International Journal of
Operations Research

International Journal of Operations Research Vol. 12, No. 3, 069–090 (2015)

Analysis of a bulk queueing system with server breakdown and vacation interruption

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Received September 2015; Revised September 2015; Accepted October 2015

Abstract **—** In this paper, the server breakdown with interrupted vacation in a $MX/G(a,b)/1$ queuing system is considered. After completing a batch of service, if the server is breakdown with probability π, then the renovation of service station will be considered. After completing the renovation of service station or there is no breakdown of the server with probability $(1-\pi)$, if the server finds at least 'a' customers waiting for service say ξ, then the server serves a batch of min (ξ, b) customers, where $b \geq a$. On the other hand, if the queue length is less than 'a', the server leaves for a secondary job (vacation) of random length. During the secondary job period, the secondary job is interrupted abruptly and the server resumes for primary service, if the queue size reaches 'a'. On completion of the secondary job, the server remains in the system (dormant period) until the queue length reaches 'a'. For the proposed model, the probability generating function of the steady state queue size distribution at an arbitrary time is obtained. Various performance measures are obtained. Numerical illustration is also given to justify the proposed model. The cost model is also developed.
Keywords — Bulk arrival, single server, batch service, vacation, breakdown.
2010 Mathematics Subject Classification: 60K25, 90B22, 68M20

1. INTRODUCTION

The roles of quality and service performance are the crucial aspects in customer perceptions, and firms must dedicate special attention to them when designing and implementing their operations. For this reason, the vacation interruption queue has received considerable attention in the recent literature. In the server vacation model, the server wishes to perform some useful internal processes during his idle time.

The roles of quality and service performance are the crucial aspects in customer perceptions, and firms must dedicate special attention to them when designing and implementing their operations. For this reason, the vacation interruption queue has received considerable attention in the recent literature. In the server vacation model, the server wishes to perform some useful internal processes during his idle time.

The motivation of this paper comes from a real life situation that exists in an industry involving the process of dyeing. The soft flow dyeing machine is used for dyeing the cloth using chemicals with 1080 kg capacity in dyeing industries. In this machine, the cloth undergoes wetting process first, then dyeing, finally washing using suitable chemicals. The dyeing machine is operated to the maximum capacity of 1080 kg which is its upper limit (b) and a minimum capacity of 800 kg which is its lower limit (a). If the supply of cloth is less than 800 kg, dyeing process is not possible. During the dyeing process (*service*), sometimes the transfer pump mechanical seal becomes worn out resulting is breakdown. Due to this, a part of water containing the chemicals leaks. It is not possible to renovate the mechanical seal within a limited time, and if the operation is stopped, heavy loss occurs. To avoid such a heavy loss, it should be operated continuously till a batch of cloth (*bulk service*) is processed fully. This could be achieved only by sending in additional amount of water containing chemicals into the machine. Thus, this additional water compensates the loss of water by leakage. Here, the only loss is some amount of water and chemicals. On completion of the dyeing process, the machine is stopped and the mechanical seal is replaced (renovation) for the next session.

Apart from dyeing, wetting and washing are the other works which can be considered as secondary jobs (*vacation*). Upon completion of the dyeing process, if the supply of cloth is less than 800kg it stops the process and performs some other work (secondary job/vacation) like wetting process, washing using suitable chemicals, etc. During this secondary job, if the required number of cloth reaches the threshold value 800 kg, then the operator returns from his secondary job

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(*vacation interruption*) and starts the dyeing process. On the other hand, after completing his secondary job, if the required number of cloths is less than the threshold value 800 kg, then the operator waits in the system till the required number of cloth arrives. *This dyeing machine can be modelled as bulk arrival batch service queueing model with breakdown and vacation interruption.*

There is an extensive literature on the $M/G/1$ queue which has been studied in various forms by numerous researchers. Borthakur and Medhi (1974) have studied a queueing system with arrival and services in batches of variable size. They have derived the queue length distribution for the bulk arrival batch service queueing system without vacation concepts. General single server vacation models have been well studied and surveyed by Doshi (1986) Tian and Zhang (2006). Detailed analysis of some bulk queueing models can be seen in the studies of Chaudhry and Templeton (1983) and Medhi (1984). Krishna Reddy et al (1998) have discussed a $M^X/G(a,b)/1$ model with N-policy, multiple vacations, and setup times. Jau-Chuan Ke and Kuo-Hsiung Wang (2006) studied a general input queue with N Policy and service rate depending on bulk size. They used an embedded Markov chain to analyze a $G/M/1$ queuing system with N-policy. A Multi-server markovian queueing system with discouraged arrivals and retention of reneged customers was studied by Rakesh Kumar and Sumeet Sharma (2012).

Arumuganathan and Jeyakumar (2005) analyzed a bulk queue with multiple vacations, setup times with N-policy and closedown times. Haridass and Arumuganathan (2011) discussed a batch arrival general bulk service queueing system with variant threshold policy for secondary jobs. Jau-Chuan Ke et al. (2010) presented a short survey on recent developments in vacation queueing models. *In all the aforesaid models with vacations, the server cannot come back (vacation interruption) to the normal working level (regular busy period), until the vacation period ends.*

For vacation interruption models, Ji-Hong Li and Nai-Shuo Tian (2007) analyzed the $M/M/1$ queue with working vacations and vacation interruptions. Zhang and Shi (2009) provided a study on the $M/M/1$ queue with Bernoulli-Schedule-Controlled vacation and vacation interruption. Mian Zhang and Zhenting Hou (2010) discussed performance analysis of $M/G/1$ queue with working vacations and vacation interruption. Haridass and Arumuganathan (2012) analyzed a bulk arrival batch service queueing system with vacation interruption. Haridass and Arumuganathan (2012) analyzed optimum cost of a Bulk Queueing system with Multiple Vacations and Restricted Admissibility of Arriving Batches. Suganya (2014) studied a single server bulk service queuing system with interrupted vacation, request for re-service and balking. *In all the aforesaid interrupted vacation models, the authors considered models without breakdown.*

Only few authors have analyzed about queue with server breakdowns. The reliability analysis of $M/G/1$ queueing system with server breakdowns and vacations were studied by Li et al (1997), Jain and Agrawal (2009) have analysed the optimal policy for bulk queue with multiple types of server breakdown. Jeyakumar and Senthilnathan (2012) have studied on the behaviour of the server breakdown without interruption in a $M^X/G(a,b)/1$ queueing system with multiple vacations and close down time. Madan et al (2003) studied steady state analysis of two $M^X/M(a,b)/1$ queue models with random breakdowns. Zhe George et al. (2011) focused on analysis of queues with an imperfectly repairable server. They considered a queueing system where the server is subjected to failures and can be repaired within a random period of time. Gautam Choudhury and Mitali Deka (2013) analyzed a batch arrival unreliable queue with two phases of service and vacation under Bernoulli vacation schedule, which consists of a breakdown period and a delay period. Madhu Jain and Anamika Jain (2014) investigated a batch arrival priority queueing model with second optional service and server breakdown. Jain and Charu Bhargava (2008) presented a bulk arrival retrial queue with unreliable server and priority subscribers. *In all the aforesaid models with breakdown, the server cannot come back (vacation interruption) to the normal working level (regular busy period), until the vacation period ends. This stimulates the authors to develop a single server bulk arrival bulk service queueing system with breakdown and vacation interruption.*

Queueing systems with bulk arrival and bulk service are common as well as essential in many practical situations for an effective and efficient utilization of resources. In the server vacation model, the server wishes to perform some useful internal processes during his idle time. The identification of the breakdown status of the service station, repair of the service station (fault status) and proper maintenance of other resources are called as renovation works of the server. Server is not interrupted for renovation before completing a batch of service, even if breakdown occurs. After finishing a service, the system either requires repair with probability (π) or does not require repair with probability $(1-\pi)$. At the end of the repair time or at the end of a service time when no repair is required, if the server finds the queue length is less than 'a', server leaves for a vacation of random length.

In the literature, all vacation models with the bulk service considered that the server can start the service only when he completes the secondary job. But in emergency, the server has to terminate the secondary job and must give priority for primary job. Once the required level is reached to start the primary service, there is no point in continuing the secondary job. This model is proposed to overcome this difficulty and to make the system operate more efficiently.

2. MODEL DESCRIPTIONS

In this section, the mathematical model for the server breakdown with interrupted vacation in a $M^X/G\,(a,\,b)/1$ queueing system is considered. After completing a batch of service, if the server is breakdown with probability π, then the renovation of service station will be considered. After completing the renovation of service station or there is no breakdown of the server with probability $(1-\pi)$, if the server finds at least 'a' customers waiting for service say ξ, then the server serves a batch of min $(\xi,\ \mathrm{b})$ customers, where $\mathrm{b} \geq \mathrm{a}$. On the other hand, if the queue length is less than 'a', the server leaves for a secondary job (vacation) of random length. It is assumed that the secondary job is interrupted abruptly and the server resumes for primary service, if the queue size reaches 'a' during the secondary job period. On completion of the secondary job, the server remains in the system (dormant period) until the queue length reaches 'a'. For the proposed model, the probability generating function of the steady state queue size distribution at an arbitrary time is obtained. Various performance measures are derived. The above system is modelled using the supplementary variable technique, by considering remaining service time of the batch in service, remaining vacation time of the server and remaining renovation time of the server as supplementary variables at an arbitrary time.

