the transportation cost $C_1$ associated with edge flows $f_1(i,j)$ is indeed the smallest possible. Draining all cyclic paths with negative cost factors leads to the smallest transportation costs in the network.

### 4.2 Examples of networks with dominated flow loops.

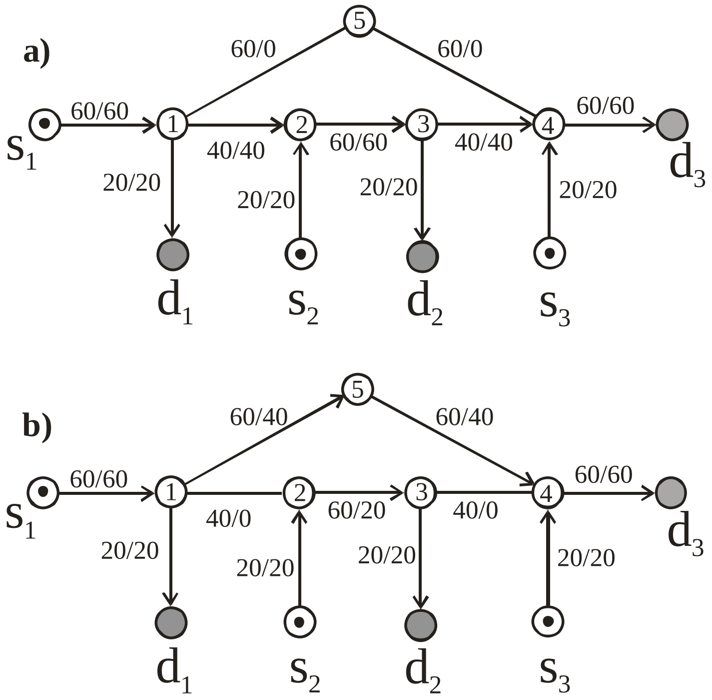


**Figure 11**. Electricity supply from three generators $s_1$, $s_2$, $s_3$, to three consumers $d_1$, $d_2$, $d_3$.

Typical examples of dominated flow loops exist in cases where a number of consumers are served by a number of sources and the commodities/services provided to the consumers are exchangeable. Here are some examples:

- A number of customers, living at different locations in a large city are serviced by a number of repairers of a particular type of equipment. The repairers belong to the same company and are also based at different locations in the city (each customer is serviced equally well by any repairer).
- Generators at different locations supply electricity to customers.
- Fuel terminals at different locations supply fuel to different petrol stations.
- A container supply company, with branches at different locations in a country, delivers the same type of containers to clients.
- Warehouses at different locations, supply supermarkets with the same type of goods.
- Students spending their industrial placement year at different locations in a country, are visited by members of staff living in different towns;
- Volunteers working for a charity and residing at different locations, visit elderly patients;
- Agents living at different locations in a city, travel to advertise the same type of commodities to prospective customers.

This list can be continued. As can be seen from the examples, dominated parasitic flow loops are present even in social support networks.

Consider the task of supplying electricity from three generators with capacities $s_1 = 60$ MW, $s_2 = 20$ MW and $s_3 = 20$ MW, to three consumers with consumption capacities $d_1 = 20$ MW, $d_2 = 20$ MW and $d_3 = 60$ MW. For the sake of simplicity, assume that the cost of transmitting 1 MW electricity for 1 hour, along each section, is *m*. One 'solution' is presented in Fig.11a. The first number on each edge/line is the line capacity (in MW) and the second number is the actual power flow transmitted through the line.

The solution from Fig.11a however, can be improved by noticing that the loop (4,3,2,1,5,4) is a dominated parasitic flow loop. Augmenting this loop with 40 MW results in the network flows from Fig.11b. The throughput flow from generators to consumers has not been affected but the transportation cost per hour has been reduced by 40*m*. For one year of continuous operation, this saved transportation cost will accumulate to a very large sum ($40 \times 365 \times 24 \times m = 350400m$). As can be verified, the solution in Fig.11a was not the optimal solution.

Another interesting application exists in the case where *n* volunteers belonging to the same organisation and living in different parts of a city are allocated to *n* patients also living in different parts of the city (Fig.12). Each volunteer must be assigned to exactly one patient. In this case, only the total length of the service routes matters, not the capacities of the routes. Consequently, a common capacity of 1 unit can be assigned to each edge. Because each volunteer must visit exactly

