International Journal of
***O**perations **R**esearch*

International Journal of Operations Research Vol. 12, No. 3, 057–068 (2015)

Joint Chance-Constrained Reliability Optimization with General Form of Distributions

Vincent Charles[1*], Saiful I. Ansari[2], and Mohammad Khodabakhshi[3]

[1]CENTRUM Católica Graduate Business School, Pontificia Universidad Católica del Perú,
Calle Daniel Alomía Robles 125-129, Los Álamos de Monterrico, Santiago de Surco, Lima 33, Peru

[2]Department of Statistics and Operations Research, Aligarh Muslim University,
Aligarh, Uttar Pradesh, India

[3]Department of Mathematics, Shahid Beheshti University, General Campus, Tehran, Iran

Received December 2014; Revised March 2015; Accepted May 2015

Abstract — Probabilistic or stochastic programming is a framework for modelling optimization problems that involve uncertainty, undoubtedly supporting many decision problems in business and management. Stochastic programming models arise as reformulations or extensions of reliability optimization problems with random parameters. Moreover, the resource elements vary and it is reasonable to consider them as stochastic variables. In this paper, we describe the chance-constrained reliability stochastic optimization (CCRSO) problem for which the objective is to maximize the system reliability for the given joint chance constraints where only the resource variables are random in nature and which follow different general form of distributions. Few numerical examples are also presented to illustrate the applicability of the methodology.

Keywords — chance-constrained programming, reliability optimization, joint constraints, general form of distributions.

1. INTRODUCTION

Stochastic programming (SP) models were first formulated by Dantzig (1955) who suggested a two-stage programming technique that involves the conversion of SP models into their equivalent deterministic programming models. However, this technique suffers from the limitation that it does not allow any constraint to be violated even at a specific probability level. This gave rise to the concept of chance-constrained programming (CCP), where constraints containing random variables are guaranteed to be satisfied with a certain probability. Charnes and Cooper (1959, 1963) developed the concept of CCP. For the interested reader, notable contributions to the field can be found in Kataoka (1963), Van De Panne and Popp (1963), Charnes, Cooper, and Thompson (1964, 1965), Charnes, Kirby, and Raike (1967), Williams (1965, 1966), Naslund (1966), Wets (1966), Symonds (1967), Ziemba (1970), Lee and Olson (1985), Olson and Swenseth (1987), Seppala (1988), Shapiro (1990), Weintraub and Vera (1991), Flam and Schult (1993), Schoen (1994), Zhao and Ziemba (2001), Beraldi and Bruni (2010), Cheng and Lisser (2012, 2013), and Kucukyavuz (2012). Joint probabilistic constraints for independent random variables were used initially by Miller and Wagner (1965) and Jagannathan (1974). Charles and Dutta (2005) also derived the deterministic equivalent of the objective function and constraint coefficients with normal random variables. The properties of stochastic programming problems and methods for obtaining an optimal solution were described in Sengupta and Fox (1969), Tintner and Sengupta (1972), Vajda (1972), Rao (1989), Kall and Wallace (1994), Prékopa (1995) and Birge and Louveaux (2011). In this regards, a bibliography was provided by Stancu-Minasian and Wets (1976).

Reliability is defined as the probability that a device or system is able to perform its intended functions satisfactorily under specified conditions for a specified period of time. However, traditional reliability assumes that a system and its components can be in either a completely working or a completely failed state only (Birnbaum, Esary, & Saunders, 1961), i.e., no intermediate states are allowed. A reliability-based methodology for the robust optimal design of uncertain linear structural systems subjected to stochastic dynamic loads was also presented by Papadimitriou, Katafygiotis, and Siu (1997) and Papadimitriou and Ntotsios (2004).

Solution methods in the literature for reliability optimization of complex systems are mainly heuristic methods. In recent years, metaheuristic algorithms such as genetic algorithm (Gen & Cheng, 1997), simulated annealing (Ravi, Murty, & Reddy, 1997), and tabu search (Glover & Laguna, 1993) have also been applied to reliability optimization of complex systems. A comprehensive review of heuristic and metaheuristic algorithms for reliability optimization can be found in a

* Corresponding author's email: vcharles@pucp.pe

relatively recent survey paper (Kuo & Prasad, 2000). Exact methods for reliability problems include the branch-and-bound methods (Nakagawa, Nakashima, & Hattori, 1978; Tillman, Hwang, & Kuo, 1980), dynamic programming methods (Ng & Sancho, 2001; Sniedovich & Vazirinejad, 1990), and implicit enumeration methods (Misra & Sharma, 1991; Prasad & Kuo, 2000). For a more comprehensive view of the research works undergone in the field of reliability optimization for the past four decades, we refer the interested readers to the following authors, whom we found to provide useful insights: Aggarwal, Gupta, and Misra (1975), Aggarwal (1976), Tillman, Hwang, and Kuo (1977a,1977b), Kuo, Hwang, and Tillman (1978), Tillman et al. (1980), Hwang, Tillman, and Lee (1981), Painton and Campbell (1995), Sung and Cho (1999), Kuo and Prasad (2000), Ravi, Reddy, and Zimmermann (2000), Kuo, Prasad, Tillman, and Hwuang (2001), Sun, Mckinnon, and Li (2001), Sun and Li (2002), Zhao and Liu (2003), Coit, Jin, and Wattanapongsakorn (2004), Cui, Kuo, Loh, and Xie (2004), Liang and Smith (2004), Ramirez-Marquez, Coit, and Konak (2004), Zafiropoulos and Dialynas (2004), Aggarwal and Gupta (2005), Marseguerra, Zio, Podofillini, and Coit (2005), Gen and Yun (2006), Ha and Kuo (2006a, 2006b), Kuo and Wan (2007), Onishi and Kimura (2007), Ramirez-Marquez and Coit (2007), Yadavalli, Malada, and Charles (2007), Zhao, Cui, and Kuo (2007), Coelho (2009), Chan and Lin (2011), Minguez, Conejo, and Garcia-Bertrand (2011), Nikolaidis and Mourelatos (2011), and Sakalli (2014).

The redundancy allocation problem (RAP) is a difficult combinatorial optimization problem (Chern, 1992). It was extensively studied in the past, and when considering binary components, it was solved as a single objective optimization problem (generally maximization of system reliability), subject to several constraints, such as cost, weight, and volume, among others. It was solved using mathematical models, such as dynamic programming (Bellman & Dreyfus, 1958; Misra, 1971; Fyffe, Hines, & Lee 1968), integer programming (Bulfin & Liu, 1985; Misra & Sharma, 1991), mixed integer and non-linear programming (Tillman et al., 1977a, 1977b), and metaheuristics, such as genetic algorithms (Coit & Smith, 1996; Ida, Gen, & Yokota, 1994; Painton & Campbell, 1995), tabu search (Kulturel-Konak, Smith, & Coit, 2003), and ant colony optimization (Liang & Smith, 2004).