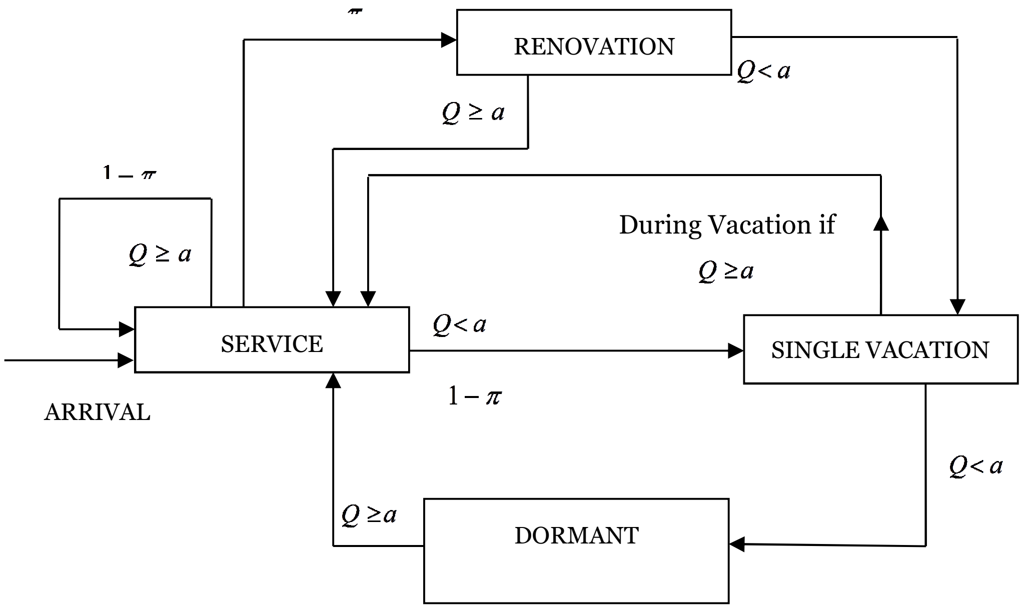

Figure 1: Schematic representation; Q - Queue length, a - minimum capacity, π - breakdown probability

2.1 Notations

Let X be the group size random variable of the arrival, λ be the Poisson arrival rate. g_k be the probability that 'k' customers arrive in a batch and $X(z)$ be its probability generating function (PGF). Let $S(x)(s(x))\{\tilde{S}(\theta)\}\left[S^0(x)\right]$ be the cumulative distribution function (probability density function) { Laplace-Stieltjes transform} [remaining service time] of service. Let $V(x)(v(x))\{\tilde{V}(\theta)\}\left[V^0(x)\right]$ be the cumulative distribution function (probability density function) { Laplace-Stieltjes transform}[remaining vacation time] of vacation. Let $R(x)(r(x))\left\{\tilde{R}(\theta)\right\}\left[R^0(x)\right]$ be the cumulative distribution function (probability density function) { Laplace-Stieltjes transform}[remaining renovation time] of renovation. $N_q(t)$ denotes the number of customers waiting for service at time t, $N_s(t)$ denotes the number of customers under the service at time t.

$$C(t) = \begin{cases} 0, & \text{when the server is busy with service} \\ 1, & \text{when the server is on vacation} \\ 2, & \text{when the server is on dormant period} \\ 3, & \text{when the server is on renovation period} \end{cases}$$

The state probabilities are defined as follows:

$$P_{ij}(x,t)dt = \Pr\left\{N_s(t)=i, N_q(t)=j,\ x \le S^0(t) \le x+dt, C(t)=0\right\},\ a \le i \le b,\ j \ge 0$$

$$Q_n(x,t)dt = \Pr\left\{N_q(t)=n,\ x \le V^0(t) \le x+dt, C(t)=1\right\},\ 0 \le n \le a-1$$

$$T_n(t) = \Pr\left\{N_q(t)=n,\ C(t)=2\right\},\ 0 \le n \le a-1$$

$$R_n(x,t)dt = \Pr\left\{N_q(t)=n,\ x \le R^0(t) \le x+dt, C(t)=3\right\},\ n \ge 0$$

3. STEADY- STATE ANALYSIS

In this section, the probability generating function (PGF) of the queue size at an arbitrary time epoch is derived. The PGF will be useful to derive the important performance measures.

3.1 Steady State Queue Size Distribution

From the above set of equations, the steady state queue size equations for the queueing model are obtained as follows:

$$0 = -\lambda T_0 + Q_0(0) \tag{1}$$

$$0 = -\lambda T_n + Q_n(0) + \sum_{k=1}^{n} T_{n-k}\lambda g_k,\ 1 \le n \le a-1 \tag{2}$$

$$-\frac{d}{dx}P_{i0}(x) = -\lambda P_{i0}(x) + (1-\pi)\sum_{m=a}^{b} P_{mi}(0)\,s(x) + \sum_{k=0}^{a-1} Q_k(x)\lambda g_{i-k}\,s(x) + \sum_{m=0}^{a-1} T_m\,\lambda g_{i-m}s(x) + R_i(0)s(x),\ a \le i \le b \tag{3}$$

$$-\frac{d}{dx}P_{ij}(x) = -\lambda P_{ij}(x) + \sum_{k=1}^{j} P_{i,j-k}(x)\lambda g_k,\ a \le i \le b-1,\ j \ge 1 \tag{4}$$

$$\begin{aligned}-\frac{d}{dx}P_{bj}(x) &= -\lambda P_{bj}(x) + (1-\pi)\sum_{m=a}^{b} P_{m,b+j}(0)\,s(x) + \sum_{k=1}^{j} P_{b,j-k}(x)\lambda g_k \\ &+ \sum_{m=0}^{a-1} T_m\,\lambda g_{b+j-m}s(x) + \sum_{k=0}^{a-1} Q_k(x)\lambda g_{b+j-k}s(x) + R_{b+j}(0)s(x),\ j \ge 1\end{aligned} \tag{5}$$

$$-\frac{d}{dx}Q_0(x) = -\lambda Q_0(x) + (1-\pi)\sum_{m=a}^{b} P_{m0}(0)\,v(x) + R_0(0)v(x) \tag{6}$$

$$-\frac{d}{dx}Q_n(x) = -\lambda Q_n(x) + (1-\pi)\sum_{m=a}^{b} P_{mn}(0)\,v(x) + \sum_{k=1}^{n} Q_{n-k}(x)\lambda g_k + R_n(0)v(x),\ 1 \le n \le a-1 \tag{7}$$

$$-\frac{d}{dx}R_0(x) = -\lambda R_0(x) + \pi\sum_{m=a}^{b} P_{m0}(0)\,r(x) \tag{8}$$

$$-\frac{d}{dx}R_n(x) = -\lambda R_n(x) + \pi\sum_{m=a}^{b} P_{mn}(0)\,r(x) + \sum_{k=1}^{n} R_{n-k}(x)\lambda g_k,\ n \ge 1 \tag{9}$$

The Laplace –Stieltjes transforms of $P_{in}(x)$, $Q_{jn}(x)$ and $R_n(x)$ are defined as

$$\tilde{P}_{in}(\theta) = \int_0^\infty e^{-\theta x}P_{in}(x)dx,\ \tilde{Q}_{jn}(\theta) = \int_0^\infty e^{-\theta x}Q_{jn}(x)dx \text{ and } \tilde{R}_n(\theta) = \int_0^\infty e^{-\theta x}R_n(x)dx$$

Taking Laplace-Stieltjes transform on both sides of the equation (3) through (9), we have

$$\lambda T_0 = Q_0(0) \tag{10}$$

$$0 = -\lambda T_n + Q_n(0) + \sum_{k=1}^{n} T_{n-k}\lambda g_k,\ 1 \le n \le a-1 \tag{11}$$

$$\theta\tilde{P}_{i0}(\theta)-\lambda P_{i0}(0)=\lambda\tilde{P}_{i0}(\theta)-(1-\pi)\sum_{m=a}^{b}P_{mi}(0)\tilde{S}(\theta)-\sum_{k=0}^{a-1}\tilde{Q}_k(\theta)\lambda g_{i-k}\tilde{S}(\theta)$$
$$-\sum_{m=0}^{a-1}T_m\lambda g_{i-m}\tilde{S}(\theta)-R_i(0)\tilde{S}(\theta),\ a\le i\le b \tag{12}$$

$$\theta\tilde{P}_{ij}(\theta)-P_{ij}(0)=\lambda\tilde{P}_{ij}(\theta)-\sum_{k=1}^{j}\tilde{P}_{i,j-k}(\theta)\lambda g_k\ ,\ \ a\le i\le b-1,\ \ j\ge 1 \tag{13}$$

$$\theta\tilde{P}_{bj}(\theta)-P_{bj}(0)=\lambda\tilde{P}_{bj}(\theta)-(1-\pi)\sum_{m=a}^{b}P_{m,b+j}(0)\tilde{S}(\theta)-\sum_{k=1}^{j}\tilde{P}_{b,j-k}(\theta)\lambda g_k$$
$$-\sum_{m=0}^{a\text{-}1}T_m\lambda g_{b+j-m}\tilde{S}(\theta)-\sum_{k=0}^{a\text{-}1}\tilde{Q}_k(\theta)\lambda g_{b+j-k}\tilde{S}(\theta)-R_{b+j}(0)\tilde{S}(\theta),\quad j\ge 1 \tag{14}$$

$$\theta\tilde{Q}_0(\theta)-Q_0(0)=\lambda\tilde{Q}_0(\theta)-(1-\pi)\sum_{m=a}^{b}P_{m0}(0)\tilde{V}(\theta)\ -R_0(0)\tilde{V}(\theta) \tag{15}$$

$$\theta\tilde{Q}_n(\theta)-Q_n(0)=\lambda\tilde{Q}_n(\theta)-(1-\pi)\sum_{m=a}^{b}P_{mn}(0)\tilde{V}(\theta)-\sum_{k=1}^{n}\tilde{Q}_{n-k}(\theta)\lambda g_k-R_n(0)\tilde{V}(\theta),1\le n\le a-1 \tag{16}$$

$$\theta\tilde{R}_0(\theta)-R_0(0)=\lambda\tilde{R}_0(\theta)-\pi\sum_{m=a}^{b}P_{m0}(0)\tilde{R}(\theta) \tag{17}$$

$$\theta\tilde{R}_n(\theta)-R_n(0)=\lambda\tilde{R}_n(\theta)-\pi\sum_{m=a}^{b}P_{m0}(0)\tilde{R}(\theta)\ -\sum_{k=1}^{n}\tilde{R}_{n-k}(\theta)\lambda g_k\quad n\ge 1 \tag{18}$$