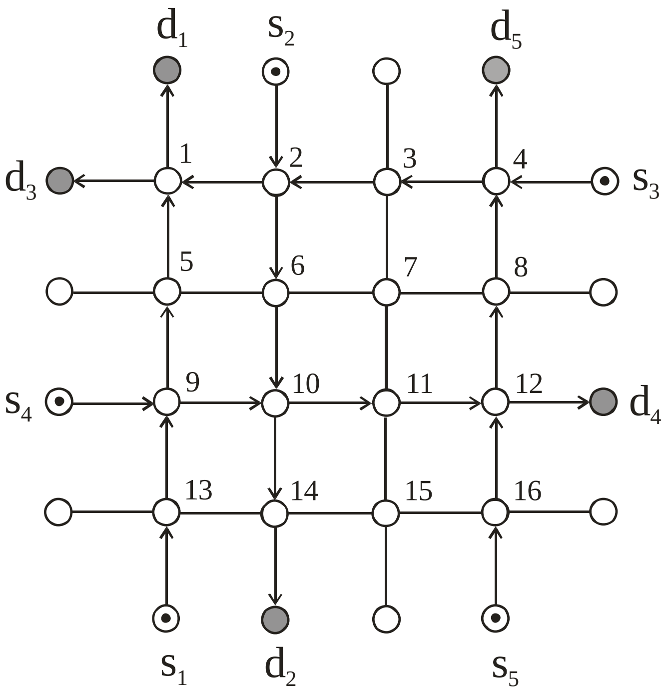


**Figure 12**. A real flow network before the optimisation.

one patient, the generation capacities of the 'sources' $s_1$, $s_2$,..., $s_5$ and the consumption capacity of the 'consumers' $d_1$, $d_2$,..., $d_5$ must also be set to be 1 unit (Fig.12). For the sake of simplicity, it has been assumed that all edges have the same length. The flow in the network is interchangeable, which means that the type of 'commodity' dispatched from any given source can satisfy any given consumer.

One way of satisfying the demand from consumers is shown in Fig.12. Each source has been connected with the corresponding destination through a direct, straight-line path. As it can be verified, this 'solution' is far from optimal.

There are dominated flow loops, which increase unnecessarily the cost of transportation of the 'commodity' in the network. For example, the network contains the parasitic flow loop (2,3,4,8,12,11,10,6,2), the dominated parasitic flow loop (14,10,9,13,14), etc. If the cost of transportation of per edge is 100 units, the cost of transportation of the commodity to all destinations amounts to 2500 units.

After augmenting the parasitic flow loop (2,3,4,8,12,11,10,6,2) with 1 unit flow and the dominated parasitic flow loop (14,10,9,13,14) with 1 unit of flow, the network from Fig.13 is obtained. No augmentable cyclic paths with negative cost factor exist, hence, according to Theorem 5, the obtained scheduling is associated with the smallest transportation cost. The resultant network is characterised by a transportation cost of 1500 units only, which constitutes a 40% reduction of the initial transportation costs.

Reducing the length of the supply paths in social support networks reduces not only the transportation costs and pollution but also the risk of delays. Suppose that the events causing delay along the supply paths follow a homogeneous Poisson process with intensity $\lambda$, and the total length of the supply paths is *L*. The probability of having a delay along the supply paths is given by the negative exponential distribution

$$p_d = 1 - \exp(-\lambda L) \tag{13}$$

Decreasing the total length $L$ of the supply paths reduces significantly the probability $p_d$ of a delay in the social support networks.

### 4.3 Algorithm for removing dominated parasitic flow loops from networks

Suppose that *n* sources supply $s_1$, $s_2$,..., $s_n$ quantities of interchangeable commodity to *m* consumers with consumptions $c_1$, $c_2$,..., $c_m$ : $\sum_{i=1}^{n} s_i = \sum_{j=1}^{m} c_j$ .

**Step 1.** The first step of the algorithm for removing dominated parasitic flow loops consists of transforming the initial flow network. A flow network with multiple sources and consumers (the *n* sources in Fig.14a with flow generation

