Batch Arrival Priority Queueing Model with Second Optional Service and Server Breakdown

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Abstract — $M_1^{X_1} M_2^{X_2} / G_1 G_2 / 1$ priority model with second optional service and server breakdown is investigated to analyze the reliability characteristics. There are two types of customers; the first type customers are priority based customers and the other ones are non priority i.e. ordinary customers. The priority and nonpriority customers arrive in batch according to Poisson fashion with rate λ_1 and λ_2 respectively. In this model, we assume that the server provides a first essential service to all the arriving customers, whereas only some of them receive second optional service. The first essential service time follows a general distribution but the second optional service is assumed to be exponentially distributed. By using the supplementary variable technique, we obtain the expressions for the availability, failure frequency and reliability function of the server, etc. The sensitivity analysis is carried out to explore the effects of different parameters on the system reliabilities indices.

Keywords — $M_1^{X_1} M_2^{X_2} / G_1 G_2 / 1$, Priority, Optional service, Breakdown, Supplementary variable, Queue size, Availability, Reliability

1. INTRODUCTION

Priority mechanism is an invaluable scheduling method that allows customers of different classes to receive preferential service from a server in a queueing system. Some papers on queueing theory are devoted to analyze priority queues, where customers are labeled and served in accordance with a priority scheme; these papers can be classified in two categories; the preemptive priority and the non-preemptive priority scheduling disciplines. For a preemptive priority scheduling discipline, the service of a customer is interrupted when a customer with higher priority arrives in the system during the service; the interrupted customer only gets hold of the server again when there are no more higher-priority customers present in the system. On the other hand, for non-preemptive scheduling, a customer's service is never interrupted. Upon departure of a customer or when the customers arrive in an empty system, the server selects a customer for service from the class with the highest priority among waiting customers.

The discipline that handles situations where preemptions are disallowed is called the non-preemptive or "head of the line" priority discipline. Typical situations where one would not want to allow preemptions include those computing applications where a lengthy service rendered prior to the point of interruption would be permanent. Consider an airline check-in counter, for instance; when an idle business class agent processes an economy-class passenger and if no business class passenger is present, it would be both confusing and bad public relations to interrupt their check-in to handle a newly arriving business-class passenger.

The priority queue has received considerable attention in the literature (cf. Miller, 1960; Takacs, 1964; Jaiswal, 1968). Takagi (1991) analyzed a vacation and priority queueing system. Takine (1996) studied a non-preemptive priority MAP/G/1 queue with two classes of customers. The batch arrival queueing model was discussed by Chaudhry and Templeton (1983), Takahashi and Takagi (1990), Takagi and Takahashi (1991), Audsin et al. (1992), Soo and Chung (2003) and many others. Krishna Reddy et al. (1993) examined a nonpreemptive priority multiserver queueing system with general bulk service and heterogeneous arrivals. Lee (2001) and Young et al. (2003) analyzed discrete-time $\text{Geo}^X/\text{G}/1$ queue with preemptive resume priority. Hassin and Haviv (2006) considered a single server queue with two classes of customers. Dimitriou and Langaris (2013) analyzed a mixed priority retrial queue with negative arrivals, unreliable server and multiple vacations. For the machining systems with spare provisioning, the transient analysis of priority queueing model with unreliable server was developed by Jain [2013] . The concept of double orbit was considered by Jain and Bhagat (2013) to propose the threshold

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recovery policy for the unreliable server retrial queue with priority. They have obtained the transient probabilities and other performance indices by using Runge-Kutta method. Vadivu et al. (2014) investigated the multi-server priority queue with retrial attempts.

A problem of interest in the practical applications of the theory of queues is the effect of interruptions to the servicing of customers due to server breakdown. One such case is when breakdowns in the service mechanism add the usual delay. Alternatively the interruptions may be caused by a queue discipline which assigns priority to a certain group of customers. For example, in a communication system, a priority customer could be an urgent message to be transmitted immediately on arrival irrespective of the state of the queue of schedule (non-priority) messages. Under this discipline, the arrival of a priority customer and a breakdown in the service device are equivalent from the point of view of the customer whose service has been interrupted. A model combining both priority discipline and breakdowns can be postulated, in which case a breakdown is formally interpreted as a priority customer with preference over all others. White and Christie (1958) and Madan (1992) discussed queueing system with priorities and breakdowns. Wang and Cao (2001) gave reliability analysis of the retrial queue with server breakdown and repairs. Classical and constant retrial policies in M/G/1 queueing model with active breakdowns of the server were discussed by Atencia et al. (2006). Fiems et al. (2008) considered queueing systems with different types of server interruptions. Wang, et al. (2010) provided a comparison of two randomized policy M/G/1 queues with second optional service, server breakdown and startup. Kalidass, and Kasturi (2012) gave a queue with working breakdowns. For the non-Markovian loss queue with priority, Vinayak et al. (2014) developed queueing model and gave performance analysis to explore the queue size distribution. Jain and Bhagat investigated (2014) the unreliable server queue to study the modified vacation policy by incorporating more realistic features including the bulk input, retrial and delayed repairs.

In some congestion situations, the queueing systems are characterized by common features such as bulk arrival and priority where the server provides a first essential service to all the arriving customers, however only some of them receive a second optional service. In the existing literature, a few papers appear on queueing system with second optional service. Madan (1994, 2000) and Choudhury (2003) studied an M/G/1 queueing system with optional service. Sapna (1996) considered an M/G/1-type queueing system with non-perfect servers and no waiting room where two types of services are offered. An arriving customer has to undergo the first service but the second service is optional. Krishnakumar et al. (2002) analyzed an M/G/1 retrial queueing system with two phase service and preemptive resume priority discipline. Artalejo and Choudhury (2004) gave steady state analysis of an M/G/1 queue with repeated attempts in which the server may provide an additional second phase of service. Wang (2004) proposed the M/G/1 queue with second optional service and server breakdowns. Wang and Zhao (2007) considered a discrete-time Geo/G/1 retrial queue with starting failures and second optional service where the two services are offered; an arrival essentially requires the first (i.e. regular) service but has the option whether to have the second service or not. Wang, and Xu (2009) analyzed the well-posedness of an M/G/1 queue with second optional service where the two services are offered; an arrival essentially requires the first (i.e. regular) service but has the option whether to have the second service or not. Wang, and Xu (2009) analyzed the well-posedness of an M/G/1 queue with second optional service and server breakdown. Ke at al. (2013) gave analysis of an infinite multi-server queue with an optional service. Rajadurai, et al. (2014) gave the performance analysis of an M[X]/(G₁, G₂)/1 retrial queueing system with balking, optional re-service under modified vacation policy and service interruption.

In this paper, we consider a batch arrival priority queueing system with second optional service and server breakdowns. Such type of queueing models can be noticed in the stochastic modeling of communication systems, local area network system, manufacturing system and many daily life congestion situations. Our model deals with more versatile congestion situations as it includes the concept of unreliable server while considering the optimal service criteria in bulk priority queue. The organization of the paper is as follows. The model is described along with requisite assumptions and notations in section 2. The queue size distribution is established in section 3. In section 4 some special cases are deduced. Section 5 provides the reliability indices of the server including the availability, failure frequency, etc. The computational results and sensitivity analysis are presented in section 6. At last conclusions have been drawn in section 7.

2. MODEL DESCRIPTION

Consider a single-server batch arrival priority based queuing system where server is subject to breakdown and also renders the second optional service. There are two types of customers, the priority (type 1) and the non-priority (type 2) customers who arrive in batches according to Poisson process. If a customer is being served at the instant of server failure, the service is interrupted. In case the server breaks down when serving the customer, he is sent for repair and the customer who has just being served should wait for the server back to complete his remaining service and restarted after repair. Immediately after returning from the repair, the server starts to serve priority/non-priority customers until the system becomes empty. A customer who arrives and finds the server busy or broken down must wait in the queue until the server is available. Although no service occurs during the repair period of a broken down server, the customers continue to arrive according to a Poisson process. The service discipline between two classes is non-preemptive priority; the priority customer is always taken for service before a non-priority customer who is undergoing service; thus the service of the priority customer begins only after the completion of service of the non-priority customer. The first essential service is needed to all 1813-713X Copyright © 2014 ORSTW

arriving customers; the essential service times are independent and identically distributed (i.i.d.) according to general distribution in each priority class. As soon as the first essential service of a customer is completed, he may opt for the second optional service; the optional service times are assumed to be exponentially distributed. To formulate the mathematical model we use the following notations:

$N_1(t), N_2(t)$:	The number of the priority and nonpriority customers in the queue at time t.
X(t), Y(t)	:	Elapsed service time and elapsed repair time at time t.
$\lambda_{_1},\lambda_{_2}$:	The arrival rate of priority and non-priority customers
X_i	:	Random variable denoting the batch size for i^{th} (<i>i</i> =1,2) type of customers.
$g_i(k)$:	$Pr(X_i=k), \ i = 1, 2; \ k \ge 1.$
$G_i(z_i)$:	Probability generating function of the batch size X_i , $i=1,2$.
$\boldsymbol{g}_{i,k}$:	k th moment of batch size for i^{th} (<i>i</i> =1,2) type of customers.
$lpha_i^{(l)}$:	Failure rate of the server during essential service $(l=1)$ and optional service $(l=2)$ while rendering service to i^{th} $(i=1,2)$ type of customers.
C(t)	:	Random variable denoting the states of the server at time t.
$B_i{}^{(l)}(x),\; b_i{}^{(1)}(x)$:	The probability distribution and the density functions of the service time for i^{th} $(i=1,2)$ type of customers.
$R_i{}^{(l)}(y), \; r_i{}^{(l)}(y)$:	The probability distribution and density functions of repair time when the server failed during l^{th} ($l=1, 2$) phase service to i^{th} ($i=1, 2$) type of customers.
${b_i}^{*(1)}(.), \; {r_i}^{*(l)}(.)$:	Laplace-Stieltjes transform of $b_i^{(1)}(.)$ and $r_i^{(l)}(.)$, $l=1, 2$.
$\mu_i^{(1)}(x) \;,\; \mu_i^{(2)}$:	The hazard rate of first essential service and second optional service, respectively for i^{th} (<i>i</i> =1,2) type of customers.
$eta_i^{(l)}(y)$:	The hazard rate of repair when server failed while rendering first essential ($l=1$) and second optional ($l=2$) service to i^{th} ($i=1,2$) type of customers.
ω_{i}	:	The probability that i^{th} $(i=1,2)$ type of customer who has completed the first essential service, opts for the second optional service; $\bar{\omega}_{i} = (1-\omega_{i})$.
${ au}_{i,k}^{(l)}, { au}_{i,k}^{(l)}$:	k th moment of service time and repair time about origin for i^{th} (<i>i</i> =1,2)type of customer; $\tau_{i,k}^{l} = (-1)^{k} b_{i,k}^{*(l)}(0), \ \gamma_{i,k}^{(l)} = (-1)^{k} r_{i,k}^{*(l)}(0)$ corresponding to l^{th}
$P_{m,n,1}^{(l)}(.),Q_{m,n,1}^{(l)}(.)$:	(l=1, 2) phase service. The probability that there are m priority and n nonpriority customers is in the system while the server is rendering the l th ($l=1, 2$) phase service.
$P_{m,n,2}^{(l)}(.),Q_{m,n,2}^{(l)}(.)$:	The probability that there are m priority and n nonpriority customers is in the system when the server is under repair after failure during the l^{th} ($l=1, 2$) phase service.