4. PROBABILITY GENERATING FUNCTION

To obtain the probability generating function (PGF) of the queue size at an arbitrary time, the following probability generating functions are defined

$$\tilde{P}_i(z,\theta)=\sum_{j=0}^{\infty}\tilde{P}_{ij}(\theta)z^j,\ P_i(z,0)=\sum_{j=0}^{\infty}P_{ij}(0)z^j,a\le i\le b$$
$$\tilde{Q}(z,\theta)=\sum_{n=0}^{a-1}\tilde{Q}_n(\theta)z^n,\ Q(z,0)=\sum_{n=0}^{a-1}Q_n(0)z^n,\ T(z)=\sum_{n=0}^{a-1}T_n z^n \tag{19}$$
$$\tilde{R}(z,\theta)=\sum_{n=0}^{\infty}\tilde{R}_n(\theta)z^n,\ R(z,0)=\sum_{n=0}^{\infty}R_n(0)z^n$$

Theorem 1. *The probability generating function P(z) of the number of customers in the queue at an arbitrary time epoch of the proposed model is given by*

$$P(z)=\frac{\left(-\lambda+\lambda\xi(z)\right)\begin{pmatrix}\sum_{i=a}^{b-1}g(\tilde{S},\tilde{R},E1,E2,E3,z)-\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\sum_{n=0}^{b\text{-}1}d_nz^n+\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\lambda E4+h(\tilde{S},\tilde{R},X(z),z)\\+\lambda\sum_{i=0}^{a-1}z^i f\left(\tilde{S},\tilde{Q},z\right)-\lambda\sum_{i=0}^{a-1}z^{i\text{-}b}\psi\left(\tilde{S},\tilde{Q},z\right)\\+\pi\left(\tilde{R}(\lambda-\lambda X(z))-1\right)\tilde{S}(\lambda-\lambda X(z))\left(\lambda E4-\sum_{n=0}^{b\text{-}1}d_nz^n+\lambda\sum_{i=0}^{a-1}\left(\tilde{Q}_i(\lambda-\lambda X(z))z^i\left(X(z)-\sum_{j=1}^{b\text{-}i\text{-}1}g_jz^j\right)\right)\right)\end{pmatrix}+\left(\left(\tilde{V}(\lambda-\lambda\xi(z))-1\right)\sum_{n=0}^{a\text{-}1}d_nz^n\right)\left(z^b-(1-\pi)\tilde{S}(\lambda-\lambda X(z))-\pi\tilde{S}(\lambda-\lambda X(z))\tilde{R}(\lambda-\lambda X(z))\right)(-\lambda+\lambda X(z))}{\left(-\lambda+\lambda\xi(z)\right)(-\lambda+\lambda X(z))\left(z^b-(1-\pi)\tilde{S}(\lambda-\lambda X(z))-\pi\tilde{S}(\lambda-\lambda X(z))\tilde{R}(\lambda-\lambda X(z))\right)} \tag{20}$$

where

$$g(\tilde{S},E1,E2,E3,z)=\left((1-\pi)z^b-(1-\pi)+\pi\widetilde{R}(\lambda-\lambda X(z))(z^b-1)\right)\tilde{S}(\lambda-\lambda X(z))\lambda E1$$
$$-\left(z^b-(1-\pi)\tilde{S}(\lambda-\lambda X(z))-\pi\tilde{S}(\lambda-\lambda X(z))\widetilde{R}(\lambda-\lambda X(z))\right)\lambda E2$$
$$+\left(\tilde{S}(\lambda-\lambda X(z))-1+\pi\tilde{S}(\lambda-\lambda X(z))\left(\widetilde{R}(\lambda-\lambda X(z))-1\right)\right)z^b E3$$

$$h\left(\tilde{S},X(z),z\right)=\left[z^b-(1-\pi)\tilde{S}(\lambda-\lambda X(z))-\pi\widetilde{R}(\lambda-\lambda X(z))\tilde{S}(\lambda-\lambda X(z)\right]\left(-\lambda+\lambda X(z)\right)T(z)$$

$$f(\tilde{S},\tilde{Q},z)=\left(\tilde{Q}_i(\lambda-\lambda X(z))\tilde{S}(\lambda-\lambda X(z))-\tilde{Q}_i(0)\right)\times\left(X(z)-\sum_{j=1}^{b-i-1}g_j z^j\right)$$

$$\psi\left(\tilde{S},\tilde{Q},z\right)=\left(\tilde{Q}_i(\lambda-\lambda X(z))-\tilde{Q}_i(0)\tilde{S}(\lambda-\lambda X(z))\right)\times\left(X(z)-\sum_{j=1}^{b-i-1}g_j z^j\right)$$

$$E1=\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))\,g_{i-k};\ E2=\sum_{k=0}^{a-1}\tilde{Q}_k(0)g_{i-k};$$
$$E3=d_i+\sum_{m=0}^{a-1}T_m\lambda g_{i-m};\ E4=T(z)X(z)-\sum_{m=0}^{a-1}T_m z^m\sum_{j=1}^{b-m-1}g_j z^j;$$

$$d_i=(1-\pi)p_i+R_i,\quad p_i=\sum_{m=a}^{b}P_{mi}(0),\ q_i=Q_i(0),\ R_i=R_i(0)$$

Proof:

Using PGF and taking Z-transforms on equations (12) – (18), we get the following equations:

$$(\theta-\lambda+\lambda\xi(z))\tilde{Q}(z,\theta)=Q(z,0)-\tilde{V}(\theta)\sum_{n=0}^{a-1}\left((1-\pi)\sum_{m=a}^{b}P_{mn}(0)z^n+R_n(0)z^n\right)$$
where $\xi(z)=\sum_{k=1}^{a-1}g_k z^k$ (21)

$$(\theta-\lambda+\lambda X(z))\tilde{P}_i(z,\theta)=P_i(z,0)-\tilde{S}(\theta)\begin{pmatrix}(1-\pi)\sum_{m=a}^{b}P_{mi}(0)+\sum_{k=0}^{a-1}\tilde{Q}_k(\theta)\lambda g_{i-k}\\+\sum_{m=0}^{a-1}T_m\lambda g_{i-m}+R_i(0)\end{pmatrix},\ a\le i\le b-1 \tag{22}$$

$$z^b(\theta-\lambda+\lambda X(z))\tilde{P}_b(z,\theta)=P_b(z,0)\left(z^b-(1-\pi)\tilde{S}(\theta)-\pi\tilde{S}(\theta)\widetilde{R}(\theta)\right)$$
$$-\tilde{S}(\theta)\begin{pmatrix}(1-\pi)\left(\sum_{m=a}^{b-1}P_m(z,0)-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{mj}(0)z^j\right)\\+\lambda\left(T(z)X(z)-\sum_{m=0}^{a-1}\left(T_m z^m\sum_{j=1}^{b-m-1}g_j z^j\right)\right)\\+\lambda\left(X(z)\sum_{i=0}^{a-1}\tilde{Q}_i(\theta)z^i-\sum_{i=0}^{a-1}\left(\tilde{Q}_i(\theta)z^i\sum_{j=1}^{b-i-1}g_j z^j\right)\right)\\+R(z,0)-\sum_{n=0}^{b-1}R_n(0)z^n\end{pmatrix} \tag{23}$$

$$(\theta-\lambda+\lambda X(z))\widetilde{R}(z,\theta)=R(z,0)-\pi\sum_{m=a}^{b}P_m(z,0)\widetilde{R}(\theta) \tag{24}$$

Substituting $\theta = \lambda - \lambda\xi(z)$ in equation (21), we get

$$Q(z,0) = \tilde{V}(\lambda - \lambda\xi(z))\sum_{n=0}^{a-1}\left((1-\pi)\sum_{m=a}^{b} P_{mn}(0)z^n + R_n(0)z^n\right) \tag{25}$$

Substituting $\theta = \lambda - \lambda X(z)$ in equations (22) – (24), we get

$$P_i(z,0) = \tilde{S}(\lambda - \lambda X(z))\left(d_i + \sum_{k=0}^{a-1}\tilde{Q}_k(\lambda - \lambda X(z))\lambda g_{i-k} + \sum_{m=0}^{a-1} T_m \lambda g_{i-m}\right), a \le i \le b-1 \tag{26}$$

$$P_b(z,0) = \frac{1}{\left(z^b - (1-\pi)\tilde{S}(\lambda - \lambda X(z)) - \pi\tilde{S}(\lambda - \lambda X(z))\tilde{R}(\lambda - \lambda X(z))\right)}\left(\tilde{S}(\lambda - \lambda X(z))\, f(z)\right) \tag{27}$$

where

$$f(z) = (1-\pi)\sum_{m=a}^{b-1} P_m(z,0) + R(z,0) - \sum_{i=0}^{b-1}\left((1-\pi)p_i + R_i\right) + \lambda\left(T(z)X(z) - \sum_{m=0}^{a-1}\left(T_m z^m \sum_{j=1}^{b-m-1} g_j z^j\right)\right)$$
$$+\lambda\left(X(z)\sum_{i=0}^{a-1}\tilde{Q}_i(\lambda - \lambda X(z))z^i - \sum_{i=0}^{a-1}\left(\tilde{Q}_i(\lambda - \lambda X(z))z^i \sum_{j=1}^{b-i-1} g_j z^j\right)\right)$$

$$R(z,0) = \pi\sum_{m=a}^{b} P_m(z,0)\,\tilde{R}(\lambda - \lambda X(z)) \tag{28}$$