This paper is organized as follows: first, the literature review is presented in section 1. In section 2, the mathematical model of a stochastic integer programming of an n-stage series system with m-joint chance constraints problem is defined and its deterministic equivalent form is derived. Moreover, some general form of distributions and their various deductions are discussed in section 3. Few numerical examples are then presented in section 4 and section 5 concludes the paper. For the interested reader, a more detailed analysis and further information in this regard can be found in the work undergone by Ansari (2011).

2. STOCHASTIC INTEGER PROGRAMMING: *N*-STAGE SERIES SYSTEM WITH *M*-JOINT CHANCE CONSTRAINTS

The chance-constrained programming problem for an *n*-stage series system with *m*-joint chance constraints can be formulated as:

$$\begin{aligned} &Max\ R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right] \\ &subject\ to \\ &\quad P\left[g_1(x)\le b_1, g_2(x)\le b_2,\ldots,g_m(x)\le b_m\right]\ge p, \\ &\quad x_j \ge 1,\ \ j=1,2,\ldots,n. \end{aligned} \tag{1}$$

where :

$R_s(X)$ - Reliability of the system

r_j, q_j - Reliability, unreliability of components j ; $r_j + q_j = 1$

x_j - Number of components used at stage j

$g_i(x)$ - Chance constraint i

b_i - Amount of resource i available (random)

$0 < p < 1$, usually close to 1.

The joint chance constraints of System (1) may also be written as $\prod_{i=1}^{m} P\left(b_i \ge y_i\right) \ge p,$ where $y_i = g_i(x)$. Hence, System (1) has the following form:

$$Max\ R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right]$$
$$subject\ to$$
$$\prod_{i=1}^{m} P(b_i \geq y_i) \geq p, \tag{2}$$
$$x_j \geq 1,\ \ j = 1, 2, ..., n.$$

3. VARIOUS SPECIAL CASES FOR JOINT CHANCE CONSTRAINTS

Case 1: In System (1), let b_i follow a general form of distributions $F(b_i) = 1-\left[A_i h(b_i) + B_i\right]^{C_i}$.

It is given that the ith random variable b_i has three known parameters $A_i\ (\neq 0)$, $B_i\ (\geq 0)$, and $C_i\ (\neq 0)$ such that $F(\alpha_i) = 0, F(\beta_i) = 1$, and $h(b_i)$ is a monotonic, continuous, and differentiable function of b_i in the interval $[\alpha_i, \beta_i]$. The probability density function (*pdf*) of the random variable b_i is given by :

$$f(b_i) = -A_i C_i \left[A_i h(b_i) + B_i\right]^{C_i - 1} h'(b_i). \tag{3}$$

Now, for the above *pdf* the joint probabilistic constraints in System (1) can be written as:

$$\prod_{i=1}^{m}\left(\int_{y_i}^{\beta_i} -A_i C_i \left[A_i h(b_i) + B_i\right]^{C_i - 1} h'(b_i)\, db_i\right) \geq p. \tag{4}$$

After integration, we have:

$$\prod_{i=1}^{m}\left[A_i h(y_i) + B_i\right]^{C_i} \geq p, \text{ as } \left[A_i h(\beta_i) + B_i\right]^{C_i} = 0.$$

Hence, for the given random variable, the joint chance constraints in System (2) are converted into joint deterministic constraints as follows:

$$\prod_{i=1}^{m}\left[A_i h(y_i) + B_i\right]^{C_i} \geq p. \tag{5}$$

The deterministic constraints may get the information from the following distributions when the latter follow the parameters $\left[A_i, B_i, C_i, h(b_i)\right]$:

Power Function distribution $\left[-\lambda_i^{-a_i}, 1, 1, b_i^{a_i}\right]$, Pareto distribution $\left[\lambda_i^{-\delta_i}, 0, -a_i\delta_i^{-1}, b_i^{a_i}\right]$, Beta distribution of first kind $\left[1, 0, a_i, (\lambda_i - b_i)(\lambda_i - \delta_i)^{-1}\right]$, Weibull distribution $\left[1, 0, \theta_i\delta_i^{-1}, e^{-\delta_i b_i^{a_i}}\right]$, Inverse Weibull distribution $\left[-1, 1, 1, e^{-\theta_i b_i^{-a_i}}\right]$, Burr Type II distribution $\left[-1, 1, 1, \left(1+e^{-b_i}\right)^{-k_i}\right]$, Burr Type III distribution $\left[-1, 1, 1, \left(1+b_i^{-\lambda_i}\right)^{-k_i}\right]$, Burr Type IV distribution $\left[-1, 1, 1, \left\{1+\left\{b_i^{-1}(\lambda_i - b_i)\right\}^{1/\lambda_i}\right\}^{-k_i}\right]$, Burr Type V distribution $\left[-1, 1, 1, \left(1+\lambda_i e^{-\tan b_i}\right)^{-k_i}\right]$, Burr Type VI distribution $\left[-1, 1, 1, \left(1+\lambda_i e^{-k_i \sinh b_i}\right)^{-k_i}\right]$, Burr Type VII distribution $\left[-2^{-k_i}, 1, 1, (1+\tanh b_i)^{k_i}\right]$, Burr Type VIII distribution $\left[-2\pi^{-1}, 1, 1, \left(\tan^{-1} e^{-b_i}\right)^{k_i}\right]$, Burr Type IX distribution $\left[0.5\lambda_i, (1-0.5\lambda_i), -1, \left(1+e^{b_i}\right)^{-k_i}\right]$, Burr Type X distribution $\left[-1, 1, 1, \left\{1-\exp\left(-b_i^2\right)\right\}^{k_i}\right]$, Burr Type XI distribution $\left[-1, 1, 1, \left\{b_i - (2\pi)^{-1}\sin 2\pi b_i\right\}^{k_i}\right]$, Burr Type XII distribution $\left[\theta_i, 1, -m_i, b_i^{a_i}\right]$, and Cauchy distribution $\left[-\pi^{-1}, 0.5, 1, \tan^{-1} b_i\right]$.