The server's state C(t) at time t is represented by

0. if the server is in idle state 1, if the server is busy in rendering essential service to priority cutomers. 2,if the server is busy in rendering optional service to priority cutomers. 3. if the server is busy in rendering essential service to nonpriority cutomers. if the server is busy in rendering optional service 4, C(t) =to nonpriority cutomers. 5,if the server is broken down while rendering essential service to priority cutomers and the server is under repair. 6, if the server is broken down while rendering optional service to priority cutomersand the server is under repair. 7, if the server is broken down while rendering essential service to nonpriority cutomersand the server is under repair. 8, if the server is broken down while rendering optional service to nonpriority cutomersand the server is under repair.

Also
$$\mu_i^{(l)}(x) = \frac{b_i^{(l)}(x)}{1 - B_i^{(l)}(x)}, \quad \beta_i^{(l)}(y) = \frac{r_i^{(l)}(y)}{1 - R_i^{(l)}(y)}, \ i = (1, 2), l = (1, 2).$$

3. QUEUE SIZE DISTRIBUTION

By introducing supplementary variables corresponding to elapsed service time and elapsed repair time, the stochastic process $[N_1(t), N_2(t), X(t), Y(t), C(t), t \ge 0]$ behaves like a Markov process. The joint probabilities for the number of customers in the queue and in the service are defined below:

$$\begin{split} &P_0\left(t\right) = \Pr\left\{N_1(t) = 0, N_2(t) = 0, C(t) = 0, \text{ no customer is in queue}\right\}.\\ &P_{m,n,1}^{(1)}\left(x,t\right) = \Pr\left\{N_1(t) = m, N_2(t) = n, C(t) = 1, x < X(t) \le x + dx\right\}, \ m \ge 1, n \ge 0.\\ &P_{m,n,1}^{(2)}\left(t\right) = \Pr\left\{N_1(t) = m, N_2(t) = n, C(t) = 2, x < X(t) \le x + dx\right\}, \ m \ge 1, n \ge 0.\\ &Q_{m,n,1}^{(1)}\left(x,t\right) = \Pr\left\{N_1(t) = m, N_2(t) = n, C(t) = 3, x < X(t) \le x + dx\right\}, \ m \ge 0, n \ge 1.\\ &Q_{m,n,1}^{(2)}\left(t\right) = \Pr\left\{N_1(t) = m, N_2(t) = n, C(t) = 4, x < X(t) \le x + dx\right\}, \ m \ge 0, n \ge 1.\\ &P_{m,n,2}^{(2)}\left(x,y,t\right) = \Pr\left\{N_1(t) = m, N_2(t) = n, C(t) = 5, X(t) = x, \ y < Y(t) \le y + dy\right\}, \ m \ge 1, n \ge 0, x \ge 0, y \ge 0.\\ &P_{m,n,2}^{(2)}\left(y,t\right) = \Pr\left\{N_1(t) = m, N_2(t) = n, C(t) = 6, X(t) = x, \ y < Y(t) \le y + dy\right\}, \ m \ge 1, n \ge 0, x \ge 0, y \ge 0.\\ &Q_{m,n,2}^{(1)}\left(x,y,t\right) = \Pr\left\{N_1(t) = m, N_2(t) = n, C(t) = 7, X(t) = x, \ y < Y(t) \le y + dy\right\}, \ m \ge 0, n \ge 1, x \ge 0, y \ge 0.\\ &Q_{m,n,2}^{(2)}\left(t,y\right)dy = \Pr\left\{N_1(t) = m, N_2(t) = n, C(t) = 8, X(t) = x, \ y < Y(t) \le y + dy\right\}, \ m \ge 0, n \ge 1, x \ge 0, y \ge 0. \end{aligned}$$

The differential equations governing the model are constructed using the appropriate transition rates as follows:

$$\frac{\partial P_{0}\left(t\right)}{\partial t} + \left(\lambda_{1} + \lambda_{2}\right)P_{0}\left(t\right) = \left(1 - \omega_{1}\right)\int_{1,0,0}^{\infty} P_{1,0,0}^{(1)}\left(x,t\right)\mu_{1}^{(1)}\left(x\right)dx + \left(1 - \omega_{2}\right)\int_{0}^{\infty} Q_{0,1,0}^{(1)}\left(x,t\right)\mu_{2}^{(1)}\left(x\right)dx \\
+ \mu_{1}^{(2)}P_{1,0,0}^{(2)}\left(t\right) + \mu_{2}^{(2)}Q_{0,1,0}^{(2)}\left(t\right), \quad for \ m \ge 0, n \ge 0$$
(1)

$$\frac{\partial P_{m,n,1}^{(1)}\left(x,t\right)}{\partial t} + \frac{\partial P_{m,n,1}^{(1)}\left(x,t\right)}{\partial x} + \left(\lambda_{1} + \lambda_{2} + \mu_{1}^{(1)}\left(x\right) + \alpha_{1}^{(1)}\right)P_{m,n,1}^{(1)}\left(x,t\right) \\
= \int_{0}^{\infty} \beta_{1}^{(1)}\left(y\right)P_{m,n,2}^{(1)}\left(x,y,t\right)dy + \lambda_{1}\sum_{k=1}^{m-1} P_{m-k,n,1}^{(1)}\left(x,t\right)g_{1}\left(m-k\right) + \lambda_{2}\sum_{k=0}^{n-1} P_{m,n-k,1}^{(1)}\left(x,t\right)g_{2}\left(n-k\right), \tag{2}$$
for $m \geq 1, n \geq 0$

$$\frac{\partial P_{m,n,1}^{(2)}\left(t\right)}{\partial t} + \frac{\partial P_{m,n,1}^{(2)}\left(t\right)}{\partial x} + \left(\lambda_{1} + \lambda_{2} + \mu_{1}^{(2)} + \alpha_{1}^{(2)}\right)P_{m,n,1}^{(2)}\left(t\right) = \int_{0}^{\infty} \beta_{1}^{(2)}\left(y\right)P_{m,n,2}^{(2)}\left(y,t\right)dy \\
+ \omega_{1}\int_{0}^{\infty} P_{m,n,1}^{(1)}\left(x,t\right)\mu_{1}^{(1)}\left(x\right)dx + \lambda_{1}\sum_{k=1}^{m-1}P_{m-k,n,1}^{(2)}\left(t\right)g_{1}\left(m-k\right) + \lambda_{2}\sum_{k=0}^{n-1}P_{m,n-k,1}^{(2)}\left(t\right)g_{2}\left(n-k\right), \quad (3)$$
for $m \geq 1, n \geq 0$

$$\frac{\partial P_{m,n,2}^{(1)}\left(x,y,t\right)}{\partial t} + \frac{\partial P_{m,n,2}^{(1)}\left(x,y,t\right)}{\partial y} + \left(\lambda_{1} + \lambda_{2} + \beta_{1}^{(1)}\left(y\right)\right)P_{m,n,2}^{(1)}\left(x,y,t\right) \\
= \lambda_{1}\sum_{k=1}^{m-1} P_{m-k,n,2}^{(1)}\left(x,y,t\right)g_{1}\left(m-k\right) + \lambda_{2}\sum_{k=0}^{n-1} P_{m,n-k,2}^{(1)}\left(x,y,t\right)g_{2}\left(n-k\right), \quad (4)$$
for $m \ge 1, n \ge 0$

$$\frac{\partial P_{m,n,2}^{(2)}\left(y,t\right)}{\partial t} + \frac{\partial P_{m,n,2}^{(2)}\left(y,t\right)}{\partial y} + \left(\lambda_{1} + \lambda_{2} + \beta_{1}^{(2)}\left(y\right)\right) P_{m,n,2}^{(1)}\left(y,t\right) \\
= \lambda_{1} \sum_{k=1}^{m-1} P_{m-k,n,2}^{(2)}\left(y,t\right) g_{1}\left(m-k\right) + \lambda_{2} \sum_{k=0}^{n-1} P_{m,n-k,2}^{(2)}\left(y,t\right) g_{2}\left(n-k\right),$$
for $m \ge 1, n \ge 0$

$$(5)$$

$$\frac{\partial Q_{m,n,1}^{(1)}\left(x,t\right)}{\partial t} + \frac{\partial Q_{m,n,1}^{(1)}\left(x,t\right)}{\partial x} + \left(\lambda_{1} + \lambda_{2} + \mu_{2}^{(1)}\left(x\right) + \alpha_{2}^{(1)}\right)Q_{m,n,1}^{(1)}\left(x,t\right) \\
= \int_{0}^{\infty} \beta_{2}^{(1)}\left(y\right)Q_{m,n,2}^{(1)}\left(x,y,t\right)dy + \lambda_{1}\sum_{k=1}^{m-1} Q_{m-k,n,1}^{(1)}\left(x,t\right)g_{1}\left(m-k\right) + \lambda_{2}\sum_{k=0}^{n-1} Q_{m,n-k,1}^{(1)}\left(x,t\right)g_{2}\left(n-k\right), \tag{6}$$
for $m \geq 0, n \geq 1$

$$\frac{\partial Q_{m,n,1}^{(2)}\left(t\right)}{\partial t} + \frac{\partial Q_{m,n,1}^{(2)}\left(t\right)}{\partial x} + \left(\lambda_{1} + \lambda_{2} + \mu_{2}^{(2)} + \alpha_{2}^{(2)}\right)Q_{m,n,1}^{(2)}\left(t\right) = \int_{0}^{\infty} \beta_{2}^{(2)}\left(y\right)Q_{m,n,2}^{(2)}\left(y,t\right)dy \\
+ \omega_{2}\int_{0}^{\infty} Q_{m,n,2}^{(1)}\left(x,t\right)\mu_{2}^{(1)}(x)dx + \lambda_{1}\sum_{k=1}^{m-1}Q_{m-k,n,1}^{(2)}\left(t\right)g_{1}\left(m-k\right) + \lambda_{2}\sum_{k=0}^{n-1}Q_{m,n-k,1}^{(2)}\left(t\right)g_{2}\left(n-k\right), \quad (7)$$
for $m \geq 0, n \geq 1$

$$\frac{\partial Q_{m,n,2}^{(1)}\left(x,y,t\right)}{\partial t} + \frac{\partial Q_{m,n,2}^{(1)}\left(x,y,t\right)}{\partial y} + \left(\lambda_{1} + \lambda_{2} + \beta_{2}^{(1)}\left(y\right)\right)Q_{m,n,2}^{(1)}\left(x,y,t\right) \\
= \lambda_{1}\sum_{k=1}^{m-1}Q_{m-k,n,2}^{(1)}\left(x,y,t\right)g_{1}\left(m-k\right) + \lambda_{2}\sum_{k=0}^{n-1}Q_{m,n-k,2}^{(1)}\left(x,y,t\right)g_{2}\left(n-k\right), \tag{8}$$
for $m \ge 0, n \ge 1$

$$\frac{\partial Q_{m,n,2}^{(2)}\left(y,t\right)}{\partial t} + \frac{\partial Q_{m,n,2}^{(2)}\left(y,t\right)}{\partial y} + \left(\lambda_{1} + \lambda_{2} + \beta_{2}^{(2)}\left(y\right)\right)Q_{m,n,2}^{(2)}\left(y,t\right) \\
= \lambda_{1}\sum_{k=1}^{m-1} Q_{m-k,n,2}^{(2)}\left(y,t\right)g_{1}\left(m-k\right) + \lambda_{2}\sum_{k=0}^{n-1} Q_{m,n-k,2}^{(2)}\left(y,t\right)g_{2}\left(n-k\right), \tag{9}$$
for $m \ge 0, n \ge 1$