From equations (21) and (25), we have

$$\tilde{Q}(z,\theta) = \frac{1}{(\theta - \lambda + \lambda\xi(z))}\left(\tilde{V}(\lambda - \lambda\xi(z)) - \tilde{V}(\theta)\right)\sum_{n=0}^{a-1}\left((1-\pi)\sum_{m=a}^{b} P_{mn}(0)z^n + R_n(0)z^n\right) \tag{29}$$

From equations (22) and (26), we have

$$\tilde{P}_i(z,\theta) = \frac{\begin{pmatrix}\left[\tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(\theta)\right]\left(d_i + \sum_{m=0}^{a-1} T_m \lambda g_{i-m}\right) \\ +\tilde{S}(\lambda - \lambda X(z))\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda - \lambda X(z))\lambda g_{i-k} - \tilde{S}(\theta)\sum_{k=0}^{a-1}\tilde{Q}_k(\theta)\lambda g_{i-k}\end{pmatrix}}{(\theta - \lambda + \lambda X(z))}, a \le i \le b-1 \tag{30}$$

From equations (23) and (27), we have

$$\tilde{P}_b(z,\theta) = \frac{\begin{pmatrix}\left(\tilde{S}(\lambda - \lambda X(z)) - \tilde{S}(\theta)\right) g(z) \\ +\lambda\sum_{i=0}^{a-1} z^i\left(\left(\tilde{Q}_i(\lambda - \lambda X(z))\tilde{S}(\lambda - \lambda X(z)) - \tilde{Q}_i(\theta)\tilde{S}(\theta)\right)\left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right)\right) \\ -\lambda\sum_{i=0}^{a-1} z^{i-b}\left(\left(\tilde{Q}_i(\lambda - \lambda X(z)) - \tilde{Q}_i(\theta)\right)\tilde{S}(\theta)\tilde{S}(\lambda - \lambda X(z))\left(X(z) - \sum_{j=1}^{b-i-1} g_j z^j\right)\right)\end{pmatrix}}{(\theta - \lambda + \lambda X(z))\left(z^b - \tilde{S}(\lambda - \lambda X(z))\right)} \tag{31}$$

where

$$g(z) = (1-\pi)\sum_{m=a}^{b-1} P_m(z,0) + R(z,0) - \sum_{n=0}^{b-1}\left((1-\pi)p_n + R_n\right) + \lambda\left(T(z)X(z) - \sum_{m=0}^{a-1}\left(T_m z^m \sum_{j=1}^{b-m-1} g_j z^j\right)\right)$$

From equations (24) and (28), we have

$$\tilde{R}(z,\theta) = \frac{\left(\tilde{R}(\lambda-\lambda X(z)) - \tilde{R}(\theta)\right)\pi \sum_{m=a}^{b} P_m(z,0)}{(\theta-\lambda+\lambda X(z))} \tag{32}$$

Let $$p_i = \sum_{m=a}^{b} P_{mi}(0), \quad q_i = Q_i(0)\,,\ R_i = R_i(0)$$

Let P(z) be the probability generating function of the queue size at an arbitrary time epoch.

Then $$P(z) = \sum_{i=a}^{b-1} \tilde{P}_i(z,0) + \tilde{P}_b(z,0) + \tilde{Q}(z,0) + \tilde{R}(z,0) + T(z)$$

Using equations (29) - (32), we get

$$P(z) = \frac{(-\lambda+\lambda\xi(z))\left(\begin{array}{l} \sum_{i=a}^{b\text{-}1} g(\tilde{S},\tilde{R},E1,E2,E3,z) - \left(\tilde{S}(\lambda-\lambda X(z))-1\right)\sum_{n=0}^{b\text{-}1} d_n z^n + \left(\tilde{S}(\lambda - \lambda X(z))-1\right)\lambda E4 + h(\tilde{S},\tilde{R},X(z),z) \\ +\lambda\sum_{i=0}^{a-1} z^i f\left(\tilde{S},\tilde{Q},z\right) - \lambda\sum_{i=0}^{a-1} z^{i-b}\psi\left(\tilde{S},\tilde{Q},z\right) \\ +\pi\left(\tilde{R}(\lambda-\lambda X(z))-1\right)\tilde{S}(\lambda-\lambda X(z))\left(\lambda E4 - \sum_{n=0}^{b\text{-}1} d_n z^n + \lambda\sum_{i=0}^{a-1}\left(\tilde{Q}_i(\lambda-\lambda X(z))z^i\left(X(z) - \sum_{j=1}^{b\text{-}i\text{-}1} g_j z^j\right)\right)\right) \end{array}\right) + \left(\left(\tilde{V}(\lambda-\lambda\xi(z))-1\right)\sum_{n=0}^{a\text{-}1} d_n z^n\right)\left(z^b - (1-\pi)\tilde{S}(\lambda-\lambda X(z)) - \pi\tilde{S}(\lambda-\lambda X(z))\tilde{R}(\lambda-\lambda X(z))\right)}{\left(-\lambda+\lambda\xi(z)\right)(-\lambda+\lambda X(z))\left(z^b - (1-\pi)\tilde{S}(\lambda-\lambda X(z)) - \pi\tilde{S}(\lambda-\lambda X(z))\tilde{R}(\lambda-\lambda X(z))\right)}$$

4.1.1 Steady State Condition

The probability generating function $P(z)$ has to satisfy $P(1)=1$. In order to satisfy this condition, applying Hospital's rule and evaluating $\lim_{z\to 1} P(Z)$ and equating the expression to 1, it is derived that, $Q<1$ is the condition to be satisfied for the existence of steady state for the model under consideration, where $\rho = \frac{\lambda E(X)\left(E(S)+\pi E(R)\right)}{b}$.

4.1.2 Computational aspects of Unknown Probabilities

Equation (20) gives the probability generating function of the number of customers in the queue, which involves the unknowns T_i and $\tilde{Q}_i$. Using the following theorems, T_i and $\tilde{Q}_i(\theta)$ are expressed in terms of d_i and the known function $\tilde{V}(\lambda)$ respectively. To find the unknown constants, Rouche's theorem of complex variables can be used. It follows that the expression $z^b - (1-\pi)\tilde{S}(\lambda-\lambda X(z)) - \pi\tilde{S}(\lambda-\lambda X(z))\tilde{R}(\lambda-\lambda X(z))$ has $b-1$ zeros inside and one on the unit circle $|z|=1$. Since $P(z)$ is analytic within and on the unit circle, the numerator of (20) must vanish at these points, which gives 'b' equations and 'b' unknowns. These equations can be solved by suitable numerical techniques. MATLAB is used for programming.

4.2

Theorem 2. *The unknown constants* q_n *involved in* T_n *are expressed in terms of* d_n *as,* $q_n = \sum_{i=0}^{n} p_{n-1}\beta_i$; $n = 0,1,2,\cdots,a-1$, *where* β_i *is the probability that* 'i' *customers arrive during the vacation.*

Proof:

From equation (25), we have

$$Q(z,0)=\tilde{V}(\lambda-\lambda\xi(z))\sum_{n=0}^{a-1}\left((1-\pi)\,p_n+R_n\right)z^n$$

$$\sum_{n=0}^{a-1}q_n z^n=\sum_{n=0}^{a-1}\left(\sum_{i=0}^{n}d_{n-i}\beta_i\right)z^n+\sum_{n=a}^{\infty}\left(\sum_{i=0}^{a-1}\beta_{n-i}p_i\right)z^n \tag{33}$$

Equating the coefficient of z^n; $n=0,1,2,3,\cdots,a-1,$ on both sides of the equation (33), we get

$$q_n=\sum_{i=0}^{n}p_{n-i}\beta_i\ . \tag{34}$$

Hence the theorem.

The unknown constants T_n involved in $P(z)$ are expressed in terms of d_n in the following theorem.

4.3

Theorem 3. *Let B_j be the collection of set of positive integers (not necessarily distinct) A, such that, sum of elements in A is j, then*

$$T_n=\frac{1}{\lambda}\left(\sum_{j=0}^{n}q_{n-j}\sum_{j=1}^{n(B_j)}\prod_{l\in A}g_l\right),\ n=0,1,2,\cdots,a-1\,, \tag{35}$$

Proof:

From the equations (1) and (2), we have $\lambda T_0=Q_0(0)=q_0$

$$\lambda T_n=Q_n(0)+\lambda\sum_{k=1}^{n}T_{n-k}g_k;\ 1\le n\le a-1$$

When $n=1$, $\lambda T_1==q_1+q_0g_1$

When $n=2$, $\lambda T_2=q_2+q_1g_1+q_0(g_1^2+g_2)$

When $n=3$, $\lambda T_3=q_3+q_2g_1+q_1(g_1^2+g_2)+q_0(g_1^3+2g_1g_2+g_3)$

$$T_3=\frac{1}{\lambda}\left(\sum_{j=0}^{3}q_{3-j}\sum_{j=1}^{n(B_j)}\prod_{l\in A}g_l\right)$$

where $B_1=\{\{1\}\}$, $B_2=\{\{1,1\},\{2\}\}$ and $B_3=\{\{3\},\{1,1,1\},\{1,2\},\{2,1\}\}$

By induction, we get $$T(z)=\sum_{n=0}^{a-1}T_nz^n=\frac{1}{\lambda}\left(\sum_{n=0}^{a-1}\left(\sum_{j=0}^{n}q_{n-j}\sum_{j=1}^{n(B_j)}\prod_{l\in A}g_l\right)z^n\right)$$

Therefore,

$$T_n=\frac{1}{\lambda}\left(\sum_{j=0}^{n}q_{n-j}\sum_{j=1}^{n(B_j)}\prod_{l\in A}g_l\right).$$

Hence the theorem.