Hence, in this case, the deterministic form of the chance-constrained programming problem for *n*-stage series with *m*-joint chance constraints is given by:

$$Max\ R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right]$$

$$\text{subject to} \tag{6}$$

$$\prod_{i=1}^{m}\left[A_i h(y_i) + B_i\right]^{C_i} \geq p,$$

$$x_j \geq 1,\ \ j = 1, 2, \ldots, n.$$

After inserting the particular value of the parameters $\left[A_i, B_i, C_i, h(b_i)\right]$ in the above deterministic constraints, we get different deterministic constraints for different distributions.

Case 2: In System (1), let b_i follow a general form of distributions $F(b_i) = A_i\left[h(b_i)\right]^{-C_i} + B_i$.

The ith random variable b_i is acknowledged to have three known parameters A_i, B_i, and C_i. These parameters are defined as $A_i\ (\neq 0)$, $B_i\ (\geq 0)$, and $C_i\ (\neq 0)$ and the following conditions apply: $F(\alpha_i) = 0, F(\beta_i) = 1$, and $h(b_i)$ is a monotonic, continuous, and differentiable function of b_i in the interval $[\alpha_i, \beta_i]$. In this context, the *pdf* of the random variable b_i is given by:

$$f(b_i) = -A_i C_i\left[h(b_i)\right]^{-C_i-1} h'(b_i). \tag{7}$$

Now, the joint chance constraints in System (2) for the above *pdf* can be written as:

$$\prod_{i=1}^{m}\left(\int_{y_i}^{\beta_i} -A_i C_i\left[h(b_i)\right]^{-C_i-1} h'(b_i)\,db_i\right) \geq p\,. \tag{8}$$

The integration leads to:

$$\prod_{i=1}^{m}\left\{A_i\left[h(y_i)\right]^{-C_i} + B_i\right\} \leq 1-p, \text{ as } A_i\left[h(\beta_i)\right]^{-C_i} + B_i = 0.$$

As such, for the given random variable, the following joint deterministic constraints can be obtained from the conversion of the joint chance constraints in System (2):

$$\prod_{i=1}^{m}\left\{A_i\left[h(y_i)\right]^{-C_i} + B_i\right\} \leq 1-p. \tag{9}$$

The deterministic constraints may obtain the information from the distributions listed below when the latter follow the parameters $\left[A_i, B_i, C_i, h(b_i)\right]$:

Power Function distribution $\left[\lambda_i^{-a_i}, 0, -a_i\delta_i^{-1}, b_i^{\delta_i}\right]$, Pareto distribution $\left[-\lambda_i^{a_i}, 1, a_i\delta_i^{-1}, b_i^{\delta_i}\right]$, Beta distribution of first kind $\left[-1, 1, -a_i\delta_i^{-1}, (1-b_i)^{\delta_i}\right]$, Weibull distribution $\left[-1, 1, \theta_i\delta_i^{-1}, e^{\delta_i b_i^{a_i}}\right]$, Inverse Weibull distribution $\left[1, 0, \theta_i\delta_i^{-1}, e^{\delta_i b_i^{-a_i}}\right]$, Burr Type II distribution $\left[1, 0, k_i\delta_i^{-1}, \left(1+e^{-b_i}\right)^{\delta_i}\right]$, Burr Type III distribution $\left[1, 0, k_i\delta_i^{-1}, \left(1+b_i^{-\lambda_i}\right)^{\delta_i}\right]$, Burr Type IV distribution $\left[1, 0, k_i\delta_i^{-1}, \left\{1+\left\{b_i^{-1}(\lambda_i - b_i)\right\}^{1/\lambda_i}\right\}^{\delta_i}\right]$, Burr Type V distribution $\left[1, 0, k_i\delta_i^{-1}, \left(1+\lambda_i e^{-\tan b_i}\right)^{\delta_i}\right]$, Burr Type VI distribution $\left[1, 0, k_i\delta_i^{-1}, \left(1+\lambda_i e^{-k_i \sin b_i}\right)^{\delta_i}\right]$, Burr Type VII distribution $\left[2^{-k_i}, 0, -k_i\delta_i^{-1}, (1+\tanh b_i)^{\delta_i}\right]$, Burr Type VIII distribution $\left[\left(2\pi^{-1}\right)^{k_i}, 0, -k_i\delta_i^{-1}, \left(\tan^{-1} e^{b_i}\right)^{\delta_i}\right]$, Burr Type IX distribution $\left[-2, 1, \delta_i^{-1}, \left\{\lambda_i\left(1+e^{b_i}\right)^{k_i} - \lambda_i + 2\right\}^{\delta_i}\right]$, Burr Type X distribution $\left[1, 0, -k_i\delta_i^{-1}, \left(1-e^{-b_i^2}\right)^{\delta_i}\right]$, Burr Type XI distribution $\left[1, 0, -k_i\delta_i^{-1}, \left\{b_i - (2\pi)^{-1}\sin 2\pi b_i\right\}^{\delta_i}\right]$, Burr Type XII distribution $\left[-1, 1, m_i\delta_i^{-1}, \left(1+\theta_i b_i^{a_i}\right)^{\delta_i}\right]$, and Cauchy distribution $\left[\pi^{-1}, 0.5, -\delta_i^{-1}, \left(\tan^{-1} b_i\right)^{\delta_i}\right]$.

Hence, in this case, the deterministic form of the chance-constrained programming problem for *n*-stage series with *m*-chance constraints is given by:

$$Max\ R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right]$$

$$\text{subject to} \tag{10}$$

$$\prod_{i=1}^{m} A_i\left[h(y_i)\right]^{-C_i} + B_i \le 1-p,$$

$$x_j \ge 1,\ \ j=1,2,\ldots,n.$$

By introducing the particular value of the parameters $\left[A_i, B_i, C_i, h(b_i)\right]$ in the above deterministic constraints, we then obtain different deterministic constraints for different distributions.

Case 3: In System (1), let b_i follow a general form of distributions $F(b_i) = 1 - B_i e^{-A_i h(b_i)}$.