Also $P_{m,n,1}^{(1)}(x,t) = 0$, for m < 1, n < 0, and $Q_{m,n,1}^{(1)}(x,t) = 0$, for m < 0, n < 1. Equations (1)–(9) are to be solved under the following boundary conditions:

$$P_{m,n,1}^{(1)}\left(0,t\right) = \left(1-\omega_{1}\right)\int_{0}^{\infty} P_{m+1,n,1}^{(1)}\left(x,t\right)\mu_{1}^{(1)}\left(x\right)dx + \left(1-\omega_{2}\right)\int_{0}^{\infty} Q_{m,n+1,1}^{(1)}\left(x,t\right)\mu_{2}^{(1)}\left(x\right)dx + \mu_{1}^{(2)} P_{m+1,n,1}^{(2)}\left(t\right) + \mu_{2}^{(2)} Q_{m,n+1,1}^{(2)}\left(t\right) + \delta_{0n} \lambda_{1} \sum_{k=0}^{m-1} P_{0}\left(t\right)g_{1}\left(m-k\right)$$

$$(10)$$

$$\begin{aligned} Q_{0,n,1}^{(1)}\left(0,t\right) = & \left(1-\omega_{1}\right) \int_{0}^{\infty} P_{1,n,1}^{(1)}\left(x,t\right) \mu_{1}^{(1)}\left(x\right) d\,x + \left(1-\omega_{2}\right) \int_{0}^{\infty} Q_{0,n+1,1}^{(1)}\left(x,t\right) \mu_{2}^{(1)}\left(x\right) d\,x \\ & + \mu_{1}^{(2)} P_{1,n,1}^{(2)}\left(x,t\right) + \mu_{2}^{(2)} Q_{0,n+1,1}^{(2)}\left(x,t\right) + \lambda_{2} \sum_{k=0}^{n-1} P_{0}\left(t\right) g_{2}\left(n-k\right) \end{aligned}$$
(11)

$$P_{m,n,2}^{(1)}\left(x,0,t\right) = \alpha_1^{(1)} P_{m,n,2}^{(1)}\left(x,t\right)$$
(12)

$$Q_{m,n,2}^{(1)}\left(x,0,t\right) = \alpha_2^{(1)} Q_{m,n,2}^{(1)}\left(x,t\right)$$
(13)

$$P_{m,n,2}^{(2)}\left(0,t\right) = \alpha_1^{(2)} P_{m,n,2}^{(2)}\left(t\right)$$
(14)

$$Q_{m,n,2}^{(2)}\left(0,t\right) = \alpha_{2}^{(2)} Q_{m,n,2}^{(2)}\left(t\right)$$
(15)

The normalization condition is

$$\begin{split} P_{0} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \int_{0}^{\infty} P_{m,n,1}^{(1)} \left(x \right) dx + \int_{0}^{\infty} \int_{0}^{\infty} P_{m,n,2}^{(1)} \left(x, y \right) dx \, dy + \int_{0}^{\infty} P_{m,n,2}^{(2)} \left(y \right) dy + P_{m,n,1}^{(2)} \right\} \\ + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \left\{ \int_{0}^{\infty} Q_{m,n,1}^{(1)} \left(x \right) + \int_{0}^{\infty} \int_{0}^{\infty} Q_{m,n,2}^{(1)} \left(x, y \right) dx \, dy + \int_{0}^{\infty} Q_{m,n,2}^{(2)} \left(y \right) dy + Q_{m,n,1}^{(2)} \right\} = 1 \end{split}$$

The initial condition is

 $P_0(0) = 1$.

Define the generating functions:

$$\begin{split} P_{1}^{(1)}\left(z_{1},z_{2},x,t\right) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} z_{1}^{m} \ z_{2}^{n} \ P_{m,n,1}^{(1)}\left(x,t\right), \quad Q_{1}^{(1)}\left(z_{1},z_{2},x,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,1}^{(1)}\left(x,t\right), \quad Q_{1}^{(2)}\left(z_{1},z_{2},t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ P_{m,n,1}^{(2)}\left(t\right), \quad Q_{1}^{(2)}\left(z_{1},z_{2},t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,1}^{(1)}\left(t\right), \quad Q_{1}^{(2)}\left(z_{1},z_{2},x,y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(1)}\left(x,y,t\right), \quad Q_{2}^{(1)}\left(z_{1},z_{2},x,y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(1)}\left(x,y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(1)}\left(x,y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ z_{2}^{n} \ Q_{m,n,2}^{(2)}\left(y,t\right), \quad Q_{2}^{(2)}\left(z_{1},z_{2},y,t\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} z_{1}^{m} \ Z_{1}^{n} \ Z_{1$$

Now we define Laplace transform of f(t), as follows:

$$f^* \equiv f^*\left(s\right) = \int_{0}^{\infty} \exp\left(-st\right) f\left(t\right) dt, \qquad \operatorname{Re}\left(s\right) > 0.$$

Also $\overline{f^*}(s) \underline{\underline{\Delta}} 1 - f^*(s)$.

Theorem 1: The Laplace transform of joint generating functions are given by

$$P_{1}^{*(1)}(z_{1}, z_{2}, x, s) = P_{p}(z_{1}, z_{2}, 0, s) \exp\left[-\phi_{1}^{(1)}(z_{1}, z_{2}, s)x\right]\overline{B}_{1}^{(1)}(x)$$
(16)

$$Q_{1}^{*(1)}(z_{1}, z_{2}, x, s) = Q_{q}(z_{1}, z_{2}, 0, s) \exp\left[-\phi_{2}^{(1)}(z_{1}, z_{2}, s)x\right]\overline{B}_{2}^{(1)}(x)$$
(17)

$$P_{1}^{*(2)}(z_{1}, z_{2}, s) = P_{p}(z_{1}, z_{2}, 0, s) \frac{\omega_{1} b_{1}^{*(1)} \left[\phi_{1}^{(1)}(z_{1}, z_{2}, s)\right]}{\phi_{1}^{(2)}(z_{1}, z_{2}, s) + \mu_{1}^{(2)}}$$
(18)

$$Q_{1}^{*(2)}(z_{1}, z_{2}, s) = Q_{q}(z_{1}, z_{2}, 0, s) \frac{\omega_{2} b_{2}^{*(1)} \left[\phi_{2}^{(1)}(z_{1}, z_{2}, s)\right]}{\phi_{2}^{(2)}(z_{1}, z_{2}, s) + \mu_{2}^{(2)}}$$
(19)

$$P_{2}^{*(1)}(z_{1}, z_{2}, x, y, s) = \alpha_{1}^{(1)} P_{p}(z_{1}, z_{2}, 0, s) \exp\left[-\left\{\lambda_{1}\left(1 - G_{1}(z_{1})\right) + \lambda_{2}\left(1 - G_{2}(z_{2})\right) + s\right\}y\right] \overline{R}_{1}^{(1)}(y) \\ \times \exp\left[-\left\{\phi_{1}^{(1)}(z_{1}, z_{2}, s)\right\}\right] \overline{B}_{1}^{(1)}(x)$$

$$(20)$$

$$\begin{aligned} Q_{2}^{*(1)}\left(z_{1}, z_{2}, x, y, s\right) &= \alpha_{2}^{(1)} Q_{q}\left(z_{1}, z_{2}, 0, s\right) \exp\left[-\left\{\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right\} y\right] \overline{R}_{2}^{(1)}\left(y\right) \\ &\times \exp\left[-\left\{\phi_{2}^{(1)}\left(z_{1}, z_{2}, s\right)\right\}\right] \overline{B}_{2}^{(1)}\left(x\right) \end{aligned}$$
(21)

$$P_{2}^{*(2)}(z_{1}, z_{2}, y, s) = \alpha_{1}^{(2)} P_{p}(z_{1}, z_{2}, 0, s) \exp\left[-\left\{\lambda_{1}\left(1 - G_{1}(z_{1})\right) + \lambda_{2}\left(1 - G_{2}(z_{2})\right) + s\right\}y\right] \overline{R}_{1}^{(2)}(y) \times P_{1}^{*(2)}(z_{1}, z_{2}, s)$$

$$(22)$$

$$Q_{2}^{*(2)}(z_{1}, z_{2}, y, s) = \alpha_{2}^{(2)} Q_{q}(z_{1}, z_{2}, 0, s) \exp\left[-\left\{\lambda_{1}\left(1 - G_{1}(z_{1})\right) + \lambda_{2}\left(1 - G_{2}(z_{2})\right) + s\right\}y\right] \overline{R}_{2}^{-(2)}(y) \times Q_{1}^{*(2)}(z_{1}, z_{2}, s)$$

$$(23)$$

where

$$\begin{split} &Q_{q}\left(z_{1},z_{2},0,s\right) = \\ &\frac{1-P_{0}^{*}\left(s\right)\left\{\lambda_{1}\left[1-G_{1}\left(h(z_{2},s)\right)\right]+\lambda_{2}\left[1-G_{2}\left(z_{2}\right)\right]+s\right\}}{1-\left[1/z_{2}\left(\phi_{2}^{(2)}\left(h(z_{2},s),z_{2},s\right)+\mu_{2}^{(2)}\right)\right]\left[\phi_{2}^{(2)}\left(h(z_{2},s),z_{2},s\right)+\mu_{2}^{(2)}\right]b_{2}^{*(1)}\left(1-\omega_{2}b_{2}^{*(1)}\right)\phi_{2}^{(1)}\left(h(z_{2},s),z_{2},s\right)\phi_{2}^{(2)}\left(h(z_{2},s),z_{2},s\right),s_{2},$$

and

$$P_{_{0}}^{*}\left(s\right) \!=\! \left[\lambda_{_{1}}\left(1 - G_{_{1}}\left(h(z_{_{2s}},s),s\right)\right) + \lambda_{_{2}}\left(1 - G_{_{2}}\left(z_{_{2s}}\right)\right) + s\right]^{-1}.$$