An important theorem, which breaks the barrier in solving vacation interruption model, is proved. The following theorem gives a compact way of representing the unknown functions $\tilde{Q}_i(\theta)$ in terms of known values.

4.4

Theorem 4. *The Laplace-Stieltjes transform of the unknown function* $\tilde{Q}_i(\theta)$; $i = 0,1,2,3,\cdots,a-1$ *are expressed in terms of* $\tilde{V}(\lambda)$ *and higher derivatives* $\tilde{V}^{(n)}(\lambda)$ *as,*

$$\tilde{Q}_i(\theta) = \sum_{j=0}^{i} \sum_{A_j \in \phi} \left((-1)^{n(A_j)} \left(\prod_{i \in A_j} \lambda g_i \right) d_{i-j}\ k_{n(A_j)} \right) \text{ where } k_l = \frac{\tilde{V}^{n(A_l)}(\lambda) + k_{l-1}}{\theta - \lambda};\ l = 1,2,3,\cdots i,\ k_0 = \frac{\tilde{V}(\lambda) - \tilde{V}(\theta)}{\theta - \lambda}$$

and ϕ *is the collection of all possible distinct sets of positive integers* A_j *such that , sum of elements in* A_j *is* j .

Proof:

Substituting $\theta = \lambda$ in equation (15), we get

$$\tilde{Q}_0(\theta) = \frac{\left(\tilde{V}(\lambda) - \tilde{V}(\theta)\right)}{\theta - \lambda} d_0 = k_0 d_0$$

where

$$k_0 = \frac{\left(\tilde{V}(\lambda) - \tilde{V}(\theta)\right)}{\theta - \lambda}$$

Substituting n=1 in equation (16), we get

$$\begin{aligned} \tilde{Q}_1(\theta) &= \frac{\left(\tilde{V}(\lambda) - \tilde{V}(\theta)\right)}{\theta - \lambda} d_1 + \frac{\lambda g_1}{\theta - \lambda} \left(\tilde{Q}_0(\lambda) - \frac{\left(\tilde{V}(\lambda) - \tilde{V}(\theta)\right)}{\theta - \lambda} d_0 \right) \\ &= k_0 d_1 - \lambda g_1 \left(\frac{\tilde{V}^1(\lambda) + k_0}{\theta - \lambda} \right) d_0 \\ &= k_0 d_1 - \lambda g_1 k_1 d_0 \text{ where } k_1 = \frac{\tilde{V}^1(\lambda) + k_0}{\theta - \lambda} \end{aligned}$$

Substituting $n = 2$ in equation (16), we get

$$\begin{array}{llllllll} \tilde{Q}_3(\theta) = k_0 d_3 & - & (\lambda g_1) k_1 d_2 & - & (\lambda g_2) k_1 d_1 & + & (\lambda g_1)(\lambda g_1) k_2 d_1 & \\ \left\{ \begin{matrix} k = 0 \\ A_0 = \varphi \end{matrix} \right\} & & \left\{ \begin{matrix} k = 1 \\ A_1 = \{1\} \end{matrix} \right\} & & \left\{ \begin{matrix} k = 2 \\ A_2 = \{2\} \end{matrix} \right\} & & \left\{ \begin{matrix} k = 2 \\ A_2 = \{1,1\} \end{matrix} \right\} & \\ - \lambda g_3 k_1 d_0 & + & (\lambda g_2)(\lambda g_1) k_2 d_0 & + & (\lambda g_1)(\lambda g_2) k_2 d_0 & - & (\lambda g_1)(\lambda g_1)(\lambda g_1) k_3 d_0 & \\ \left\{ \begin{matrix} k = 3 \\ A_3 = \{3\} \end{matrix} \right\} & & \left\{ \begin{matrix} k = 3 \\ A_3 = \{2,1\} \end{matrix} \right\} & & \left\{ \begin{matrix} k = 3 \\ A_3 = \{1,2\} \end{matrix} \right\} & & \left\{ \begin{matrix} k = 3 \\ A_3 = \{1,1,1\} \end{matrix} \right\} & \end{array}$$

Where

$$k_3 = \frac{\tilde{V}^3(\lambda) + k_2}{\theta - \lambda}$$

Using appropriate notations, $\tilde{Q}_3(\theta)$ is expressed in a compact form as

$$\tilde{Q}_3(\theta) = \sum_{j=0}^{3} \sum_{A_j \in \phi} \left((-1)^{n(A_j)} \left(\prod_{i \in A_j} \lambda g_i \right) d_{3-j}\ k_{n(A_j)} \right)$$

Generalizing by recursive approach, we get

$$\tilde{Q}_i(\theta) = \sum_{j=0}^{i} \sum_{A_j \in \phi} \left((-1)^{n(A_j)} \left(\prod_{i \in A_j} \lambda g_i \right) d_{i-j} \; k_{n(A_j)} \right); \; i = 0,1,2,\cdots,a-1 \quad (36)$$

Where $$k_l = \frac{\tilde{V}^{n(A_l)}(\lambda) + k_{l-1}}{\theta - \lambda}; \; l = 1,2,3,\cdots,i \text{ and } k_0 = \frac{\tilde{V}(\lambda) - \tilde{V}(\theta)}{\theta - \lambda}$$

5. PERFORMANCE MEASURES

In a waiting line, it is customary to access the mean number of waiting units and mean waiting time. In this section, some useful performance measures of the proposed model like expected number of customers in the queue $E(Q)$, expected length of busy period $E(B)$, expected length of idle period $E(I)$ are derived, which are useful to find the total average cost of the system. Also, expected waiting time in the queue $E(W)$, probability that the server is on vacation $P(V)$ and the probability that the server is busy $P(B)$, are derived.

5.1 Expected Queue Length

The mean queue length $E(Q)$ (i.e. mean number of customers waiting in the queue) at an arbitrary time epoch is obtained by differentiating $P(z)$ at $z = 1$ and is given by

$$E(Q) = \lim_{z \to 1} P'(z)$$

$$E(Q) = \frac{\left[\begin{array}{c} \sum_{i=a}^{b-1} \left[2f1(T13) - 2(f10)(T14) - 4(f18)f(14) \right] - \sum_{j-0}^{b-1} d_j \left[2T13(f2 + f7) \right] \\ -T14(2S2 + T11S9\pi) - 4f18S1 + \sum_{n=0}^{a-1} 2T13\lambda \left[f4 - f5 \right] \\ -2\left[f12 \right] T14\lambda + 2T13f3 \\ +2\left[f6 + f7 + f8 + f9 \right] T13 - 2T14\left[f11 + f13 \right] - 4(f15)(f18) \end{array} \right]}{24(T13)^2} + \frac{\sum_{n=0}^{a-1} d_n (f16 + f17)}{(T15)^2} \quad (37)$$

where

$$\begin{array}{ll}
S1 = \lambda E(X) E(S) & S2 = \lambda X''(1)E(S) + \lambda^2 E^2(X)E(S^2) \\
S3 = \lambda^3 E^3(X)E(S^3) & S4 = \lambda^2 E(X^2)E(X)E(S^2) \\
S5 = \lambda E(S)E(X^3) & S6 = \lambda^2 E(X^2)E(X) \\
S7 = \lambda^2 E(S^3)E(X) & S8 = \lambda E(S)E(X^2) \\
S9 = \lambda E(X) E(R) & S10 = S9E(S) \\
S11 = \lambda^2 E(R^2)E^2(X) & S12 = \lambda^2 E(R^3)E^2(X) \\
S13 = S10E(X^2) & S14 = \lambda^2 E(R^2)E^3(X) \\
S15 = E^2(X)S10 & S16 = S9E^3(X) \\
S17 = S9E^2(X) & S18 = E(R)\lambda^3 E(S^2))E^3(X) \\
S19 = \lambda E(R)E(X^2) & S20 = \lambda^3 E(R^3)E^3(X)E(X)
\end{array}$$

$$R1 = \lambda X''(1)E(R) + \lambda^2 E^2(X)E(R^2)$$

$$G1 = 1 - \sum_{j=1}^{b-i-1} g_i \qquad G2 = E(X) - \sum_{j=1}^{b-i-1} j g_j \qquad G3 = E(X^2) - \sum_{j=1}^{b-i-1} j(j-1) g_j$$

$$G4 = T(1) - \sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1} T_m g_j \qquad G5 = \sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1}(m+j)(m+j-1)T_m g_j \qquad G6 = \sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1}(m+j)T_m g_j$$

$$G7 = E(X)T'(1) \qquad G8 = E(X)T(1) \qquad G9 = E(X^2)T'(1)$$

$$G10 = d_i + \lambda\sum_{m=0}^{a-1} T_m g_{i-m}, \; d_i = (1-\pi)?+R_i$$

$$T1 = \left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))g_{i-k}\right)_{z=1} \qquad T2 = \left(\frac{d}{dz}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))g_{i-k}\right)\right)_{z=1}$$

$$T3 = \left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))\right)_{z=1} \qquad T4 = \left(\frac{d}{dz}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))\right)\right)_{z=1}$$

$$T5 = \left(\frac{d^2}{dz^2}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))\right)\right)_{z=1} \qquad T6 = \sum_{k=0}^{a-1}\tilde{Q}_k(0)g_{i-k}$$

$$T7 = \frac{d}{dz}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(0)\right) \qquad T8 = \sum_{k=0}^{a-1}\tilde{Q}_k(0)$$

$$T9 = \left(\frac{d^3}{dz^3}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))\right)\right)_{z=1} \qquad T10 = \left(\frac{d^2}{dz^2}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))g_{i-k}\right)\right)_{z=1}$$

$$D1 = \left(\frac{d^3}{dz^3}\left(\sum_{k=0}^{a-1}\tilde{Q}_k(\lambda-\lambda X(z))g_{i-k}\right)\right)_{z=1} \qquad T11 = (b - S1 - \pi S9)$$