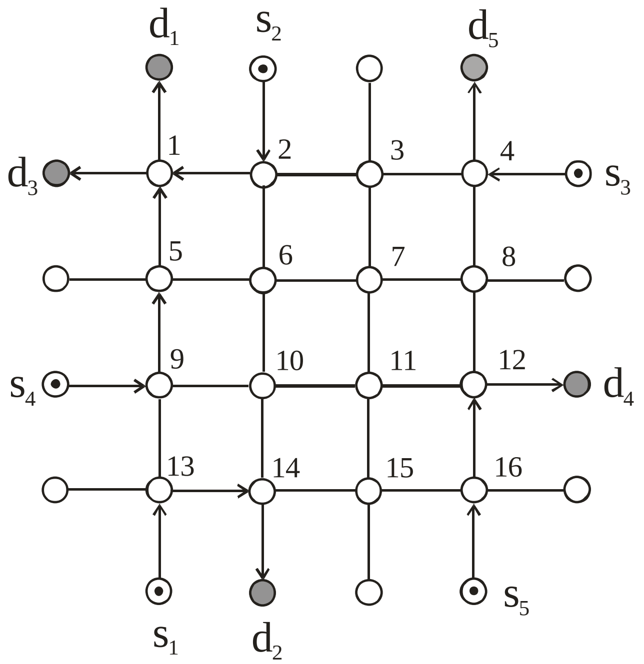


**Figure 13**. The flow network after the optimisation.

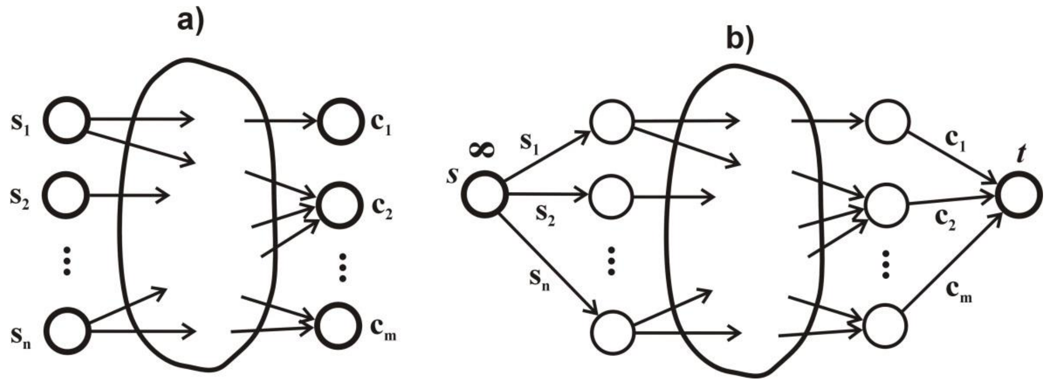


**Figure 14** a) An example of a flow network with *n* sources of interchangeable commodity and *m* consumers. b) The network (a) has been reduced to a flow network with a single super-source *s*, and a single super-consumer t.

quantities $s_1, s_2, ..., s_n$ and the *m* consumers with consumption quantities $c_1$, $c_2, ..., c_m$ ) is first reduced to a flow network with a single super-source *s* (Fig.14b) and a single super-consumer *t*.

The multiple sources in Fig.14a are replaced by a single super-source with infinite generation feeding each of the initial sources through lines with throughput flow capacities $s_1, s_2, ..., s_n$, equal to the amounts of commodity initially supplied by the sources. In a similar fashion, the multiple consumers in Fig.14a are replaced by a single consumer with infinite consumption capacity. The super-consumer is connected through lines with throughput flow capacities $c_1$, $c_2, ..., c_m$ equal to the amount of consumed quantities (Fig.14b).

The sources of flow and the consumers become ordinary throughput edges, with flow capacities equal to the flows generated by the sources and the flows consumed by the consumers. As a result, the sources of interchangeable commodity and consumers 'disappear', and instead, ordinary throughput edges appear.

**Step 2**. Assign zero transportation costs to the edges connecting the super-source *s* and the super-consumer *t* with the rest of the network. For the rest of the edges, assign cost of transportation proportional to the length of each edge.

**Step 3**. This step consists of maximising the flow in the transformed network (Fig.14b), from the super-source *s* to the super-consumer *t*, at a minimum transportation cost. A number of algorithms have already been proposed for maximising the flow at a minimum cost (Klein, 1967; Bennington, 1973; Friesdorf and Hamacher 1982; Tardos 1985; Goldberg and

Tarjan 1987,1989; Ahuja et al. 1992; Orlin 1993), some of which are characterised by a strictly polynomial running time (Tardos 1985; Orlin 1993).

The successive shortest path method with modified weights could for example be used in Step 3 for maximising the throughput flow at a minimum cost.

**Step 4**. The super-source *s*, the super-consumer *t* and their connecting edges are removed from the transformed network. The nodes corresponding to the initial sources and consumers become sources and consumers again. The edge flows in the resultant network define the optimal solution.

### 4.3.1 Proof of the correctness of the proposed algorithm.

After conducting Step 4, it can be shown that there are no dominated parasitic flow loops left in the original network.

Indeed, suppose that the specified throughput flow has been guaranteed and the sum of the costs of transportation along the edges is the smallest possible. Suppose that there exists a dominated parasitic flow loop (i, i + 1,..., i + c, i) in the network, starting and ending at a particular node 'i'.

The bottleneck residual capacity Δ of the loop is then determined and the loop is augmented with the bottleneck flow Δ. Because the augmentation is along a cyclic path, it does not affect the throughput flow in the network. The throughput flow from sources to destinations will remain unchanged after the augmentation. During the augmentation (draining) of the parasitic flow loop, the backward flows along the loop are reduced (the bottleneck flow is prevented from being circulated), and the flow along the closing edges is increased. Because the sum of the lengths of the backward edges is larger than the sum of the lengths of the closing edges, the overall cost of transportation will be reduced. However, this contradicts the assumption that the transportation cost associated with the initial edge flows is the smallest possible. Consequently, there can be no dominated parasitic flow loops in the transformed network, where the specified throughput flow from sources to destinations has been achieved at a minimum cost. If no dominated parasitic flow loops exist in the transformed network, no dominated parasitic flow loops will exist in the network after removing the super-source *s*, the super-consumer *t* and their connecting edges.