Also $h(z_2,s)$ is the root of the equation

$$z_{1} = b_{1}^{*(1)} \left\{ \phi_{1}^{(1)} \left(h(z_{2}, s), z_{2}, s \right) \right\} - \frac{\omega_{1} b_{1}^{*(1)} \left\{ \phi_{1}^{(1)} \left(h(z_{2}, s), z_{2}, s \right) \right\} \phi_{1}^{(2)} \left(h(z_{2}, s), z_{2}, s \right)}{\left\{ \phi_{1}^{(2)} \left(h(z_{2}, s), z_{2}, s \right) + \mu_{1}^{(2)} \right\}}$$
(24)

inside $\left|z_{1}\right| = 1$ for Re (s) >0. Again $h(z_{2},s)$ is the root of the equation

$$z_{2s} = b_2^{*(1)} \left\{ \phi_2^{(1)} \left(h(z_{2s}, s), z_{2s}, s \right) \right\} - \frac{\omega_2 b_2^{*(1)} \left\{ \phi_2^{(1)} \left(h(z_{2s}, s), z_{2s}, s \right) \right\} \phi_2^{(2)} \left(h(z_{2s}, s), z_{2s}, s \right)}{\left\{ \phi_2^{(2)} \left(h(z_{2s}, s), z_{2s}, s \right) + \mu_2^{(2)} \right\}}$$
(25)

 $inside\left|z_{_{2}}\right| \!=\! 1 \;\; {\rm for \; Re \; (s) >} 0.$

Proof: Taking Laplace transforms of equations (1)-(9) and boundary equations (10)-(15), and then after some algebraic manipulation, we obtain the results as given in equations (16) to (25). \Box

Corollary 1:

(i) The marginal generating functions of the priority and non-priority customers when the server is busy in first essential service and second optimal service, are given by

$$P_{1}^{*(1)}(z_{1},z_{2},s) = P_{p}(z_{1},z_{2},0,s) \frac{1 - b_{1}^{*(1)}\left[\phi_{1}^{(1)}(z_{1},z_{2},s)\right]}{\phi_{1}^{(1)}(z_{1},z_{2},s)}, \quad Q_{1}^{*(1)}(z_{1},z_{2},s) = Q_{q}(z_{1},z_{2},0,s) \frac{1 - b_{2}^{*(1)}\left[\phi_{2}^{(1)}(z_{1},z_{2},s)\right]}{\phi_{2}^{(1)}(z_{1},z_{2},s)} \quad (26)$$

$$P_{1}^{*(2)}\left(z_{1},z_{2},s\right) = P_{p}\left(z_{1},z_{2},0,s\right) \frac{\omega_{1}b_{1}^{*(1)}\left[\phi_{1}^{(1)}\left(z_{1},z_{2},s\right)\right]}{\phi_{1}^{(2)}\left(z_{1},z_{2},s\right) + \mu_{1}^{(2)}}, \ Q_{1}^{*(2)}\left(z_{1},z_{2},s\right) = Q_{q}\left(z_{1},z_{2},0,s\right) \frac{\omega_{2}b_{2}^{*(1)}\left[\phi_{2}^{(1)}\left(z_{1},z_{2},s\right)\right]}{\phi_{2}^{(2)}\left(z_{1},z_{2},s\right) + \mu_{2}^{(2)}}$$
(27)

(ii) The marginal generating functions of the priority and non-priority customers when the server is down during the first essential service and second optimal service are given by

$$P_{2}^{*(1)}(z_{1}, z_{2}, s) = \alpha_{1}^{(1)}P_{p}(z_{1}, z_{2}, 0, s) \frac{1 - b_{1}^{*(1)}\left[\phi_{1}^{(1)}(z_{1}, z_{2}, s)\right]}{\phi_{1}^{(1)}(z_{1}, z_{2}, s)} \times \frac{1 - r_{1}^{*(1)}\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]}{\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]}$$
(28)

$$Q_{2}^{*(1)}\left(z_{1}, z_{2}, s\right) = \alpha_{2}^{(1)}Q_{q}\left(z_{1}, z_{2}, 0, s\right) \frac{1 - b_{2}^{*(1)}\left[\phi_{2}^{(1)}\left(z_{1}, z_{2}, s\right)\right]}{\phi_{2}^{(1)}\left(z_{1}, z_{2}, s\right)} \times \frac{1 - r_{2}^{*(1)}\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]}{\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]}$$
(29)

$$P_{2}^{*(2)}\left(z_{1}, z_{2}, s\right) = \alpha_{1}^{(2)} P_{p}\left(z_{1}, z_{2}, 0, s\right) \frac{\omega_{1} b_{1}^{*(1)}\left[\phi_{1}^{(1)}\left(z_{1}, z_{2}, s\right)\right]}{\phi_{1}^{(2)}\left(z_{1}, z_{2}, s\right) + \mu_{1}^{(2)}} \times \frac{1 - r_{1}^{*(2)}\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]}{\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]}$$
(30)

$$Q_{2}^{*(2)}\left(z_{1}, z_{2}, s\right) = \alpha_{2}^{(2)}Q_{q}\left(z_{1}, z_{2}, 0, s\right) \frac{\omega_{2}b_{2}^{*(1)}\left[\phi_{2}^{(1)}\left(z_{1}, z_{2}, s\right)\right]}{\phi_{2}^{(2)}\left(z_{1}, z_{2}, s\right) + \mu_{2}^{(2)}} \times \frac{1 - r_{2}^{*(2)}\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]}{\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]}$$
(31)

Proof: Solving (16) to (23), we obtain the results given in equations (26) to (28).

Theorem 2: The joint probability generating function for the number of customers in the system is

$$\begin{split} P^*\left(z_1, z_2, s\right) &= P_p\left(z_1, z_2, 0, s\right) \frac{\left|1 - b_1^{*(1)}\left\{\phi_1^{(1)}\left(z_1, z_2, s\right)\right\}\right| \left|\phi_1^{(2)}\left(z_1, z_2, s\right) + \mu_1^{(2)}\right| + \omega_1 b_1^{*(1)}\left\{\phi_1^{(1)}\left(z_1, z_2, s\right)\right\} \phi_1^{(2)}\left(z_1, z_2, s\right)}{\left[\lambda_1\left(1 - G_1\left(z_1\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right] \left[\phi_1^{(2)}\left(z_1, z_2, s\right) + \mu_1^{(2)}\right]} \\ &+ Q_q\left(z_1, z_2, 0, s\right) \frac{\left[1 - b_2^{*(1)}\left\{\phi_2^{(1)}\left(z_1, z_2, s\right)\right\}\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right] + \omega_2 b_2^{*(1)}\left\{\phi_2^{(1)}\left(z_1, z_2, s\right)\right\} \phi_2^{(2)}\left(z_1, z_2, s\right)}{\left[\lambda_1\left(1 - G_1\left(z_1\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &- \left[\lambda_1\left(1 - G_1\left(z_1\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &+ \left[\lambda_1\left(1 - G_1\left(z_1\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &- \left[\lambda_1\left(1 - G_1\left(z_1\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &+ \left[\lambda_1\left(1 - G_1\left(z_1\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &+ \left[\lambda_1\left(1 - G_1\left(z_1\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &+ \left[\lambda_1\left(1 - G_1\left(z_1\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &+ \left[\lambda_1\left(1 - G_1\left(z_1\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &+ \left[\lambda_1\left(1 - A_1\left(z_1\right)\right) + \lambda_2\left(1 - A_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &+ \left[\lambda_1\left(1 - A_1\left(z_1\right)\right) + \lambda_2\left(1 - A_2\left(z_2\right)\right) + s\right] \left[\phi_2^{(2)}\left(z_1, z_2, s\right) + \mu_2^{(2)}\right]} \\ &+ \left[\lambda_1\left(z_1, z_2, z_2\right) + s\right] \left[\lambda_1\left(z_1, z_2\right)$$

Proof: For the proof of theorem, we use

$$P^{*}(z_{1}, z_{2}, s) = \sum_{i=1}^{2} \sum_{l=1}^{2} \left[P_{i}^{*(l)}(z_{1}, z_{2}, s) + Q_{i}^{*(l)}(z_{1}, z_{2}, s) \right]$$

$$(32)$$

Theorem 3: If the system is in steady state, then

(i) The long run probability of server being in idle state is

$$P_{0} = 1 - \sum_{i=1}^{2} \left\{ \rho_{i}^{(1)} \left(1 + \alpha_{i}^{(1)} \gamma_{i,1}^{(1)} \right) + \omega_{i} \rho_{i}^{(2)} \left(1 + \alpha_{i}^{(2)} \gamma_{i,1}^{(2)} \right) \right\}$$
(33)

(ii) The long run probability of the server being busy

$$P(B) = \sum_{i=1}^{2} \left(\rho_i^{(1)} + \omega_i \rho_i^{(2)} \right)$$
(34)

(iii) The long run probability that the server under repair state is

$$P(R) = \sum_{i=1}^{2} \left[\rho_i^{(1)} \alpha_i^{(1)} \gamma_{i,1}^{(1)} + \omega_i \rho_i^{(2)} \alpha_i^{(2)} \gamma_{i,1}^{(2)} \right]$$
(35)

where $\rho_{i}^{(l)} = \lambda_{i} g_{i,1} \tau_{i,1}^{(l)}, i = 1,2; l = 1,2.$

Proof: Applying Abel's theorem, $P(z_1, z_2) = \lim_{s \to 0} s P^*(z_1, z_2, s)$, we obtain the results given equations (33) to (35).

Theorem 4: The average number of the priority customers (L_p) and non priority customers (L_q) in the system are

$$L_{p} = \left[\frac{d\left(P\left(z_{1}, z_{2}\right)\right)}{dz_{1}}\right]_{z_{1}=z_{2}=1} = \frac{\left(B_{1}'A_{1}'' - A_{1}'B_{1}''\right)}{2\left(B_{1}'\right)^{2}} + \frac{\left(E_{1}'G_{1}'\right)\left(D_{1}'F_{1}' + D_{1}'F_{1}'\right) - \left(D_{1}'F_{1}'\right)\left(E_{1}'G_{1}' + E_{1}'G_{1}''\right)}{2\left(E_{1}'G_{1}'\right)^{2}} + \frac{\left(B_{1}'E_{1}'G_{1}'\right)\left(A_{1}'C_{1}'F_{1}' + A_{1}'C_{1}'F_{1}' + A_{1}'C_{1}'F_{1}'\right) - \left(A_{1}'C_{1}'F_{1}'\right)\left(B_{1}''E_{1}'G_{1}' + B_{1}'E_{1}'G_{1}' + B_{1}'E_{1}'G_{1}''\right)}{2\left(B_{1}'E_{1}'G_{1}'^{2}\right)} \\ L_{q} = \left[\frac{d\left(P\left(z_{1}, z_{2}\right)\right)}{dz_{2}}\right]_{z_{1}=z_{2}=1} = \frac{\left(B_{2}'A_{2}'' - A_{2}'B_{2}''\right)}{2\left(B_{2}'^{2}\right)^{2}} + \frac{\left(E_{2}'G_{2}'\right)\left(D_{2}''F_{2}' + D_{2}'F_{2}''\right) - \left(D_{2}'F_{2}'\right)\left(E_{2}''G_{2}' + E_{2}'G_{2}''\right)}{2\left(E_{2}'G_{2}'} + E_{2}'G_{2}''\right)} \\ + \frac{\left(B_{2}'E_{2}'G_{2}'\right)\left(A_{2}''C_{2}'F_{2}' + A_{2}'C_{2}''F_{2}' + A_{2}'C_{2}'F_{2}''\right) - \left(A_{2}'C_{2}'F_{2}'\right)\left(B_{2}''E_{2}'G_{2}' + B_{2}'E_{2}''G_{2}' + B_{2}'E_{2}'G_{2}''}\right)}{2\left(B_{2}'E_{2}'G_{2}'^{2}\right)^{2}} \right]^{(36)}$$