$$T12 = (b\ (b-1) - S2 - R1\pi - 2\pi S1S9) \qquad T13 = \lambda E(X)T11$$

$$T14 = 3T11\lambda X''(1) + 3T12\lambda E(X) \qquad T15 = (-\lambda + \lambda\xi(1)), \; \xi(1) = \sum_{k=1}^{a-1} g_k, \; \xi'(1) = \sum_{k=1}^{a-1} k g_k$$

$$\begin{aligned} f1\ (\lambda,\ b,\ S,\ G,\tilde{Q}) = {} & (3S4 - S1S1 + 2bS8 - 2bS1 + 3S3 + 6\pi(S16 - S17) - \pi\ S11 + bS2 + 2S3 \\ & + 2b(b-1)S1 + b(b-1)(b-2) + (2-\pi)S3 + 3\pi S18 + 2\ \pi S11S1 - \pi S11 - S12 \\ & + X'''(1)\left[2E(S) + \pi E(R)\right])\ T1\lambda + T2\lambda(3S2 + 6bS1 + 3b(b-1) + (1-\pi)S2 \\ & + 2(1-\pi)(S8 - S1) - 2(1-\pi)S1 + 3\pi S2 + 6\pi S1S8 + 3\pi(S19 - S1) + \pi(2S1 - S9) \\ & + (3S1 + 3b + 3(1-\pi k) + 3\pi k(S1 + S10))T10\pi f + 2D1 + (b(b-1)(b-2) - E(S)\lambda X'''(1) \\ & - 6S4 + (1-\pi)S3 - 2\pi S9S2 - 2\pi R1S1 - 2\pi S1S9 - S11 - S18)T6\lambda \\ & + G10\left[b(b-1)(3S1 + 2S2) + S3 + 3S4 + S5 + bS2\right] \end{aligned}$$

$$f2\ (\lambda,\ X,\ S) = 3i\left(S2 + (j-1)\ S1\right) + S3 + 3S4 + S5$$

$$\begin{aligned} f3(\lambda, b, S, G) = {} & 3\lambda S1[G9 + 2G7 + T''(1) - G5] + 3\lambda S2[G8 + [T'(1) - G6] + \lambda[S3 + 3S4 + S7]G4 \\ & + (b - S1 - S9)\left[6G7 + G9 - G8\right] - 3\left[b(b-1) - S2 - R1\right]G8 \end{aligned}$$

$$\begin{aligned} f4(\lambda, S, G, \tilde{Q}) = {} & (T4 + T3S1)\left(3(i-1)iG1 + G3 + G2(2+4i)\right) + \left[T3S2 + 2T4S1 + T5\right]\left[2G1(2i+1) + G2\right] \\ & + \left[(S3 + 3S4 + S5)T3 + 3T4S2 + 3T5S1 + T9\right]G1 \end{aligned}$$

$$f5(\lambda,S,G,\tilde{Q}) = 3(i-b)((i-b-1)G1T4 + 2T4[G2+(S1)(G1)+(T5)(S1)])$$
$$+3(T4)\left[(G3+2(S1)(G2)+(S2)(G1)\right]+(3T5)\left[G2+S1G1\right]+(T9)(G1)$$

$$f6(\lambda,b,S,G,\tilde{Q}) = 2S9\pi\left[b(b-1)-S2-2S17\pi-\pi R1\right]+(\pi[G10+\lambda T1]\left[(S9)(S9)+S9+2(S19-S9)\right]$$
$$+4S9\pi\ [S1G10+S1T1+\lambda T2])T11$$

$$f7(\lambda,\ X,S) = 3(T11)R1+3(T12)S1,$$

$$f8(\lambda,\ X,G) = \pi\lambda(6(T11)(\pi S9)(G5)+\left[3(T11)R1+3(T12)S9\right]G4)$$

$$f9(\lambda,\ S,\ G,\tilde{Q}) = \ 6(T11)S9\pi\left[T3(G2+iG1)+T4G1)\right]$$

$$f10(\lambda,\ b,\ S,\ G,\tilde{Q}) = \left[S2+bS1\right]G10+\lambda T1\left[2S2+(1-\pi)\ S2+R1+b(b-1)+S1[2b+\pi S9]\right]$$
$$+\lambda T2[4S1+2b+2(1-\pi)\ S1+2(1-\pi)S1+\pi(S9+1)]+(2-\ \pi)\lambda T10+(T12)(T6)$$

$$f11(\lambda,S,G) = 2\lambda S1[G8-T'(1)-G6+\lambda(S2)G4]+(2T11)G8$$

$$f12(\lambda,S,G,\tilde{Q}) = \left[(S1)(T8)+T3\right]+(G1)(2i)+(G1)(T8)(S2)+2(T3)(S1)+T5-2(T4)(G1)\left[i-b+S1\right]$$
$$-2(T4)G2-(T5)G1$$

$$f13\ (\lambda,S,G,\tilde{Q}) = 2S9\pi[G10+\lambda T1]\ T11+2(T11)(S9)\pi\lambda(T8)(G1)$$

$$f14\ (\lambda,S,G,\tilde{Q}) = T1\lambda\left[2S1+b+S9\right]+2\ \lambda T2+S1(G10)+T6\lambda(T11)$$

$$f15(\lambda,S,G) = S1\lambda G4$$

$$f16(\lambda,\xi,\tilde{V}) = [\tilde{V}(\lambda-\ \lambda\xi(1))-1][n(T15)-\lambda\xi'(1)]$$

$$f17(\lambda,\xi'\tilde{V}') = (T15)[-\ \lambda\xi'(1)\tilde{V}'(\lambda-\ \lambda\xi(1))]$$

$$f18(\lambda,b,X,S) = [b(b-1)(b-2)-E(S)\lambda X'''(1)-6S4+(1-\pi)S3-2\pi(S9)S2-2\pi(R1)S1-2\pi(S9)(S1)-\mathrm{S}11-\mathrm{S}18]$$

5.2 Expected Waiting Time in the Queue

The mean waiting time of the customers in the queue $E(W)$ can be easily obtained using Little's formula

$$E(W) = \frac{E(Q)}{\lambda E(X)} \tag{38}$$

5.3 Expected Length of Idle Period

The time period from the vacation initiation epoch to the busy period initiation epoch is called the idle time period. Let I be the random variable for 'idle period'.

$\pi_j,\ j = 0,1,2,\cdots,a-1$, is the probability that the system state (number of customers in the system) visits 'j' during an idle period.

Let $I_j = \begin{cases} 1 & \text{if the state } 'j' \text{ is visited during an idle period} \\ 0 & \text{otherwise} \end{cases}$

Conditioning on the queue size at service completion epoch, we have $\pi_0 = \alpha_0$

$$\pi_j = P(I_j = 1) = \alpha_j + \sum_{k=0}^{j-1} \alpha_k\ P(I_{j-k}^1 = 1)\ ;\ \ j = 1,2,3,\cdots,a-1,$$

where $P(I_j^1 = 1)$ is the probability that the system state becomes j during an idle period of $M^X / G(a,b)/1$ queueing system without vacation and α_j is the probability that 'j' customers in the queue at a service completion epoch.

$P(I_j^1 = 1)$ is obtained as $P(I_j^1 = 1) = \phi_j$; where $\phi_0 = 1$, $\phi_n = \sum_{i=1}^{n} g_i \phi_{n-i}$.

Thus the expected length of the idle period is obtained as

$$E(I) = \frac{1}{\lambda}\sum_{j=0}^{a-1}\pi_j \ . \tag{39}$$

where $\frac{1}{\lambda}$ is the expected staying time in the state 'j' during an idle period.

5.4 Expected Length of Busy Period

Let B be the busy period random variable. Let T be the residence time that the server is rendering service or under repair. Then $E(T) = E(S) + E(R)$, where $E(S)$ is the expected service time and $E(R)$ is the expected renovation time.

A random variable J is defined as, $J = 0$, if the server finds less than 'a' customers after the residence time and $J = 1$, if the server finds 'a' or more customers after the residence time. Then

$$E(B) = E(B/J=0)P(J=0) + E(B/J=1)P(J=1)$$
$$= E(T)P(J=0) + \big(E(T)+E(B)\big)P(J=1)$$

And since $P(J=0)+P(J=1)=1$, solving for $E(B)$,

we get

$$E(B) = \frac{E(T)}{P(J=0)} = \frac{E(T)}{\sum_{i=0}^{a-1}(d_i)} \tag{40}$$

5.5 Probability that the Server is Busy

Let $P(B)$ be the probability that the server is in the busy period at time t .