In this case, it is assumed that that the ith random variable b_i has two known parameters A_i, B_i. A_i and B_i are such that $A_i\ (\neq 0)$ and $B_i\ (\neq 0)$ and $F(\alpha_i) = 0, F(\beta_i) = 1$. Moreover, in the interval $[\alpha_i, \beta_i]$, $h(b_i)$ is a monotonic, continuous, and differentiable function of b_i. The *pdf* of the random variable b_i is then defined by:

$$f(b_i) = A_i B_i e^{-A_i h(b_i)} h'(b_i). \tag{11}$$

Now, the below represent the joint chance constraints for the above *pdf*:

$$\prod_{i=1}^{m}\left(\int_{y_i}^{\beta_i} A_i B_i e^{-A_i h(b_i)} h'(b_i)\, db_i\right) \ge p. \tag{12}$$

The following is obtained after integration:

$$\prod_{i=1}^{m} B_i e^{-A_i h(y_i)} \ge p,\ \text{ as } \left[A_i h(\beta_i) + B_i\right]^{C_i} = 0.$$

The below joint deterministic constraints are then derived from the conversion of the joint chance constraints in System (2) for the given random variable:

$$\prod_{i=1}^{m} B_i e^{-A_i h(y_i)} \ge p. \tag{13}$$

When the subsequent specified distributions follow the parameters $\left[A_i, B_i, h(b_i)\right]$, the deterministic constraints may contract the information from these distributions:

Exponential distribution $[\theta_i, 1, b_i]$, Rayleigh distribution $\left[\theta_i, 1, b_i^2\right]$, Weibull distribution $\left[\theta_i, 1, b_i^{a_i}\right]$, Pareto distribution $\left[a_i, \lambda_i^{a_i}, \ln(b_i)\right]$, Lomax distribution $\left[a_i, 1, \ln\left(1+b_i\lambda_i^{-1}\right)\right]$, Beta distribution of first kind $\left[-a_i, 1, \ln(\lambda_i - b_i)(\lambda_i - \delta_i)^{-1}\right]$, Beta distribution of second kind $\left[1, 1, \ln(1+b_i)\right]$, Extreme Value I distribution $\left[1, 1, e^{b_i}\right]$, Log logistic distribution $\left[1, 1, \ln(1-b_i)\right]$, Burr Type IX distribution $\left[1, 1, \ln\left\{0.5\lambda_i\left(1+e^{b_i}\right)^{k_i} - 0.5\lambda_i + 1\right\}\right]$, and Burr Type XII distribution $\left[\lambda_i, 1, \ln\left(1+\theta_i b_i^{a_i}\right)\right]$.

In consequence, the chance-constrained programming problem for *n*-stage series with *m*-chance constraints can be found in its deterministic form, defined as follows:

$$Max\ R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right]$$

$$\text{subject to} \tag{14}$$

$$\prod_{i=1}^{m} B_i e^{-A_i h(y_i)} \ge p,$$

$$x_j \ge 1,\ \ j=1,2,\ldots,n.$$

Finally, different deterministic constraints for different distributions can be obtained after inserting the particular value of the parameters $\left[A_i, B_i, h(b_i)\right]$ in the above deterministic constraints.

4. NUMERICAL EXAMPLES

Example 1: (for case 1)

We have the following stochastic problem:

$$Max\ R_s(X) = \left(1-0.20^{x_1}\right)\left(1-0.10^{x_2}\right)\left(1-0.15^{x_3}\right)$$

$$\begin{aligned} &\textit{subject to} \\ &P\begin{pmatrix} 4.0x_1+2.0x_2+5.0x_3 \le b_1, 3.0x_1+2.0x_2+6.0x_3 \le b_2, \\ 6.0x_1+3.0x_2+2.0x_3 \le b_3 \end{pmatrix} \ge 0.90, \\ &x_j \ge 1,\ \ j=1,2,3. \end{aligned} \tag{15}$$

where b_1 follows a Weibull distribution with parameters $\theta = 1/15, a = 1/4$; b_2 follows a Beta distribution of first kind with parameters $\lambda = 10,\ a = 15,\ \delta = 5$; and b_3 follows a Power Function distribution with parameters $\lambda = 12,\ a = 10.$

The deterministic model of the above stochastic problem is as follows:

$$Max\ R_s(X) = \left(1-0.20^{x_1}\right)\left(1-0.25^{x_2}\right)\left(1-0.15^{x_3}\right)$$

$$\begin{aligned} &\textit{subject to} \\ &\left[e^{-y_1^{1/2}/20}\right]\left[\left(\frac{70-y_2}{10}\right)^{1/18}\right]\left[1-\left(\frac{y_3}{95}\right)^{85}\right] \ge 0.90, \\ &x_j \ge 1,\ \ j=1,2,3. \end{aligned} \tag{16}$$

In order to obtain the reliability of the system, the above 3-stage series with 3-chance constraints problem is solved using the LINGO software. The results show that the reliability of the system is R_s = 0.9998, at x_1 =6, x_2 =5, and x_3 =5.

Example 2: (for case 2)

This second example builds upon the following stochastic model:

$$Max\ R_s(X) = \left(1-0.10^{x_1}\right)\left(1-0.15^{x_2}\right)\left(1-0.05^{x_3}\right)$$

$$\begin{aligned} &\textit{subject to} \\ &P\begin{pmatrix} 4.0x_1+7.0x_2+8.0x_3 \le b_1, 5.0x_1+4.0x_2+7.0x_3 \le b_2, \\ 7.0x_1+3.0x_2+8.0x_3 \le b_3, 6.0x_1+4.0x_2+2.0x_3 \le b_4 \end{pmatrix} \ge 0.90, \\ &x_j \ge 1,\ \ j=1,2,3. \end{aligned} \tag{17}$$

where, in this case, b_1 follows a Pareto distribution with parameters $\lambda = 10,\ a = 5$; furthermore, b_2 follows a Weibull distribution with parameters $\theta = 1/20,\ a = 1/3$; b_3 follows a Burr type IV distribution with parameters $k = 6,\ \lambda = 10$; and b_4 follows a Burr type IX distribution with parameters $\lambda = 1/20,\ k = 1/3$.

In this case, the deterministic model of the above stochastic problem becomes as follows:

$$Max\ R_s(X) = \left(1-0.10^{x_1}\right)\left(1-0.15^{x_2}\right)\left(1-0.20^{x_3}\right)$$

$$\begin{aligned} &\textit{subject to} \\ &\left[1-\left(\frac{60}{y_1}\right)^{1/5}\right]\left[1-e^{-\left(y_2^{1/10}/20\right)}\right]\left[\frac{1}{1+\left\{\left(75/y_3\right)-1\right\}^{1/75}}\right]\left[107-\frac{2\left(1+e^{y_4}\right)^{1/15}}{55}\right] \le 0.05, \\ &x_j \ge 1,\ \ j=1,2,3. \end{aligned} \tag{18}$$

We now have a 3-stage series but with 4-chance constraints problem that we solve by means of employing the LINGO software and we obtain the reliability of the system, that is, R_s = 0.9983, at x_1 =3, x_2 =4, and x_3 =3.