where

$$\begin{split} &A_i \equiv A_i(z_1, z_2) = \left[\left[1 - b_1^{*(1)} \left\{ \phi_1^{(1)} \left(z_1, z_2, 0 \right) \right\} \right] \left[\phi_1^{(2)} \left(z_1, z_2, 0 \right) + \mu_1^{(2)} \right] + \omega_1 b_1^{*(1)} \left\{ \phi_1^{(1)} \left(z_1, z_2, 0 \right) \right\} \phi_1^{(2)} \left(z_1, z_2, 0 \right) \right] \\ &B_i \equiv B_i(z_1, z_2) = \left[\left\{ \phi_1^{(2)} \left(z_1, z_2, 0 \right) + \mu_1^{(2)} \right\} \left\{ b_1^{*(1)} \left\{ \phi_1^{(1)} \left(z_1, z_2, 0 \right) \right\} - z_1 \right\} - \omega_1 b_1^{*(1)} \left\{ \phi_1^{(1)} \left(z_1, z_2, 0 \right) \right\} \phi_1^{(2)} \left(z_1, z_2, 0 \right) \right] \\ &C_i \equiv C_i(z_1, z_2) = \left[\left\{ z_2 - b_2^{*(1)} \left\{ \phi_2^{(1)} \left(z_1, z_2, 0 \right) \right\} \right] \left[\phi_2^{(2)} \left(z_1, z_2, 0 \right) + \mu_2^{(2)} \right] + \omega_2 b_2^{*(1)} \left\{ \phi_2^{(1)} \left(z_1, z_2, 0 \right) \right\} \phi_2^{(2)} \left(z_1, z_2, 0 \right) \right] \\ &D_i \equiv D_i(z_1, z_2) = \left[\left\{ 1 - b_2^{*(1)} \left\{ \phi_2^{(1)} \left(z_1, z_2, 0 \right) \right\} \right] \left[\phi_2^{(2)} \left(z_1, z_2, 0 \right) + \mu_2^{(2)} \right] + \omega_2 b_2^{*(1)} \left\{ \phi_2^{(1)} \left(z_1, z_2, 0 \right) \right\} \phi_2^{(2)} \left(z_1, z_2, 0 \right) \right] \\ &E_i \equiv E_i(z_1, z_2) = \left[\phi_2^{(2)} \left(z_1, z_2, 0 \right) + \mu_2^{(2)} \right] \left\{ \lambda_1 \left(1 - G_1 \left(z_1 \right) \right) + \lambda_2 \left(1 - G_2 \left(z_2 \right) \right) \right\} \\ &F_i \equiv F_i(z_1, z_2) = \left[\phi_2^{(2)} \left(h(z_2), z_2, 0 \right) + \mu_2^{(2)} \right] \left\{ \lambda_1 \left(1 - G_1 \left(h(z_2), z_2, 0 \right) \right\} - \lambda_2 \left(1 - G_2 \left(z_2 \right) \right) \right\} \\ &G_i \equiv G_i(z_1, z_2) = \left[\left\{ \phi_2^{(2)} \left(h(z_2), z_2, 0 \right) + \mu_2^{(2)} \right\} \left\{ b_2^{*(1)} \left\{ \phi_2^{(1)} \left(h(z_2), z_2, 0 \right) \right\} - z_2 \right\} \\ &- \omega_2 b_2^{*(1)} \left\{ \phi_2^{(1)} \left(h(z_2), z_2, 0 \right) \right\} \phi_2^{(2)} \left(h(z_2), z_2, 0 \right) \right\} \\ &D_i = D_i \left[\left\{ \phi_2^{(2)} \left(h(z_2), z_2, 0 \right) + \mu_2^{(2)} \right\} \left\{ b_2^{*(1)} \left\{ \phi_2^{(1)} \left(h(z_2), z_2, 0 \right) \right\} - z_2 \right\} \\ &- \omega_2 b_2^{*(1)} \left\{ \phi_2^{(1)} \left(h(z_2), z_2, 0 \right) \right\} dz_2 \right] \\ &D_i = D_i \left[\left\{ b_2^{(2)} \left(h(z_2), z_2, 0 \right) + \mu_2^{(2)} \right\} \left\{ b_2^{*(1)} \left\{ b_2^{*(1)} \left(h(z_2), z_2, 0 \right) \right\} - z_2 \right\} \\ &- \omega_2 b_2^{*(1)} \left\{ \phi_2^{(1)} \left(h(z_2), z_2, 0 \right) \right\} dz_2 \right] \\ &D_i = D_i \left[\left\{ b_2^{(2)} \left(h(z_2), z_2, 0 \right) + \mu_2^{(2)} \right\} dz_2 \right] dz_2 \right] \\ \\ &D_i = D_i \left\{ b_2^{*(1)} \left\{ b_2^{*(1$$

Proof: Differentiating $A(z_1, z_2)$ to $G(z_1, z_2)$ with respect to z_1 for priority customers and z_2 for nonpriority customers and applying L' Hospital rule, then putting $z_1 = z_2 = 1$, we get the desired results.

Corollary 2: Using Little's formula, we obtain the expected waiting time for the priority and non-priority customers, respectively as given below

$$W_{p} = \frac{L_{p}}{\lambda_{1}g_{1,1}} \text{ and } W_{q} = \frac{L_{q}}{\lambda_{2}g_{2,1}}$$
 (37)

4. SPECIAL CASES

In this section, we examine whether by setting appropriate parameters, our results are consistent with known results for some specific cases.

Case 1: If batch size $X_1 = 1$, and $X_2 = 0$, $z_1 = G_1(z_1) = z$, $z_2 = G_2(z_2) = 0$, $\alpha_2^{(1)} = \alpha_2^{(2)} = 0$, $\lambda_1 = \lambda$, $\lambda_2 = 0$, $\mu_2^{(1)} = \mu_2^{(2)} = 0$, $\omega_2 = 0$, $Q_q(z_1, z_2, 0, s) = 0$ then our results reduce to the model studied by Wang(2004). Now the joint probability generating function for the number of customers in the system is expressed as

$$P^{*}\left(z,s\right) = \frac{\left[\left\{1 - b_{1}^{*(1)}\left\{\phi_{1}^{(1)}\left(z,s\right)\right\}\right\}\left[\phi_{1}^{(2)}\left(z,s\right) + \mu_{1}^{(2)}\right] + \omega_{1}b_{1}^{*(1)}\left\{\phi_{1}^{(1)}\left(z,s\right)\right\}\phi_{1}^{(2)}\left(z,s\right)\right]\left[P_{0}^{*}\left(s\right) - 1/\left\{\lambda\left(1-z\right)+s\right\}\right]}{\left[\left\{\phi_{1}^{(2)}\left(z,s\right) + \mu_{1}^{(2)}\right\}\left\{b_{1}^{*(1)}\left\{\phi_{1}^{(1)}\left(z,s\right)\right\} - z\right\} - \omega_{1}b_{1}^{*(1)}\left\{\phi_{1}^{(1)}\left(z,s\right)\right\}\phi_{1}^{(2)}\left(z,s\right)\right]}$$

$$\text{where } \phi_{i}^{(l)}\left(z,s\right) = \left\{s + \alpha_{i}^{(l)} + \lambda\left(1-z\right) + \alpha_{i}^{(l)}\overline{r_{i}}^{*(l)}\left\{s + \lambda\left(1-z\right)\right\}\right\}, \quad (i = 1; l = 1, 2) \quad .$$

$$(38)$$

Case 2: If batch size $X_1 = 1$, and $X_2 = 0$, $z_1 = G_1(z_1) = z$, $z_2 = G_2(z_2) = 0$, $\alpha_i^{(l)} = 0$, (i=1, 2; l=1, 2),

 $\lambda_1 = \lambda$, $\lambda_2 = 0$, $\mu_2^{(1)} = \mu_2^{(2)} = 0$, $\omega_2 = 0$, $Q_q(z_1, z_2, 0, s) = 0$ then our model coincides to the model analyzed by Madan(2000). Now the joint probability generating function for the number of customers in the system is expressed as

$$P(z) = \frac{\left[\left\{1 - b_{1}^{(1)}\left\{\lambda\left(1 - z\right)\right\}\right\}\left[\lambda\left(1 - z\right) + \mu_{1}^{(2)}\right] + \omega_{1}\lambda\left(1 - z\right)b_{1}^{*(1)}\left\{\lambda\left(1 - z\right)\right\}\right]P_{0}}{b_{1}^{(1)}\left\{\lambda\left(1 - z\right)\right\}\left[\left(1 - \omega_{1}\right)\lambda\left(1 - z\right) + \mu_{1}^{(2)}\right] - \left\{\lambda\left(1 - z\right) + \mu_{1}^{(2)}\right\}z}$$
(39)

Case 3: If $\alpha_i^{(j)} = 0$, (i=1, 2; *l*=1, 2), $\mu_1^{(2)} = \mu_2^{(2)} = 0$, $\omega_1 = \omega_2 = 0$, $\phi_1^{(2)}(z_1, z_2, s) = \phi_2^{(2)}(z_1, z_2, s) = 1$, then our results tally with those obtained by Chaudhry & Templeton (1983) for $M_1^{X_1} M_2^{X_2} / G_1 G_2 / 1$ queueing system. The joint probability generating function for the number of customers in the system is expressed as

$$P^{*}\left(z_{1}, z_{2}, s\right) = P_{p}\left(z_{1}, z_{2}, 0, s\right) \frac{\left[1 - b_{1}^{*(1)}\left\{\phi_{1}^{(1)}\left(z_{1}, z_{2}, s\right)\right\}\right]}{\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]} + Q_{q}\left(z_{1}, z_{2}, 0, s\right) \frac{\left[1 - b_{2}^{*(1)}\left\{\phi_{2}^{(1)}\left(z_{1}, z_{2}, s\right)\right\}\right]}{\left[\lambda_{1}\left(1 - G_{1}\left(z_{1}\right)\right) + \lambda_{2}\left(1 - G_{2}\left(z_{2}\right)\right) + s\right]}$$

$$(40)$$
The

where

$$\begin{split} \phi_i^{(l)}\left(z_1, z_2, s\right) &= \left\{s + \sum_{i=1}^2 \lambda_i \left(1 - G_i \left(z_i\right)\right)\right\}, \ (i = 1, 2; \ l = 1) \\ Q_q\left(z_1, z_2, 0, s\right) &= \frac{1 - P_0^*\left(s\right) \left[\lambda_1 \left(1 - G_1 \left(h(z_2, s)\right)\right) + \lambda_2 \left(1 - G_2 \left(z_2\right)\right) + s\right]}{1 - \left(1/z_2\right) \left[b_2^{*(1)} \left\{\phi_2^{(1)} \left(h(z_2, s), z_2, s\right)\right\}\right]} , \\ P_p\left(z_1, z_2, 0, s\right) &= \frac{1 - P_0^*\left(s\right) \left[\lambda_1 \left(1 - G_1 \left(z_1\right)\right) + \lambda_2 \left(1 - G_2 \left(z_2\right)\right) + s\right]}{1 - \left(1/z_1\right) \left[b_1^{*(1)} \left\{\phi_1^{(1)} \left(z_1, z_2, s\right)\right\}\right]} - Q_q\left(z_1, z_2, 0, s\right) \frac{1 - \left(1/z_2\right) \left[b_2^{*(1)} \left\{\phi_1^{(1)} \left(z_1, z_2, s\right)\right\}\right]}{1 - \left(1/z_1\right) \left[b_1^{*(1)} \left\{\phi_1^{(1)} \left(z_1, z_2, s\right)\right\}\right]} \end{split}$$

5. RELIABILITY ANALYSIS

Since the breakdown and repair processes are independent of the servicing processes, then the server reliability and availability metrics are defined in the usual way. In this section we derive the reliability indices of the server namely, availability, failure frequency, mean time to failure, etc. Let A(t) be the availability of the server at time t. The steady state availability of the server defined as $A = \lim_{t \to \infty} A(t)$.