$$P(B) = \lim_{z\to 1}\left(\sum_{i=a}^{b}\tilde{P}_i(z,0) + \tilde{R}(z,0)\right)$$

$$= \frac{\left(\begin{array}{l} \sum_{i=a}^{b-1}\left(f_1(X,S,\tilde{Q}) - f_2(X,S,\tilde{Q})\right) - \sum_{i=0}^{b-1}\left((d_i)(2i\ S1+S2)\right) \\ +f_3(X,S,\tilde{Q}) + \lambda\sum_{i=0}^{a-1}\left(f_4(X,S,\tilde{Q}) - f_5(X,\tilde{Q})\right) + f_6(X,S,\tilde{Q}) \\ -\pi\sum_{i=0}^{b-1}\left((d_i)(2i+2(S9)(S1)+R1)\right) + f_8(X,S,\tilde{Q}) - f_9(X,S,\tilde{Q}) \end{array}\right)}{2\lambda E(X)(b - S1 - \pi S9)} \tag{41}$$

where

$$f_1(X,S,\tilde{Q}) = 2\lambda b\big(T2 + (S1)(T1) + \pi(S9)(T1)\big) + \lambda b(b-1)(T1)$$
$$+\left(d_i + \lambda\sum_{m=0}^{a-1}T_m g_{i-m}\right)(2bS1+\pi S9)\big(S2+\pi(R1+2(S1)(S9)\big),$$

$$f_2(X,S,\tilde{Q}) = \lambda T6\big(b(b-1) - S2 - \pi(R1+2(S1)(S9)\big),$$

$$f_3(X,S,\tilde{Q}) = 2\lambda S1\left(T(1)E(X)+T'(1)-\sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1}(m+j)T_m g_j\right)+\lambda S2\left(T(1)-\sum_{m=0}^{a-1}\sum_{j=1}^{b-m-1}T_m g_j\right)$$

$$f_4(X,S,\tilde{Q}) = 2(T8.S1+T4)(iG1+G2)+G1.T8.S2\,,\quad f_5(X,\tilde{Q}) = 2T4((i-b)G1+G2)$$

$$f_6(X,S,\tilde{Q}) = \lambda\pi(G4)(2(S1)(S9)+R1)+2\lambda\pi(S9)(G8+T'(1)-G6)$$

$$f_8(X,S,\tilde{Q}) = 2\lambda\pi(S9)(T4+E(X)T3+(S1)(T3))+\pi\lambda(T3)(R1)$$

$$f_9(X,S,\tilde{Q}) = \lambda\pi(G11)(T8)(2(S1)(S9)+R1)+2\lambda\pi(S9)((T8)(G12)(1+i)+(T4)(G12))$$

$$G11 = \sum_{j=1}^{b-k-1} g_j \quad \text{and} \quad G12 = \sum_{j=1}^{b-k-1} jg_j$$

5.6 Probability that the Server is on Vacation

Let $P(V)$ be the probability that the server is on vacation at time t.

$$P(V) = \lim_{z\to 1}\tilde{Q}(z,0) = \frac{\left(\tilde{V}(\lambda-\lambda\xi(1))-1\right)}{-\lambda+\lambda\xi(1)}\sum_{n=0}^{a\text{-}1} d_n \tag{42}$$

Where $d_n = (1-\pi)p_n + R_n$ and $\xi(z) = \sum_{k=1}^{a\text{-}1} g_k z^k$

6. PARTICULAR CASES

In this section, some of the existing models as a particular case of the proposed model are derived.

Case (i):

When there is no server breakdown (i.e., $\tilde{R}(\lambda-\lambda X(z))=1$ and $\pi=0$), then the equation (20) reduces to

$$P(z) = \frac{(\text{-}\lambda+\lambda\xi(z))\begin{pmatrix}\sum_{i=a}^{b\text{-}1} g(\tilde{S},E1,E2,E3,z)-\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\sum_{j=0}^{b\text{-}1}p_j z^j+\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\lambda E4+h(\tilde{S},X(z),z)\\ +\lambda\sum_{i=0}^{a\text{-}1}z^i f\left(\tilde{S},\tilde{Q},z\right)-\lambda\sum_{i=0}^{a\text{-}1}z^{i\text{-}b}\psi\left(\tilde{S},\tilde{Q},z\right)\end{pmatrix} +\left((-\lambda+\lambda X(z))\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)\left(\tilde{V}(\lambda-\lambda\xi(z))-1\right)\sum_{n=0}^{a\text{-}1}p_n z^n\right)}{\left(-\lambda+\lambda\xi(z)\right)(-\lambda+\lambda X(z))\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)} \tag{43}$$

where

$$g(\tilde{S},E1,E2,E3,z) = \left(\left(z^b-1\right)\tilde{S}(\lambda-\lambda X(z))\lambda E1+\left(\tilde{S}(\lambda-\lambda X(z))-1\right)z^b E3-\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)\lambda E2\right),$$

$$h(\tilde{S},X(z),z) = \left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)\left(-\lambda+\lambda X(z)\right)T(z)\,,$$

$$f(\tilde{S},\tilde{Q},z) = \left(\tilde{Q}_i(\lambda-\lambda X(z))\tilde{S}(\lambda-\lambda X(z))-\tilde{Q}_i(0)\right)\left(X(z)-\sum_{j=1}^{b-i-1}g_j z^j\right),$$

$$\psi(\tilde{S},\tilde{Q},z) = \left(\tilde{Q}_i(\lambda-\lambda X(z))-\tilde{Q}_i(0)\right)\tilde{S}(\lambda-\lambda X(z))\left(X(z)-\sum_{j=1}^{b-i-1}g_j z^j\right),$$

$$E1=\sum_{k=0}^{a\text{-}1}\tilde{Q}_k(\lambda-\lambda X(z))\,g_{i-k}\,,\; E2=\sum_{k=0}^{a\text{-}1}\tilde{Q}_k(0)\,g_{i\text{-}k}\,,\; E3=p_i+\lambda\sum_{m=0}^{a\text{-}1}T_m g_{i\text{-}m}\;\text{ and }\; E4=T(z)X(z)-\sum_{m=0}^{a-1}\left(T_m z^m\sum_{j=1}^{b-m-1}g_j z^j\right)$$

Equation (43) is the PGF of queue size distribution of $M^X/G(a,b)/1$ queueing system with vacation interruption and it coincides with the result of analysis of a $M^X/G(a,b)/1$ queueing system with vacation interruption of Haridass and Arumuganathan (2012)

Case (ii):

If the server doesn't avail any vacation (i.e., $\tilde{V}(\lambda-\lambda\xi(z))=1$),then the equation (20) reduces to

$$P(z)=\frac{\left(\begin{array}{l}\sum_{i=a}^{b-1}\left(\tilde{S}(\lambda\text{-}\lambda X(z))-1\right)\left(z^b-z^i\right)p_i+\lambda\sum_{i=a}^{b-1}\left(\left(\tilde{S}(\lambda-\lambda X(z))-1\right)z^b\left(\sum_{m=0}^{a-1}T_m g_{i-m}\right)\right)\\ -\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\sum_{j=0}^{a-1}p_j z^j+\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)\left(-\lambda+\lambda X(z)\right)T(z)\\ +\left(\tilde{S}(\lambda-\lambda X(z))-1\right)\lambda\left(T(z)(X(z)-\sum_{m=0}^{a-1}T_m z^m\sum_{j=1}^{b-m-1}g_j z^j\right)\end{array}\right)}{(-\lambda+\lambda X(z))\left(z^b-\tilde{S}(\lambda-\lambda X(z))\right)} \tag{44}$$

Equation (44) is the PGF of queue size distribution of $M^X/G(a,b)/1$ queueing system without vacation and it coincides with the result of $M^X/G(a,b)/1$ queueing system without modified vacation and constant arrival rate of Balasubramanian and Arumuganathan (2011).

7. COST MODEL

Cost analysis is the most important phenomenon in any practical situation at every stage. Cost involves startup cost, operating cost, holding cost, renovation cost and reward cost (if any). It is quite natural that the management of the system desires to minimize the total average cost. Addressing this, in this section, the cost model for the proposed queueing system is developed and the total average cost is obtained with the following assumptions:

C_h : holding cost per customer , C_O : operating cost per unit time

C_S : startup cost per cycle , C_r : reward cost per cycle due vacation

C_{rp} : renovation cost per unit time

Since the length of the cycle is the sum of the idle period and busy period, from equations (39) and (40), the expected length of cycle, $E(T_c)$ is obtained as

$$E(T_c)=E(\text{length of the Idle Period})+E(\text{length of the Busy Period})$$

$$E(T_c)=\frac{1}{\lambda}\sum_{j=0}^{a-1}\pi_j+\frac{E(T)}{\sum_{i=0}^{a-1}\left((1-\pi)p_i+R_i\right)}$$

Now, the total average cost per unit time is obtained as

Total average cost=Start-up cost per cycle + holding cost of number of customers in the queue per unit time + Operating cost per unit time$\times\rho$ − reward due to vacation per cycle + Renovation cost per cycle

$$TAC=\left[C_s+C_{rp}\pi E(R)-C_r E(I)\right]\frac{1}{E(T_c)}+C_h E(Q)+C_o\rho \tag{45}$$

where $\rho = \frac{\lambda E(X)\left(E(S) + \pi E(R)\right)}{b}$.

8. NUMERICAL ILLUSTRATION

The theoretical results obtained in this paper are justified numerically with the following assumptions and notations:

Service time distribution is 2-Erlang with parameter	μ
Batch size distribution of the arrival is geometric with mean	2
Vacation time is exponential with parameter	ε
Renovation time is exponential with parameter	η
Minimum service capacity	a
Maximum service capacity	b

8.1 Effects of various parameters on the performance measures

In this section, the effects of various parameters such as arrival rate, service rate on the performance measures like expected queue length, expected busy and idle period, probability that the server is on vacation, probability that the server is busy, and total average cost are analyzed numerically. The results are presented in tables and figures. The numerical results are obtained using MATLAB software.

8.1.1 Effects of arrival rate and service rate on the performance measures

The effects of the expected queue length, expected length of busy and idle period, the probability that the server is busy and on vacation, for various arrival rates, and service rates are presented in the table 8.1 and 8.2 and represented in figures 2,3 and 4. From the table, it is clear that if the arrival rate increases, then

- the mean queue size, the probability that the server is busy and expected busy period increases
- the probability that the server is on vacation and expected idle period decreases.

Also that, from the tables and figures 2,3 and 4, it is clear that if the service rate (μ) increases, then

- the mean queue size, the probability that the server is busy and expected busy period decreases
- the probability that the server is on vacation and expected idle period increases.

8.1.2 Effects of arrival rates on the total average cost

The effects of different arrival rates on the total average cost for a fixed breakdown probability with respect to various service rates are discussed numerically and these values are reported in table 8.1. A graphical representation is also shown in figures 5 and 6. From the table and the figures, the following points are observed:

As arrival rate increases, the total average cost increases

- As service rate increases, the total average cost decreases
- When the server is assigned for secondary job, the total average cost decreases

Thus, the theoretical development of the model is justified with the numerical results which are consistent with the fact that when the server is allotted to secondary job, the idle time is properly utilized and hence the total average cost is minimized.