Example 3: (for case 3)

For our final example, we have the following stochastic problem:

$$Max\ R_s(X) = \left(1-0.15^{x_1}\right)\left(1-0.05^{x_2}\right)\left(1-0.07^{x_3}\right)\left(1-0.10^{x_4}\right)\left(1-0.12^{x_5}\right)$$

subject to

$$P\begin{pmatrix} 3.0x_1+5.0x_2+4.0x_3+6.0x_4+2.0x_5 \le b_1, 2.0x_1+4.0x_2+4.0x_3+5.0x_4+6.0x_5 \le b_2, \\ 4.0x_1+3.0x_2+7.0x_3+2.0x_4+5.0x_5 \le b_3, 1.0x_1+2.0x_2+5.0x_3+3.0x_4+4.0x_5 \le b_4 \end{pmatrix} \ge 0.99, \tag{19}$$

$$x_j \ge 1,\ \ j=1,2,3,4,5.$$

where b_1 follows a Burr type XII distribution with parameters $\lambda = 1/15, \theta = 1/10, a = 2/9$; b_2 , on the other hand, follows a Beta distribution of first kind with parameters $\lambda = 15,\ a = 10,\ \delta = 7$; b_3 follows a Lomax distribution with parameters $\lambda = 6,\ \ a = 1/5$; and, finally, b_4 follows a Pareto distribution with parameters $\lambda = 9,\ \ a = 5$.

The transformation of the above stochastic problem leads to obtaining the following deterministic problem:

$$Max\ R_s(X) = \left(1-0.20^{x_1}\right)\left(1-0.25^{x_2}\right)\left(1-0.15^{x_3}\right)\left(1-0.10^{x_4}\right)\left(1-0.25^{x_5}\right)$$

subject to

$$\left[1\Big/\left(1+\frac{y_2^6}{12}\right)^{1/18}\right]\left[1\Big/\left(\frac{85-y_3}{10}\right)^{60}\right]\left[1\Big/\left(1+\frac{y_4}{40}\right)^{1/12}\right]\left[\left(\frac{65}{y_5}\right)^{1/15}\right] \ge 0.99, \tag{20}$$

$$x_j \ge 1,\ \ j=1,2,3,4,5.$$

The LINGO software is used once again to solve the above 5-stage series with 4-chance constraints problem and the reliability of the system is obtained, that is, R_s = 0.9979, at x_1 =5, x_2 =3, x_3 =7, x_4 =4, and x_5 =3.

5. CONCLUSION

In this paper, we formulate the chance-constrained reliability stochastic optimization problem for optimal solution to an n-stage series system with m-joint chance constraints in which only resource variables are random in nature. Various cases have been discussed with different general form of distributions when resource variables are random in nature and have different general form of distributions. After formulating the problem, we solved it using the LINGO software. One of the limitations of the study is that the current approach to tackle the problem assumes that only the right-hand side of the constraints is random in nature; simultaneously studying the case in which one can introduce randomness on the left- hand side of the joint chance constraints, as well as in the objective function, separately or combined, which is used to measure system performances, such as mean system-life time, α -system lifetime, and system reliability; many real life management problems actually do have multiple objectives, i.e., minimizing the cost, maximizing the performance, maximizing the reliability, and so on, subject to satisfying several requirements. Taking the lead from this, and in line with Charles and Udhayakumar (2012) and Charles, Udhayakumar, and Rhymend Uthariaraj (2010), the present work may be extended to multi-objective reliability optimization problems with constraints having the finite probability being violated, as well as it may be extended to solve the proposed systems using hybrid algorithms.

ACKNOWLEDGEMENTS

The authors are grateful to Prof. M. M. Khalid, Department of Statistics and Operations Research, A.M.U., Aligarh, India, for his valuable help and to the Editor-in-Chief and the anonymous reviewers for their useful insights and comments on the previous drafts of this research paper.

REFERENCES

1. Aggarwal, K. K., Gupta, J. S., & Misra, K. B. (1975). A new heuristic criterion for solving a redundancy optimization problem. *IEEE Transactions on Reliability*, *24*(1), 86–87.
2. Aggarwal, K. K. (1976). Redundancy optimization in general system. *IEEE Transactions on Reliability*, *25*(5), 330–332.
3. Aggarwal, K. K., & Gupta, J. S. (2005). Penalty function approach in heuristic algorithms for constrained redundancy reliability optimization. *IEEE Transactions on Reliability*, *54*(3), 549–558.
4. Ansari, S. I. (2011). Study of some developments in stochastic programming and their applications (*Doctoral dissertation*). Retrieved from http://hdl.handle.net/10603/28650
5. Bellman, R., & Dreyfus, S. (1958). Dynamic programming and the reliability of multi component devices. *Operations Research*, *6*(2), 200–206.
6. Beraldi, P., & Bruni, M. (2010). An exact approach for solving integer problem under probabilistic constraints with random technological matrix. *Annals of Operations Research*, *177*(1), 127–137.