Theorem 5: The Laplace-Stieltjes transform of A(t) is given by

$$\begin{aligned} A^{*}(s) &= \frac{1}{\left\{\lambda_{1}\left(1-G_{1}\left(h(z_{2,s},s)\right)\right)+\lambda_{2}\left(1-G_{2}\left(z_{2,s}\right)\right)+s\right\}} \\ &+ \frac{\left[\left\{1-b_{1}^{*(1)}\left\{\phi_{1}^{(1)}\left(1,1,s\right)\right\}\right\}\left[\phi_{1}^{(2)}\left(1,1,s\right)+\mu_{1}^{(2)}\right]+\omega_{1}b_{1}^{*(1)}\left\{\phi_{1}^{(1)}\left(1,1,s\right)\right\}\sigma_{1}^{(1)}\left(1,1,s\right)\right]\left[1-s/\left\{\lambda_{1}\left(1-G_{1}\left(h(z_{2,s},s)\right)\right)+\lambda_{2}\left(1-G_{2}\left(z_{2,s}\right)\right)+s\right\}\right]}{\left[\phi_{1}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\right]\left[1-s/\left\{\lambda_{1}\left(1-G_{1}\left(h(z_{2,s},s)\right)\right)+\lambda_{2}\left(1-G_{2}\left(z_{2,s}\right)\right)+s\right\}\right]}{\left[\left\{\phi_{2}^{(2)}\left(h(1),1,s\right)+\mu_{2}^{(2)}\right]\left[1-s/\left\{\lambda_{1}\left(1-G_{1}\left(h(z_{2,s},s)\right)\right)+\lambda_{2}\left(1-G_{2}\left(z_{2,s}\right)\right)+s\right\}\right]}{\left[\left\{\phi_{2}^{(2)}\left(h(1),1,s\right)+\mu_{2}^{(2)}\right]\left\{b_{2}^{*(1)}\left\{\phi_{2}^{(1)}\left(h(1),1,s\right)\right\}-1\right\}-\omega_{2}b_{2}^{*(1)}\left\{\phi_{2}^{(1)}\left(h(1),1,s\right)\right\}\phi_{2}^{(2)}\left(h(1),1,s\right)\right]} \\ &\times \left\{\frac{\left[\left\{1-b_{1}^{*(1)}\left\{\phi_{1}^{(1)}\left(1,1,s\right)\right\}\right\}\left[\phi_{1}^{(2)}\left(1,1,s\right)+\mu_{1}^{(2)}\right]+\omega_{1}b_{1}^{*(1)}\left\{\phi_{1}^{(1)}\left(1,1,s\right)\right\}\phi_{1}^{(2)}\left(1,1,s\right)\right]}{\left[\left\{\phi_{1}^{(2)}\left(1,1,s\right)\right\}\right]\left\{b_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\right]+\omega_{2}b_{1}^{*(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}\phi_{2}^{(1)}\left(1,1,s\right)\right]}{\left[\phi_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\right]\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}} \\ &+ \frac{\left[\left\{1-b_{2}^{*(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}\right\}\left\{b_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\right]+\omega_{2}b_{1}^{*(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}\phi_{2}^{(1)}\left(1,1,s\right)\right]}{\left[\phi_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\right]\left\{\phi_{2}^{(2)}\left(1,1,s\right)\right\}} \\ \\ &+ \frac{\left[\left\{1-b_{2}^{*(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}\right\}\left\{b_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\right]+\omega_{2}b_{1}^{*(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}\phi_{2}^{(1)}\left(1,1,s\right)\right]}{\left[\phi_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\right]\left\{\phi_{2}^{(2)}\left(1,1,s\right)\right\}} \\ \\ &+ \frac{\left[\left\{1-b_{2}^{*(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}\right\}\left\{b_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\right]+\omega_{2}b_{1}^{*(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}\phi_{2}^{(1)}\left(1,1,s\right)\right\}}\right\}}{\left[\phi_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\right]\left\{\phi_{2}^{(2)}\left(1,1,s\right)\right\}} \\ \\ &+ \frac{\left[\left\{1-b_{2}^{*(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}\right\}\left\{b_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}}\right\}}{\left[\phi_{2}^{(2)}\left(1,1,s\right)+\mu_{2}^{(2)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}}\right]} \\ \\ &+ \frac{\left[\left\{1-b_{2}^{*(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}\right\}\left\{b_{2}^{(1)}\left\{\phi_{2}^{(1)}\left(1,1,s\right)\right\}}\right\}}{\left\{1-b_{2}^{(2)}\left\{0,1,s\right)+\mu_{2}^{(2)}\left\{0,1,s\right)}\right\}} \\ \\ \\ &+ \frac{\left[\left\{1-b_{$$

where $h(z_2)$ is the root of the equation (24), inside the |z| = 1, Re(s) > 0. **Proof:** Laplace transform of the system availability is obtained using

$$A^{*}(s) = \left[P_{0}^{*}(s) + P_{1}^{*(1)}\left(z_{1}, z_{2}, s\right) + P_{1}^{*(2)}\left(z_{1}, z_{2}, s\right) + Q_{1}^{*(1)}\left(z_{1}, z_{2}, s\right) + + Q_{1}^{*(2)}\left(z_{1}, z_{2}, s\right) + Q_{1}^{*$$

Thus, we obtain the results given in equation (41).

Corollary 3: The steady state availability of the system is given by

$$A = 1 - \sum_{i=1}^{2} \left(\rho_i^{(1)} \alpha_i^{(1)} \gamma_{i,1}^{(1)} + \omega_i \rho_1^{(2)} \alpha_i^{(2)} \gamma_{i,1}^{(2)} \right)$$
(42)

Proof: The steady state availability is obtained using

$$A = \lim_{z_1, z_2 \to 1} \lim_{s \to 0} s \left[P_0^*(s) + \left\{ \int_0^\infty P_1^{*(1)}\left(z_1, z_2, x, s\right) dx + \int_0^\infty Q_1^{*(1)}\left(z_1, z_2, x, s\right) dx \right\} + \left\{ P_1^{*(2)}\left(z_1, z_2, s\right) + Q_1^{*(2)}\left(z_1, z_2, s\right) \right\} \right] \quad \Box$$

Theorem 6: The steady state failure frequency of the system is given by

$$F_f = \sum_{i=1}^{2} \left(\rho_i^{(1)} \alpha_i^{(1)} + \omega_i \rho_i^{(2)} \alpha_i^{(2)} \right)$$
(43)

Proof: The steady state failure frequency of the system is obtained by using

$$\begin{split} F_{f} &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \int_{0}^{\infty} \alpha_{1}^{(1)} P_{m,n,1}^{(1)} \left(x,s\right) dx + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_{0}^{\infty} \alpha_{2}^{(1)} Q_{m,n,1}^{(1)} \left(x,s\right) dx + \alpha_{1}^{(2)} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} P_{m,n,1}^{(2)} \left(s\right) + \alpha_{2}^{(2)} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} Q_{m,n,1}^{(2)} \left(s\right) \\ &= \lim_{z_{1}, z_{2} \to 1} \left[\left\{ \int_{0}^{\infty} \alpha_{1}^{(1)} P_{1}^{(1)} \left(z_{1}, z_{2}, x, s\right) dx + \int_{0}^{\infty} \alpha_{2}^{(1)} Q_{1}^{(1)} \left(z_{1}, z_{2}, x, s\right) dx \right\} + \left\{ \alpha_{1}^{(2)} P_{1}^{(2)} \left(z_{1}, z_{2}, s\right) + \alpha_{2}^{(2)} Q_{1}^{(2)} \left(z_{1}, z_{2}, s\right) \right\} \right] \end{split}$$

Let τ be the time to the first failure of the server, then the reliability function of the server is $R(t) = P(\tau > t)$.

Theorem 7: The Laplace transform of reliability function R(t) is given by

$$\begin{split} R^*\left(s\right) &= \frac{1}{\left\{\lambda_1\left(1 - G_1\left(\hbar(s)\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right\}} \\ &\times \left[\frac{\left[\left\{1 - b_1^{*(1)}\left\{s + \alpha_1^{(1)}\right\}\right\}\left[s + \alpha_1^{(2)} + \mu_1^{(2)}\right] + \omega_1 b_1^{*(1)}\left\{s + \alpha_1^{(1)}\right\}\left(s + \alpha_1^{(1)}\right)\right]}{\left\{s + \alpha_1^{(1)}\right\}\left[\left\{s + \alpha_1^{(2)} + \mu_1^{(2)}\right\}\left\{b_1^{*(1)}\left\{s + \alpha_1^{(1)}\right\} - 1\right\} - \omega_1 b_1^{*(1)}\left\{s + \alpha_1^{(1)}\right\}\left(s + \alpha_1^{(2)}\right)\right]} \right] \\ &- \frac{\left[s + \alpha_2^{(2)} + \mu_2^{(2)}\right]\left\{b_2^{*(1)}\left\{s + \alpha_2^{(1)}\right\} - 1\right\} - \omega_2 b_2^{*(1)}\left\{s + \alpha_2^{(1)}\right\}\left(s + \alpha_2^{(2)}\right)\right]}{\left[\left\{s + \alpha_1^{(2)} + \mu_2^{(2)}\right\}\left\{b_2^{*(1)}\left\{s + \alpha_2^{(1)}\right\} - 1\right\} - \omega_2 b_2^{*(1)}\left\{s + \alpha_2^{(1)}\right\}\left(s + \alpha_2^{(2)}\right)\right]}\right\}} \\ &\times \left\{\frac{\left[\left\{1 - b_2^{*(1)}\left\{s + \alpha_1^{(1)}\right\}\right\}\left\{b_1^{*(1)}\left\{s + \alpha_1^{(1)}\right\} - 1\right\} - \omega_1 b_1^{*(1)}\left\{s + \alpha_1^{(1)}\right\}\left(s + \alpha_1^{(2)}\right)\right]}{\left[\left\{s + \alpha_1^{(2)} + \mu_1^{(2)}\right\}\right\}\left[s + \alpha_2^{(2)} + \mu_2^{(2)}\right] + \omega_2 b_2^{*(1)}\left\{s + \alpha_1^{(1)}\right\}\left(s + \alpha_1^{(2)}\right)\right]}\right\}} \\ &\times \left\{\frac{\left[\left\{1 - b_2^{*(1)}\left\{s + \alpha_2^{(1)}\right\}\right\}\left[s + \alpha_2^{(2)} + \mu_2^{(2)}\right] + \omega_2 b_2^{*(1)}\left\{s + \alpha_2^{(1)}\right\}\left(s + \alpha_2^{(1)}\right)\right]}\right\}}{\left\{s + \alpha_2^{(2)}\right\}\left[s + \alpha_2^{(2)} + \mu_2^{(2)}\right] + \omega_2 b_2^{*(1)}\left\{s + \alpha_2^{(1)}\right\}\left(s + \alpha_2^{(1)}\right)\right]}\right\}} \\ &\times \left\{1 - s/\left\{\lambda_1\left(1 - G_1\left(\hbar(s\right)\right)\right) + \lambda_2\left(1 - G_2\left(z_2\right)\right) + s\right\}\right] \end{split}$$
(44)