After conducting Step 3, it can also be shown that the supplied and drained interchangeable commodity through the edges connecting the super-source *s* and the super-sink *t*, correspond one-to-one to the supplied and consumed quantities of interchangeable commodity in the initial network.

Because in the initial flow network, the throughput flow from the sources to the consumers is a feasible flow, the maximum possible flow in the transformed flow network cannot be smaller than the initially existing throughput flow. Simultaneously, the maximum possible flow in the transformed network cannot exceed the existing feasible flow in the initial network, because the sum of the capacities of the edges connecting the super-source are equal to the total generated interchangeable commodity by the initial sources. Consequently, the maximum throughput flow in the transformed network is equal to the sum of the flows from the sources in the initial network. In addition, in the transformed network, the amount of flow passing through each source (now an ordinary node) is exactly equal to the generated by the source flow in the initial network. As a result, after conducting Step 3, the supplied quantities through the edges connecting the super-source in the transformed network correspond one-to-one to the generated quantities in the initial network. In a similar fashion, it can be proved that the drained quantities through the edges connecting the super-consumer in the transformed network correspond one-to-one to the consumed quantities in the initial network. □

The presented algorithm goes beyond discovering dominated parasitic flow loops. It removes from the network not only the dominated parasitic flow loops but also directed and almost directed parasitic flow loops.

## CONCLUSIONS

1. Similar to the directed and almost-directed parasitic flow loops, augmentable broken loops with dominating flow in a particular direction of traversing, are also associated with transportation losses, congestion and increased pollution of the environment.
2. Dominated parasitic flow loops occur naturally in real networks by following directed straight-line paths from sources of flow to destinations. The probability of a dominated parasitic flow loop in real networks is unexpectedly large and increases very quickly with increasing the number of intersecting flow paths.
3. An important concept referred to as 'cost factor of a cyclic path' has been introduced, with fundamental importance to optimising the flows in real flow networks. The cost factor of a cyclic path is the difference between the sum of the lengths of the closing edges and the sum of the lengths of the edges with backward flow.
4. The dominated parasitic flow loops are augmentable cyclic paths characterised by a negative cost factor. Dominated parasitic flow loops can be augmented with flow, which is essentially a process of draining the flow

loops. The result is an overall decrease of the transportation costs without affecting the quantity of the delivered commodity from sources to consumers.

5. The necessary and sufficient condition for nonexistence of dominated flow loops in a network is the non-existence of cyclic paths with negative cost factor.
6. An algorithm has been proposed for eliminating dominated parasitic flow loops in networks. The algorithm involves transforming the network into a single source/single sink network and a step which involves maximising the flow in the transformed network at a minimal transportation cost. The proposed algorithm has the potential to save significant amount of resources to the world economy.
7. If more than one zero flow paths and non-zero flow paths are present, dominated flow loops are absent if and only if all random nz-vectors reside in one of the sectors defined by the zero flow paths.
8. A number of important applications have been identified, related to optimising flow networks by removing dominated parasitic flow loops.

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## APPENDIX A. Algorithm for determining the probability of a dominated parasitic flow loop.

This is the algorithm in pseudo-code for determining the probability of a parasitic flow loop. The variable 'count' counts the number of times a dominated parasitic flow loop has not been found and the variable 'Probability' contains the probability of a dominated parasitic flow loop. The statements within braces '{...}' are treated as a compound statement.

*// The number n of non-zero flow paths is n>=2;*

IJOR Vol. 11, No. 1, 001−017 (2014)

```
// The number m of zero flow paths is m>=0;

    counter = 0;

if (m=0) then   // there are no zero flow paths
  {
    if (n=2) then Probability = 0;
    else if (n>=3) then Probability = 1 − n / 2^(n−1) ;
  }
else  // there are zero flow paths (m>0)
{
  for i=1 to number_of_trials do
  {
  Generate 'n' random nz-vectors corresponding to the directions of the flows in the non-zero flow paths;
  Sort their orientation angles in ascending order;

  Generate 'm' random directions which correspond to the zero flow paths; Sort in ascending order the
  angles defining the sectors formed by the directions of the zero flow paths;

  Check if all nz-random directions reside in a single sector;  Set the variable Y to 1 if this is the case and
  to 0, otherwise;

  if (Y=1) then count = count + 1;
  }

  Probability = 1 - count/number_of_trials;
   }
```