7. Birge, J. R., & Louveaux, F. V. (2011). *Introduction to Stochastic Programming* (2nd Edition). New York, NY: Springer
8. Birnbaum, Z. W., Esary, J. D., & Saunders, S. C. (1961). Multi-component systems and structures and their reliability. *Technometrics*, *3*(1), 55–77.
9. Bulfin, R. L., & Liu, C. Y. (1985). Optimal allocation of redundant components for large systems. *IEEE Transactions on Reliability*, *34*(3), 241–247.
10. Chan, K.-Y., & Lin, D.-S. (2011). Algorithm developments for optimization problem with joint reliability constraints. *International Journal for Numerical Methods in Engineering*, *85*(6), 768–783.
11. Charles, V., & Dutta, D. (2005). *Linear stochastic fractional programming with sum-of probabilistic fractional objectives*. Retrieved from http://www.optimization-online.org/DB_FILE/2005/06/1142.pdf
12. Charles, V., & Udhayakumar, A. (2012). Genetic algorithm for chance constrained reliability stochastic optimization problems. *International Journal of Operational Research*, *14*(4), 417–432.
13. Charles, V., Udhayakumar, A., & Rhymend Uthariaraj, V. (2010). Stochastic simulation-based genetic algorithm for chance constrained fractional programming problem. *International Journal of Operational Research*, *9*(1), 23–38.
14. Charnes, A., & Cooper, W. W. (1959). Chance constrained programming. *Management Science*, *6*(1), 73–79.
15. Charnes, A., & Cooper, W. W. (1963). Deterministic equivalents for optimizing and satisficing under chance constraints. *Operations Research*, *11*(1), 18–39.
16. Charnes, A., Cooper, W. W., & Thompson, G. L. (1964). Critical path analyses via chance constrained and stochastic programming. *Operations Research*, *12*(3), 460–470.
17. Charnes, A., Cooper, W. W., & Thompson, G. L. (1965). Constrained generalized medians and hypermedians as deterministic equivalents for two stage linear programs under uncertainty. *Management Science*, *12*(1), 83–112.
18. Charnes, A., Kirby, M. J. L., & Raike, W. M. (1967). Solution theorems in probabilistic programming: a linear programming approach. *Journal of Mathematical Analysis and Applications*, *20*(3), 565–582.
19. Cheng, J., & Lisser, A. (2012). A second order cone programming approach for linear programming with joint constraints. *Operations Research Letters*, *40*(5), 325–328.
20. Cheng, J., & Lisser, A. (2013). A completely positive representation of 0-1 linear programs with joint constraints. *Operations Research Letters*, *41*(6), 597–601.
21. Chern, M. S. (1992). On the computational complexity of reliability redundancy allocation in a series system. *Operations Research Letters*, *11*(5), 309–315.
22. Coelho, L. D. S. (2009). An efficient particle swarm approach for mixed-integer programming in reliability-redundancy optimization applications. *Reliability Engineering and System Safety*, *94*(4), 830–837.
23. Coit, D. W., & Smith, A. E. (1996). Reliability optimization of series-parallel systems using a genetic algorithm. *IEEE Transactions on Reliability*, *45*(2), 254–266.
24. Coit, D. W., Jin, T., & Wattanapongsakorn, N. (2004). System optimization with component reliability estimation uncertainty: A multi-criteria approach. *IEEE Transactions on Reliability*, *53*(3), 369–380.
25. Cui, L., Kuo, W., Loh, H. T., & Xie, M. (2004). Optimal allocation of minimal & perfect repairs under resource constraints. *IEEE Transactions on Reliability*, *53*(2), 193–199.
26. Dantzig, G. B. (1955). Linear programming under uncertainty. *Management Science*, *1*(3- 4), 197-206.
27. Flam, S. D., & Schult, R. (1993). A new approach to stochastic linear programming. *Numerical Functional Analysis and Optimization*, *14*(5-6), 545–554.
28. Fyffe, D. E., Hines, W. W., & Lee, N. K. (1968). System reliability allocation and a computational algorithm. *IEEE Transactions on Reliability*, *17*(2), 64–69.
29. Gen, M., & Cheng, R. (1997). *Genetic Algorithms and Engineering Design*. New York, NY: Wiley
30. Gen, M., & Yun, Y. S. (2006). Soft computing approach for reliability optimization: State-of-the-art survey. *Reliability Engineering & System Safety*, *91*(9), 1008–1026.
31. Glover, F., & Laguna, M. (1993). Tabu Search. In *Modern Heuristic Techniques for Combinatorial Problems* (Ed. C. Reeves). Blackwell Scientific Publications, Oxford, pp. 70–150.
32. Ha, C., & Kuo, W. (2006a). Reliability redundancy allocation: An improved realization for nonconvex nonlinear programming problems. *European Journal of Operational Research*, *171*(1), 24–38.
33. Ha, C., & Kuo, W. (2006b). Multi-path heuristic for redundancy allocation: the tree heuristic. *IEEE Transactions on Reliability*, *55*(1), 37–43.
34. Hwang, C. L., Tillman, F. A., & Lee, M. H. (1981). System reliability evaluation techniques for complex/large systems-a review. *IEEE Transactions on Reliability*, *30*(5), 416–423.
35. Ida, K., Gen, M., & Yokota, T. (1994). System reliability optimization of series-parallel systems using a genetic algorithm. *Proceedings of the 16th International Conference of Computers and Industrial Engineering*, pp. 349–352.
36. Jagannathan, R. (1974). Chance-constrained programming with joint constraints. *Operations Research*, *22*(2), 358–372.
37. Kall, P., & Wallace, S. W. (1994). *Stochastic Programming*. New York, NY: Wiley.
38. Kataoka, S. (1963). A stochastic programming model. *Econometrica*, *31*(1–2), 181–196.