where $\hbar(s)$ is the root of the equation

$$z_{2} = b_{2}^{*(1)} \left\{ \lambda_{1} \left(1 - G_{1} \left(\hbar(z_{2}) \right) \right) + \lambda_{2} \left(1 - G_{2} \left(z_{2} \right) \right) + \alpha_{1}^{(1)} + s \right\} \left[1 - \frac{\omega_{2} \left\{ \lambda_{1} \left(1 - G_{1} \left(\hbar(z_{2}) \right) \right) + \lambda_{2} \left(1 - G_{2} \left(z_{2} \right) \right) + \alpha_{1}^{(2)} + s \right\}}{\left\{ \lambda_{1} \left(1 - G_{1} \left(\hbar(z_{2}) \right) \right) + \lambda_{2} \left(1 - G_{2} \left(z_{2} \right) \right) + \alpha_{2}^{(2)} + s \right\}} \right]$$

inside the $|z_2|$, Re(s) > 0.

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Proof: Using

$$R^{*}(s) = P_{0}^{*}(s) + \lim_{z_{1}, z_{2} \to 1} \left[\left\{ \int_{0}^{\infty} P_{1}^{*(1)}\left(z_{1}, z_{2}, x, s\right) d\, x + \int_{0}^{\infty} Q_{1}^{*(1)}\left(z_{1}, z_{2}, x, s\right) d\, x \right\} + \left\{ P_{1}^{*(2)}\left(z_{1}, z_{2}, s\right) + Q_{1}^{*(2)}\left(z_{1}, z_{2}, s\right) \right\} \right],$$

we obtain the result given in equation (44).

Corollary 4: The mean time to the first failure (MTTFF) of the server is given by

$$MTTFF = P^{*}(0) + \left(\rho_{1}^{(1)} + \omega_{1}\rho_{1}^{(2)} + \rho_{2}^{(1)} + \omega_{2}\rho_{2}^{(2)}\right) \left[\frac{\left\{1 - b_{1}^{*(1)}\left(\alpha_{1}^{(1)}\right)\right\}\left\{\alpha_{1}^{(2)} + \mu_{1}^{(2)}\right\} + \omega_{1}\alpha_{1}^{(1)}b_{1}^{*(1)}\left(\alpha_{1}^{(1)}\right)}{\alpha_{1}^{(1)}\left[\left\{\alpha_{1}^{(2)} + \mu_{1}^{(2)}\right\}\left\{1 - b_{1}^{*(1)}\left(\alpha_{1}^{(1)}\right)\right\} + \omega_{1}\alpha_{1}^{(2)}b_{1}^{*(1)}\left(\alpha_{1}^{(1)}\right)\right]} - \frac{\left[\left\{1 - b_{2}^{*(1)}\left(\alpha_{2}^{(1)}\right)\right\}\left\{\alpha_{2}^{(2)} + \mu_{2}^{(2)}\right\} + \omega_{2}\alpha_{2}^{(1)}b_{2}^{*(1)}\left(\alpha_{2}^{(1)}\right)\right]}{\alpha_{2}^{(2)}\left[\left\{\alpha_{2}^{(2)} + \mu_{2}^{(2)}\right\}\left\{b_{2}^{*(1)}\left(\alpha_{2}^{(1)}\right) - 1\right\} - \omega_{2}\alpha_{2}^{(2)}b_{2}^{*(1)}\left(\alpha_{2}^{(1)}\right)\right]} \times \left\{\frac{\omega_{1}b_{1}^{*(1)}\left(\alpha_{1}^{(1)}\right)\left(\alpha_{1}^{(1)} - \alpha_{1}^{(2)}\right)}{\left[\left\{\alpha_{1}^{(2)} + \mu_{1}^{(2)}\right\}\left\{b_{1}^{*(1)}\left(\alpha_{1}^{(1)}\right) - 1\right\} - \omega_{1}\alpha_{1}^{(2)}b_{1}^{*(1)}\left(\alpha_{1}^{(1)}\right)\right]}\right\}\right\}$$
(45)

Proof: The MTTFF is obtained by

$$MTTFF = \int_{0}^{\infty} R(t) dt = R^{*}(s) |_{s=0} \text{ and } \lim_{s \to 0} s P_{0}^{*}(s) = P_{0}$$

6. SENSITIVITY ANALYSIS

In this section, to demonstrate the computational tractability of analytical results we provide some numerical results which are displayed in tables 1-8 and graphs 1-6. By taking numerical example, we illustrate the influence of system parameters on the probabilities for different states namely idle state P₀, busy state P(B) and repair state P(R), the expected waiting time for priority customers (W_p) and nonpriority customers (W_q), failure frequency (F_t), average queue length of priority customers (L_p) and nonpriority customers (L_q), availability (A) of the system. We set the default parameters as $\lambda_1=0.2, \lambda_2=0.05, \mu_1^{(1)}=10, \mu_1^{(2)}=5, \mu_2^{(1)}=3, \mu_2^{(2)}=2, \beta_1^{(1)}=5, \beta_1^{(2)}=3, \beta_2^{(1)}=2, \beta_2^{(2)}=1.5, \alpha_1^{(1)}=0.5, \alpha_1^{(2)}=0.3, \alpha_2^{(1)}=0.2, \alpha_2^{(2)}=0.1.$

The batch size is taken to follow the geometric distribution with parameter p=1/5, so that the mean batch size and the second moment of the batch size for ith type of customers are given as $g_{i,1}=q/p$, $g_{i,2} = q(2q+p)/p^2$, where q=1-p. The service time and repair time distributions follow Erlangian distributions with mean $\tau_{i,1}^{(1)} = 1/\mu_i^{(1)}$ and $\gamma_{i,1}^{(l)} = 1/\beta_i^{(l)}$,

respectively. Second moment corresponding to service time and repair time are given as $\tau_{i,2}^{(1)} = (k+1) / k \left(\mu_i^{(1)}\right)^2$,

$$\gamma_{i,2}^{(l)} = (k+1) / k (\beta_i^{(l)})^2$$
, where k=5, i = 1, 2, l = 1, 2.

Tables 1-2 demonstrate the effect of arrival rates λ_1 and λ_2 , on the probabilities of the server being idle P_0 , busy P(B) and under repair P(R), for different sets of ω_1 for priority customers and different sets of ω_2 for non priority customers. We observe that as λ_1 and λ_2 increase, P_0 decreases but P(B) and P(R) increase; also on increasing ω_1 and ω_2 , P_0 decreases but P(B) and P(R) show the increasing trends. In tables 3-4, we have shown the effect of batch size g_1 for priority customers and g_2 for nonpriority customers on P_0 , P(B) and P(R) by varying arrival rates λ_1 and λ_2 . With respect to λ_1 and λ_2 similar pattern has been found as obtained in tables 1-2. As we increase g_1 and g_2 , P_0 decreases while P(B) and P(R) increase.

Tables 5-6 demonstrate the effect of service rates $\mu_1^{(1)}$ and $\mu_1^{(2)}$ on the waiting time of priority customers (W_p), non priority customers (W_q) and failure frequency (F_f) for different sets of (ω_1 , ω_2). When we increase the essential service rate ($\mu_1^{(1)}$) and second optional service rate ($\mu_1^{(2)}$) for the priority customers then the waiting time of priority customers (W_p) and failure frequency (F_f) decrease while waiting time of nonpriority customers (W_q) increases. The similar pattern has been noticed in tables 7-8 with regard to service rates $\mu_2^{(1)}$ and $\mu_2^{(2)}$ on the waiting time of priority customers (W_p), non priority (W_q) customers and failure frequency (F_f) for different sets of (ω_1 , ω_2).

Figures 1(a) and 1(b) depict the results for the expected number of the priority (L_p) and nonpriority (L_q) customers for different sets of g_1 and g_2 by varying arrival rates λ_1 and λ_2 , respectively. In these figures we see that as we increase arrival rates λ_1 and λ_2 , the expected number of the priority (L_p) and non priority (L_q) customers increase. Also as we increase g_1 and g_2 , initially L_p and L_q increase gradually and then after increase sharply.

Figures 2(a) and 2(b) exhibit the expected number of priority (L_p) and nonpriority (L_q) customers for different values of ω_1 and ω_2 by varying arrival rates λ_1 and λ_2 , respectively. On increasing the value of ω_1 and ω_2 , initially L_p and L_q

increase gradually and then after increase sharply. Same pattern has been observed with respect to λ_1 and λ_2 as in figs. 1(a) and 1(b).

The effects of failure rates $\alpha_1^{(1)}$, $\alpha_1^{(2)}$, $\alpha_2^{(1)}$, $\alpha_2^{(2)}$ and repair rates $\beta_1^{(1)}$, $\beta_1^{(2)}$, $\beta_2^{(1)}$, $\beta_2^{(2)}$ on L_p and L_q by varying the traffic intensity $\rho_1^{(1)}$ and $\rho_1^{(2)}$ have been shown in figures 3(a-b) and 4(a-b), respectively. In these figures, we observe that L_p and L_q increase with the increase in traffic intensity $\rho_1^{(1)}$ and $\rho_1^{(2)}$. A sharp increasing (decreasing) trend in L_p and L_q can be easily seen by increasing (decreasing) failure rate (repair rate) during the essential service time compared to second optional service time.

The results for availability (A) are plotted against the failure rate $\alpha_1^{(1)}$, $\alpha_1^{(2)}$, $\alpha_2^{(1)}$, $\alpha_2^{(2)}$ and repair rate $\beta_1^{(1)}$, $\beta_1^{(2)}$, $\beta_2^{(1)}$, $\beta_2^{(2)}$ separately for different sets of (λ_1, λ_2) in figures 5(a-d) and 6(a-d). Figures 5(a)-5(d) show a linearly sharp decreasing trend in A with the increase in arrival rate of nonpriority customers (λ_2) compared to arrival rate of priority customers (λ_1). Figures 6(a)–6(d) show an increasing trend in A on increasing λ_1 and λ_2 . Finally, we conclude that

- On increasing traffic intensity, the average queue length of priority and nonpriority customers increases. The increasing (decreasing) trends of L_p and L_q with failure (repair) rate match with physical situations.
- The waiting time of the priority (nonpriority) customers decreases (increases) with the increase in essential and second optional service rates of priority customers but increases (decreases) with the essential and second optional service rates of nonpriority customers.
- The failure frequency decreases with the increase in the service rate. As expected, the availability decreases (increases) with the increase in failure (repair) rate of the server.