39. Kucukyavuz, S. (2012). On mixing sets arising in chance constrained programming. *Mathematical Programming*, *132*(1–2), 31–56.
40. Kulturel-Konak, S., Smith, A. E., & Coit, D. W. (2003). Efficiently solving the redundancy allocation problem using Tabu search. *IIE Transactions*, *35*(6), 515–526.
41. Kuo, W., Hwang, C. L., & Tillman, F. A. (1978). A note on heuristic method for in optimal system reliability. *IEEE Transaction on Reliability*, *27*(5), 320–324.
42. Kuo, W., & Prasad, V. R. (2000). An annotated overview of system-reliability optimization. *IEEE Transactions on Reliability*, *49*(2), 176–187.
43. Kuo, W., Prasad, V. R., Tillman, F. A., & Hwang, C. L. (2001). *Optimal Reliability Design: Fundamentals and Application*. Cambridge, United Kingdom: Cambridge University Press.
44. Kuo, W., & Wan, R. (2007). Recent advances in optimal reliability allocation. *IEEE Transactions on System, Man, and Cybernetics, Part A: System and Humans*, *37*(2), 143–156.
45. Lee, S. M., & Olson, D. L. (1985). A gradient algorithm for chance constrained non-linear goal programming. *European Journal of operational Research*, *22*(3), 359–369.
46. Liang, Y.-C., & Smith, A. E. (2004). An ant colony optimization algorithm for the redundancy allocation problem (RAP). *IEEE Transactions on Reliability*, *53*(3), 417–423.
47. Marseguerra, M., Zio, E., Podofillini, L., & Coit, D. W. (2005). Optimal design of reliable network systems in presence of uncertainty. *IEEE Transactions on Reliability*, *54*(2), 243–253.
48. Miller, B. L., & Wagner, H. M. (1965). Chance-constrained programming with joint constraints. *Operations Research*, *13*(6), 930–945.
49. Minguez, R., Conejo, A. J., & Garcia-Bertrand, R. (2011). Reliability and decomposition techniques to solve certain class of stochastic programming problems. *Reliability Engineering & System Safety*, *92*(2), 314–323.
50. Misra, K. B. (1971). Dynamic programming formulation of the redundancy allocation problem. *International Journal of Mathematical Education in Science and Technology*, *2*(3), 207–215.
51. Misra, K. B., & Sharma, U. (1991). An efficient algorithm to solve integer-programming problems arising in system-reliability design. *IEEE Transactions on Reliability*, *40*(1), 81–91.
52. Nakagawa, Y., Nakashima, K., & Hattori, Y. (1978). Optimal reliability allocation by branch-and-bound techniques. *IEEE Transactions on Reliability*, *27*(1), 31–38.
53. Naslund, B. (1966). A model of capital budgeting under risk. *The Journal of Business*, *39*(2), 257–271.
54. Ng, K. Y. K., & Sancho, N. G. F. (2001). A hybrid 'dynamic programming/depth-first search' algorithm, with an application to redundancy allocation. *IIE Transactions*, *33*(12), 1047–1058.
55. Nikolaidis, E., & Mourelatos, Z. P. (2011). Imprecise probability assessment when the type of the probability distribution of random variable is unknown. *International Journal of Reliability and Safety*, *5*(2), 140–157.
56. Onishi, J., & Kimura, S. (2007). Solving the redundancy allocation problem with a mix of components using the improved surrogate constraint method. *IEEE Transactions on Reliability*, *56*(1), 94–101.
57. Olson, D. L., & Swenseth, S. R. (1987). A linear approximation for chance-constrained programming. *Journal of the Operational Research Society*, *38*(3), 261–267.
58. Painton, L., & Campbell, J. (1995). Genetic algorithms in optimization of system reliability. *IEEE Transactions on Reliability*, *44*(2), 172–178.
59. Papadimitriou, C., Katafygiotis, L. S., & Siu, K. A. (1997). Effects of structural uncertainties on TMD design: a reliability-based approach. *Journal of Structural Control*, *4*(1), 65–88.
60. Papadimitriou, C., & Ntotsios, E. (2004). Robust reliability-based optimization in structural dynamics using evolutionary algorithms. *9th ASCE Joint Specialty Conference on Probabilistic Mechanics and Structural Reliability*, Albuquerque, New Mexico.
61. Prasad, V. R., & Kuo, W. (2000). Reliability optimization of coherent systems. *IEEE Transactions on Reliability*, *49*(3), 323–330.
62. Prékopa, A. (1995). *Stochastic Programming*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
63. Ramirez-Marquez, J. E., Coit, D. W., & Konak, A. (2004). Redundancy allocation for series-parallel systems using a max-min approach. *IIE Transactions*, *36*(9), 891–898.
64. Ramirez-Marquez, J.E., & Coit, D.W. (2007). Optimization of system reliability in the presence of Common cause failures. *Reliability Engineering and System Safety*, *92*(10), 1421–1434.
65. Rao, S. S. (1989). *Optimization: Theory and Applications* (4th Edition). New York, NY: Wiley Eastern Limited,
66. Ravi, V., Murty, B. S. N., & Reddy, P. J. (1997). Nonequilibrium simulated-annealing algorithm applied to reliability optimization of complex systems. *IEEE Transactions on Reliability*, *46*(2), 233–239.
67. Ravi, V., Reddy, J. P., & Zimmermann, H.-J. (2000). Fuzzy global optimization of complex system reliability. *IEEE Transactions on Fuzzy Systems*, *8*(3), 241–248.
68. Sakalli, U. S. (2014). A simulated annealing approach for reliability-based chance constrained programming. *Applied*

Stochastic Models in Business and Industry, *30*(4), 497–508.
69. Schoen, F. (1994). On a new stochastic global optimization algorithm based on censored observation. *Journal of Global Optimization*, *4*(1), 17–35.
70. Sengupta, J. K., & Fox, K. A. (1969). *Economic Analysis and Operation Research Optimization Techniques in Quantitative Economic Models*. Amsterdam, The Netherlands: North-Holland.
71. Seppala, Y. (1988). On accurate approximations for chance constrained programming. *Journal of the Operational Research Society*, *39*(7), 693–694.
72. Shapiro, A. (1990). On differential stability in stochastic programming. *Mathematical Programming*, *47*(1–3), 107–116.
73. Sniedovich, M., & Vazirinejad, S. (1990). A solution strategy for a class of nonlinear knapsack problems. *American Journal of Mathematical and Management Sciences*, *10*(1–2), 51–71.
74. Stancu-Minasian, I. M., & Wets, M. J. (1976). A research bibliography in stochastic programming(1955–1975). *Operations Research*, *24*(6), 1078–1119.
75. Sung, C. S., & Cho, Y. K. (1999). Branch and bound redundancy optimization for a series system with multiple-choice constraints. *IEEE Transactions on Reliability*, *48*(2), 108–117.
76. Sun, X. L., Mckinnon, K. I. M., & Li, D. (2001). A convexification method for a class of global optimization problems with application to reliability optimization. *Journal of Global Optimization*, *21*(2), 185–199.
77. Sun, X., & Li, D. (2002). Optimality condition and branch and bound algorithm for constrained redundancy optimization in series system. *Optimization and Engineering*, *3*(1), 53–65.
78. Symonds, G. H. (1967). Deterministic solutions for a class of chance-constrained programming problems. *Operations Research*, *15*(3), 495–512.
79. Tillman, F. A., Hwang, C. L., & Kuo, W. (1977a). Optimization techniques for system reliability with redundancy–a review. *IEEE Transactions on Reliability*, *26*(3), 148–155.
80. Tillman, F. A., Hwang, C. L., & Kuo, W. (1977b). Determining component reliability and redundancy for optimum system reliability. *IEEE Transactions on Reliability*, *26*(3), 162–165.
81. Tillman, F. A., Hwang, C. L., & Kuo, W. (1980). *Optimization of System Reliability*. New York, NY: Marcel Dekker Inc.
82. Tintner, G., & Sengupta, J. K. (1972). *Stochastic Economics: Stochastic Processes, Control and Programming*. New York, NY: Academic Press.
83. Vajda, S. (1972). *Probabilistic Programming*. New York, NY: Academic Press.
84. Van De Panne, C., & Popp, W. (1963). Minimum-cost cattle feed under probabilistic protein constraints. *Management Science*, *9*(3), 405–430.
85. Weintraub, A., & Vera, J. (1991). A cutting plane approach for chance constrained linear programs. *Operations Research*, *39*(5), 776–785.
86. Wets, R. J.-B. (1966). Programming under uncertainty: the solution set. *SIAM Journal on Applied Mathematics*, *14*(5), 1143–1151.
87. Williams, A. C. (1965). On stochastic linear programming. *Journal of the Society for Industrial and Applied Mathematics*, *13*(4), 927–940.
88. Williams, A. C. (1966). Approximation formulas for stochastic linear programming. *SIAM Journal on Applied Mathematics*, *14*(4), 668–677.
89. Yadavalli, V. S. S., Malada, A., & Charles, V. (2007). Reliability stochastic optimization for an n-stage series system with m-chance constraints. *South African Journal of Sciences*, *103*(11-12), 502–504.
90. Zafiropoulos, E. P., & Dialynas, E. N. (2004). Reliability and cost optimization of electronic devices considering the component failure rate uncertainty. *Reliability Engineering & System Safety*, *84*(3), 271–284.
91. Zhao, Y., & Ziemba, W. T. (2001). A stochastic programming model using an endogenously determined worst case risk measure for dynamic asset allocation. *Mathematical Programming*, *89*(2), 293–309.
92. Zhao, R., & Liu, B. (2003). Stochastic Programming models for general redundancy optimization problems. *IEEE Transactions on Reliability*, *52*(2), 181–191.
93. Zhao, X., Cui, L., & Kuo, W. (2007). Reliability for sparsely connected consecutive-k systems. *IEEE Transactions on Reliability*, *56*(3), 516–524.
94. Ziemba, W.T. (1970). Computational algorithms for convex stochastic programs with simple recourse. *Operations Research*, *18*(3), 414–431.

APPENDIX

***Proposition 1*:** System (A_1) is a complement of System (6).

In System (1), if b_i follows a general form of distributions $F(b_i) = \left[A_i h(b_i) + B_i\right]^{C_i}$, then with the similar argument in Case 1, we shall obtain the following System (A_1):

$$Max\ R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right]$$
$$subject\ to$$
$$\prod_{i=1}^{m}\left[A_i h(y_i) + B_i\right]^{C_i} \le 1-p, \tag{A1}$$
$$x_j \ge 1,\ \ j = 1,2,\ldots,n.$$

After inserting the particular value of the parameters $\left[A_i, B_i, C_i, h(b_i)\right]$ in the above deterministic constraints, we get different deterministic constraints for the various distributions listed below: Power Function distribution $\left[\lambda_i^{-\delta_i^{-1}}, 0, a_i\delta_i^{-1}, b_i^{\delta_i}\right]$, Pareto distribution $\left[-\lambda_i^{a_i}, 1, 1, b_i^{-a_i}\right]$, Beta distribution of first kind $\left[-1,1,1,\left(1-b_i\right)^{a_i}\right]$, Weibull distribution $\left[-1,1,1,\exp\left(-\theta_i b_i^{a_i}\right)\right]$, Inverse Weibull distribution $\left[1,0,1,e^{-\theta_i b_i^{-a_i}}\right]$, Burr Type II distribution $\left[1,1,-k_i,e^{-b_i}\right]$, Burr Type III distribution $\left[1,1,-k_i,b_i^{-\lambda_i}\right]$, Burr Type IV distribution ,$\left[1,1,-k_i,\left\{b_i^{-1}\left(\lambda_i-b_i\right)\right\}^{1/\lambda_i}\right]$, Burr Type V distribution ,$\left[\lambda_i,1,-k_i,e^{-\tan b_i}\right]$, Burr Type VI distribution $\left[\lambda_i,1,-k_i,e^{-k_i\sinh b_i}\right]$, Burr Type VII distribution $\left[0.5,0.5,k_i,\tanh b_i\right]$, Burr Type VIII distribution $\left[\left(2\pi^{-1}\right)^{\delta_i},0,\ k_i\delta_i^{-1},\left(\tan^{-1}e^{-b_i}\right)^{\delta_i}\right]$, Burr Type IX distribution $\left[-2,1,1,\left\{\lambda_i\left(1+e^{b_i}\right)^{k_i}-\lambda_i+2\right\}^{-1}\right]$, Burr Type X distribution $\left[-1,1,k_i,e^{-b_i^2}\right]$, Burr Type XI distribution $\left[1,0,k_i\delta_i^{-1},\left(b_i-\left(2\pi\right)^{-1}\sin 2\pi b_i\right)^{\delta_i}\right]$, Burr Type XII distribution $\left[-1,1,1,\left(1+\theta_i b_i^{a_i}\right)^{-m_i}\right]$, and Cauchy distribution $\left[\pi^{-1},0.5,1,\tan^{-1}b_i\right]$.

Proposition 2: System (A2) is a complement of System (10).

In System (1), if b_i follows a general form of distributions $F(b_i) = B_i e^{-A_i h(b_i)}$, then with the similar argument in Case 2, we shall obtain the following System (A2):

$$Max\ R_s(X) = \prod_{j=1}^{n}\left[1-\left(1-r_j\right)^{x_j}\right]$$
$$subject\ to$$
$$\prod_{i=1}^{m} B_i e^{-A_i h(y_i)} \le 1-p, \tag{A2}$$
$$x_j \ge 1,\ \ j = 1,2,\ldots,n.$$

After inserting the particular value of the parameters $\left[A_i, B_i, C_i, h(b_i)\right]$ in the above deterministic constraints, we get different deterministic constraints for the various distributions listed below: Inverse Weibull distribution $\left[\theta_i, 1, b_i^{a_i}\right]$, Power Function distribution $\left[-a_i, \lambda_i^{a_i}, \ln(b_i)\right]$, Logistic distribution $\left[1,1,\ln\left(1+e^{-b_i}\right)\right]$, Burr Type II distribution $\left[\theta_i,1,\ln\left(1+e^{-b_i}\right)\right]$, Burr Type III distribution $\left[k_i,1,\ln\left(1+b_i^{-\lambda_i}\right)\right]$, Burr Type IV distribution $\left[k_i,1,\ln\left\{1+\left\{b_i^{-1}\left(\lambda_i-b_i\right)\right\}^{1/\lambda_i}\right\}\right]$, Burr Type V distribution $\left[k_i,1,\ln\left(1+\lambda_i e^{-\tan b_i}\right)\right]$, Burr Type VI distribution $\left[k_i,1,\ln\left(1+\lambda_i e^{-k_i\sinh b_i}\right)\right]$, Burr Type VII distribution $\left[-k_i,1,\ln\left(0.5+0.5\tanh b_i\right)\right]$, Burr Type VIII distribution $\left[-k_i,1,\ln\left(2\pi^{-1}\tan^{-1}e^{b_i}\right)\right]$, Burr Type X distribution $\left[-k_i,1,\ln\left(1-e^{-b_i^2}\right)\right]$, Burr Type XI distribution $\left[-k_i,1,\ln\left\{b_i-\left(2\pi\right)^{-1}\sin 2\pi b_i\right\}\right]$, Gumble distribution $\left[1,1,e^{-b_i}\right]$, and Extreme Value II distribution $\left[\theta_i^{a_i},1,b_i^{-a_i}\right]$.