7. CONCLUSION

In this paper we have provided the queue size distribution and other performance indices by employing the generating functions and the supplementary variables for the priority queueing model. We have established some reliability indices such as the availability of the server, failure frequency, etc. In the model developed, we have incorporated many novel features namely bulk input, non-preemptive priority, unreliable server, optional service etc simultaneously which make our results applicable to more versatile congestion situations encountered in computer and communication systems, distribution and service sectors, production and manufacturing systems, and many more real world queueing problems. In order to justify computational tractability, we have performed extensive numerical experiments. In future, the work done in the present investigation can be extended by incorporating the multiple server, batch service, etc..

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	P ₀				P(B)			P(R)		
λ1	$\omega_1 = .1$	$\omega_1 = .1$	$\omega_1 = .2$	ω ₁ =.1	$\omega_1 = .1$	$\omega_1 = .2$	$\omega_1 = .1$	$\omega_1 = .1$	ω ₁ =.2	
	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	
0.5	0.771	0.776	0.732	0.208	0.204	0.244	0.021	0.020	0.024	
0.7	0.695	0.699	0.638	0.277	0.273	0.329	0.027	0.027	0.033	
0.9	0.619	0.623	0.544	0.347	0.343	0.415	0.034	0.034	0.041	
1.1	0.543	0.547	0.450	0.416	0.412	0.500	0.041	0.041	0.050	
1.3	0.466	0.471	0.356	0.485	0.481	0.585	0.048	0.048	0.058	
1.5	0.390	0.394	0.262	0.555	0.551	0.671	0.055	0.055	0.067	

Table 1: Effect of (ω_1, ω_2) on long run probabilities of server's state by varying λ_1

	P ₀				P(B)			P(R)		
λ2	ω ₁ =.1	$\omega_1 = .1$	$\omega_1 = .2$	ω ₁ =.1	$\omega_1 = .1$	$\omega_1 = .2$	ω ₁ =.1	$\omega_1 = .1$	ω ₁ =.2	
	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	
0.05	0.715	0.725	0.681	0.260	0.250	0.290	0.025	0.025	0.029	
0.07	0.677	0.692	0.648	0.295	0.281	0.321	0.029	0.028	0.032	
0.09	0.639	0.658	0.614	0.329	0.311	0.351	0.032	0.031	0.035	
0.11	0.601	0.625	0.581	0.364	0.342	0.382	0.035	0.033	0.037	
0.13	0.563	0.591	0.547	0.399	0.373	0.413	0.038	0.036	0.040	
0.15	0.525	0.557	0.513	0.433	0.403	0.443	0.041	0.039	0.043	

Table 2: Effect of (ω_1, ω_2) on long run probabilities of server's state by varying λ_2

	P ₀				P(B)			P(R)		
λ1	g ₁ = 3	$g_1 = 6$	$g_1 = 6$	g ₁ = 3	$g_1 = 6$	$g_1 = 6$	g ₁ = 3	$g_1 = 6$	$g_1 = 6$	
	$g_2 = 6$	g ₂ = 3	$g_2 = 6$	$g_2 = 6$	g ₂ = 3	$g_2 = 6$	$g_2 = 6$	g ₂ = 3	$g_2 = 6$	
0.5	0.807	0.689	0.664	0.176	0.283	0.306	0.017	0.028	0.030	
0.7	0.749	0.574	0.549	0.228	0.387	0.410	0.023	0.039	0.041	
0.9	0.692	0.460	0.435	0.280	0.491	0.514	0.028	0.049	0.051	
1.1	0.635	0.346	0.320	0.332	0.595	0.618	0.033	0.059	0.062	
1.3	0.578	0.231	0.206	0.384	0.699	0.722	0.038	0.070	0.072	
1.5	0.521	0.117	0.092	0.436	0.803	0.826	0.043	0.080	0.082	

Table 3: Effect of (g_1, g_2) on long run probabilities of server's state by varying λ_1

	P ₀				P(B)			P(R)		
λ2	g ₁ = 3	$g_1 = 6$	$g_1 = 6$	g ₁ = 3	$g_1 = 6$	g ₁ = 6	g ₁ = 3	$g_1 = 6$	g ₁ = 6	
	g ₂ = 6	g ₂ = 3	$g_2 = 6$	g ₂ = 6	g ₂ = 3	g ₂ = 6	$g_2 = 6$	g ₂ = 3	$g_2 = 6$	
0.05	0.731	0.651	0.588	0.245	0.318	0.375	0.024	0.032	0.037	
0.07	0.681	0.626	0.538	0.291	0.341	0.421	0.028	0.034	0.041	
0.09	0.630	0.601	0.487	0.337	0.364	0.467	0.033	0.036	0.046	
0.11	0.580	0.575	0.437	0.383	0.387	0.513	0.037	0.038	0.050	
0.13	0.529	0.550	0.386	0.429	0.410	0.559	0.042	0.040	0.055	
0.15	0.479	0.525	0.336	0.475	0.433	0.605	0.046	0.043	0.059	

Table 4: Effect of (g_1, g_2) on long run probabilities of server's state by varying λ_2

		Wp			W _q		F _f		
$\mu_1^{(1)}$	ω ₁ =.1	ω1 =.1	ω ₁ =.2	ω ₁ =.1	ω ₁ =.1	ω ₁ =.2	ω ₁ =.1	ω1 =.1	ω ₁ =.2
	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$
3	2.335	2.367	2.799	0.403	0.283	0.135	0.079	0.079	0.083
3.5	1.955	1.987	2.397	0.522	0.399	0.271	0.070	0.070	0.074
4	1.683	1.714	2.110	0.601	0.475	0.360	0.063	0.062	0.066
4.5	1.478	1.510	1.893	0.655	0.527	0.422	0.057	0.057	0.061
5	1.319	1.350	1.725	0.696	0.566	0.468	0.053	0.052	0.056
5.5	1.192	1.223	1.591	0.727	0.596	0.504	0.049	0.049	0.053

 $\begin{array}{l} \textbf{Table 5:} \ \text{Effect of } (\omega_1, \omega_2) \ \text{on waiting time of priority customers } (W_p), \ \text{nonpriority customers } (W_q) \\ & \text{and failure frequency } (F_f) \ \text{by varying } \mu_1^{(1)} \end{array}$

	W _p				W _q			F _f		
$\mu_1^{(2)}$	ω ₁ =.1	ω ₁ =.1	$\omega_1 = .2$	$\omega_1 = .1$	ω ₁ =.1	ω ₁ =.2	ω ₁ =.1	ω ₁ =.1	$\omega_1 = .2$	
	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	ω ₂ =. 1	
3	0.328	0.359	0.570	0.903	0.763	0.727	0.025	0.024	0.027	
3.5	0.298	0.329	0.508	0.908	0.768	0.738	0.024	0.024	0.026	
4	0.275	0.306	0.462	0.912	0.772	0.747	0.024	0.024	0.026	
4.5	0.258	0.289	0.426	0.915	0.775	0.753	0.024	0.024	0.025	
5	0.244	0.275	0.398	0.918	0.777	0.758	0.024	0.023	0.025	
5.5	0.233	0.264	0.375	0.920	0.779	0.762	0.024	0.023	0.025	

 $\begin{array}{l} \textbf{Table 6:} \ Effect \ of \ (\omega_1, \ \omega_2) \ on \ waiting \ time \ of \ priority \ customers \ (W_p), \ nonpriority \ customers \ (W_q) \\ & and \ failure \ frequency \ (F_f) \ by \ varying \ \mu_1^{(2)} \end{array}$

		W _p			W _q			F _f		
$\mu_2^{(1)}$	ω ₁ =.1	$\omega_1 = .1$	$\omega_1 = .2$	ω ₁ =.1	ω ₁ =.1	$\omega_1 = .2$	ω ₁ =.1	$\omega_1 = .1$	$\omega_1 = .2$	
	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	
3	0.437	0.468	0.795	0.881	0.743	0.681	0.026	0.026	0.030	
3.5	0.466	0.497	0.825	0.744	0.607	0.548	0.025	0.025	0.029	
4	0.488	0.519	0.847	0.642	0.505	0.449	0.024	0.024	0.028	
4.5	0.505	0.535	0.863	0.563	0.427	0.373	0.023	0.023	0.027	
5	0.519	0.549	0.877	0.500	0.365	0.312	0.023	0.023	0.027	
5.5	0.530	0.560	0.888	0.449	0.315	0.263	0.022	0.022	0.026	

 $\begin{array}{l} \textbf{Table 7:} \ \text{Effect of } (\omega_1, \omega_2) \ \text{on waiting time of priority customers } (W_p), \ \text{nonpriority customers } (W_q) \\ & \text{and failure frequency } (F_f) \ \text{by varying } \mu_2^{(1)} \end{array}$

		Wp			W _q			F _f		
$\mu_2^{(2)}$	ω ₁ =.1	ω ₁ =.1	$\omega_1 = .2$	$\omega_1 = .1$	ω ₁ =.1	ω ₁ =.2	ω ₁ =.1	ω ₁ =.1	$\omega_1 = .2$	
	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	$\omega_2 = .2$	$\omega_2 = .1$	$\omega_2 = .1$	
3	0.528	0.542	0.866	0.584	0.523	0.484	0.026	0.026	0.030	
3.5	0.552	0.563	0.886	0.499	0.455	0.422	0.026	0.026	0.030	
4	0.569	0.577	0.900	0.437	0.403	0.374	0.026	0.026	0.030	
4.5	0.582	0.588	0.910	0.388	0.361	0.336	0.026	0.026	0.030	
5	0.592	0.597	0.919	0.349	0.327	0.305	0.026	0.025	0.029	
5.5	0.600	0.604	0.925	0.317	0.299	0.279	0.026	0.025	0.029	

 $\begin{array}{l} \textbf{Table 8:} \ \text{Effect of } (\omega_1, \omega_2) \ \text{on waiting time of priority customers } (W_p), \ \text{nonpriority customers } \\ (W_q) \ \text{and failure frequency } (F_f) \ \text{by varying } \mu_2^{(2)} \end{array}$



Figure 2: Effect of (ω_1, ω_2) on (a) L_p vs λ_1 and (b) L_q vs λ_2 .



Figure 3: Expected number of priority customers by varying $\rho_1^{(1)}$ for different values of (a) failure rate ($\alpha_1^{(1)}, \alpha_1^{(2)}$) and (b) repair rate ($\beta_1^{(1)}, \beta_1^{(2)}$)







Figure 5: Effect of (λ_1, λ_2) on availability by varying failure rate (a) $\alpha_1^{(1)}$; (b) $\alpha_1^{(2)}$; (c) $\alpha_2^{(1)}$; (d) $\alpha_2^{(2)}$.



Figure 6: Effect of (λ_1, λ_2) on availability by varying repair rate (a) $\beta_1^{(1)}$; (b) $\beta_1^{(2)}$; (c) $\beta_2^{(1)}$; (d) $\beta_2^{(2)}$.