Table 8.1 Arrival Rate (Vs) Total Average Cost

(For $a=2$, $b=5$, $\varepsilon=10$, $\eta=8$, $\pi=0.2$, $C_S=3.0$, $C_h=0.50$, $C_O=8.0$, $C_r=1.0$, $C_{rp}=1.5$)

μ	Arrival rate λ	E(Q)	E(W)	E(B)	E(I)	TAC when server is assigned for secondary job (in rupees)	TAC when server is not assigned for secondary job (in rupees)
service rate 2.5	2	0.695	0.173	2.405	0.587	6.4465	6.6426
	2.3	0.869	0.188	3.118	0.494	7.2107	7.3475
	2.6	1.198	0.230	4.955	0.423	7.9491	8.0278
	2.9	1.387	0.239	15.421	0.367	8.5187	8.5419
service rate 3.0	2	0.569	0.142	1.432	0.609	5.9010	6.1994
	2.3	0.819	0.178	1.605	0.515	6.6901	6.9331
	2.6	0.975	0.187	1.921	0.443	7.3395	7.5269
	2.9	1.271	0.219	2.545	0.387	7.9582	8.0902

$E(Q)$ – Expected orbit length; $E(W)$ – Expected waiting time; $E(B)$ – Expected busy period;

$E(I)$ – Expected Idle period; TAC – Total average cost

Table 8.2. Arrival rate (Vs) Servers state (For $a=2$, $b=5$, $\varepsilon=10$, $\eta=8$, $\pi=0.2$)

Arrival ratd λ	Service rate $\mu=2$		Service rate $\mu=2.5$		Service rate $\mu=3$	
	$P(B)$	$P(V)$	$P(B)$	$P(V)$	$P(B)$	$P(V)$
1.0	0.044	0.033	0.032	0.040	0.025	0.049
1.2	0.072	0.032	0.050	0.039	0.037	0.048
1.4	0.112	0.029	0.072	0.039	0.053	0.047
1.6	0.175	0.026	0.103	0.038	0.073	0.047
1.8	0.279	0.021	0.145	0.035	0.098	0.046
2.0	0.480	0.015	0.204	0.031	0.129	0.044

$P(B)$ – Probability that the server is busy; $P(V)$ – Probability that the server is on vacation

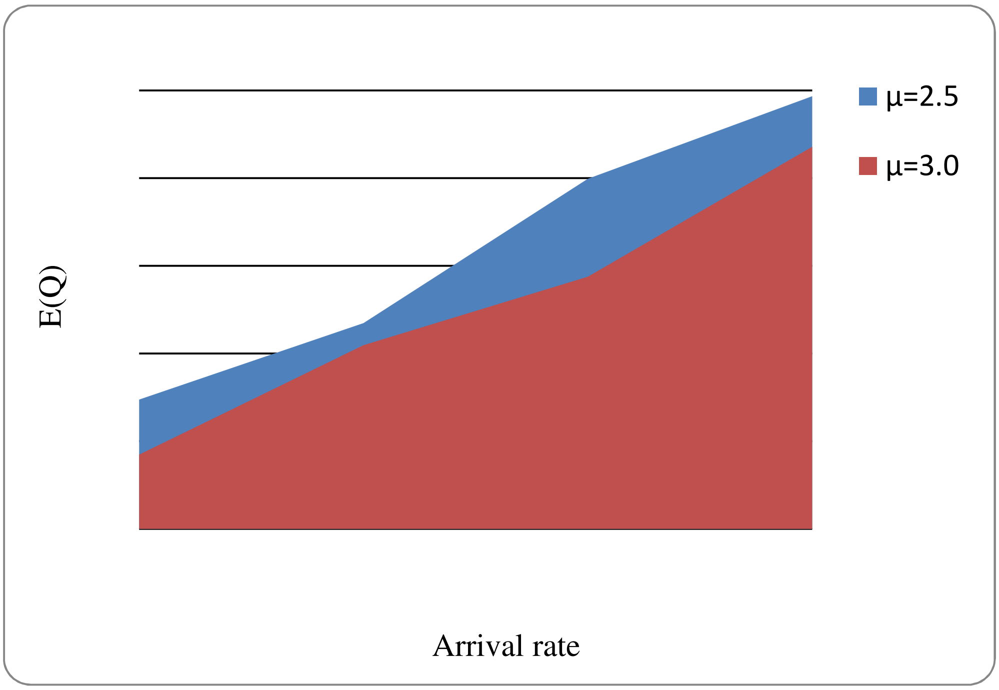

Figure 2: Arrival rate (Vs1) Expected Queue Length

(For $a = 2,\ b = 5,\ \varepsilon = 10,\ \eta = 8,\ \pi = 0.2$)

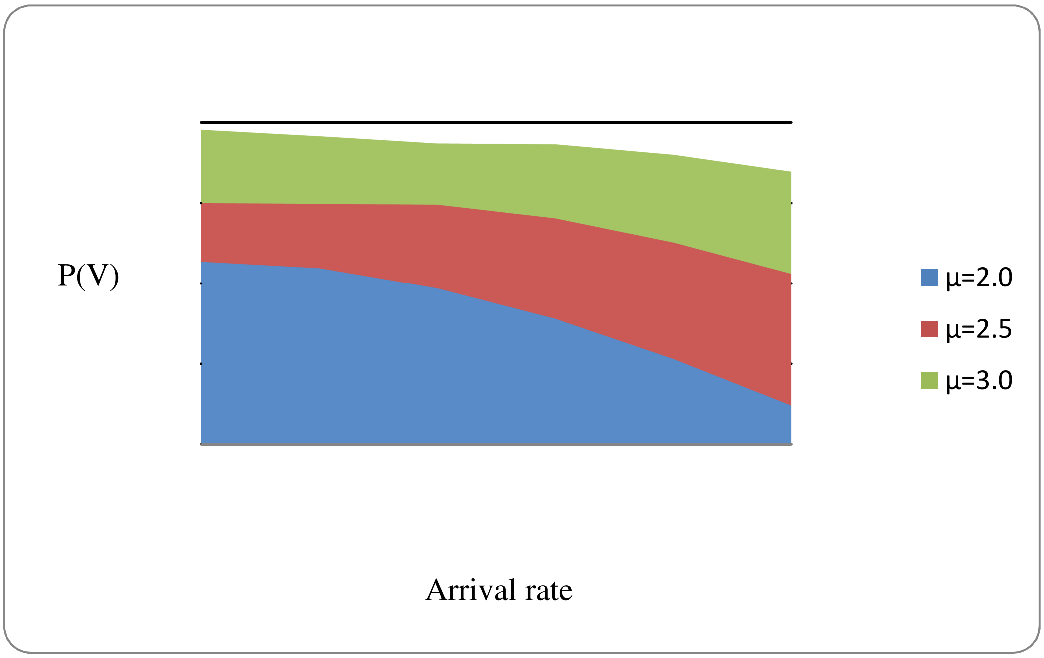

Figure 3: Arrival rate (Vs) Probability that the server is on vacation

(For $a = 2,\ b = 5,\ \varepsilon = 10,\ \eta = 8,\ \pi = 0.2$)

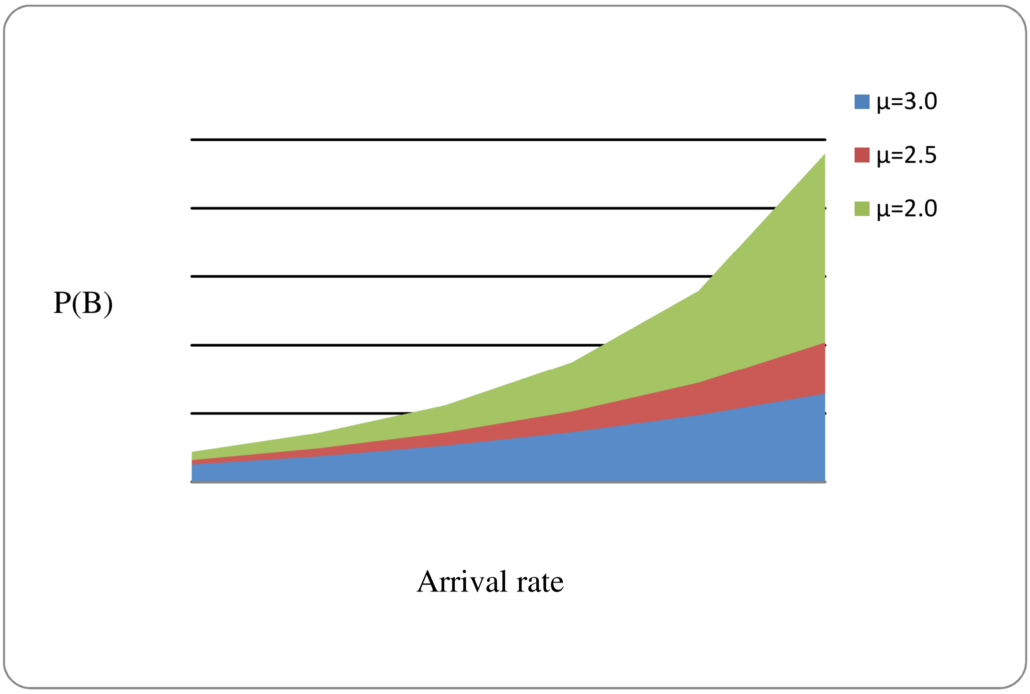

Figure 4: Arrival rate (Vs) Probability that the server is busy

(For $a = 2,\ b = 5,\ \varepsilon = 10,\ \eta = 8,\ \pi = 0.2$)

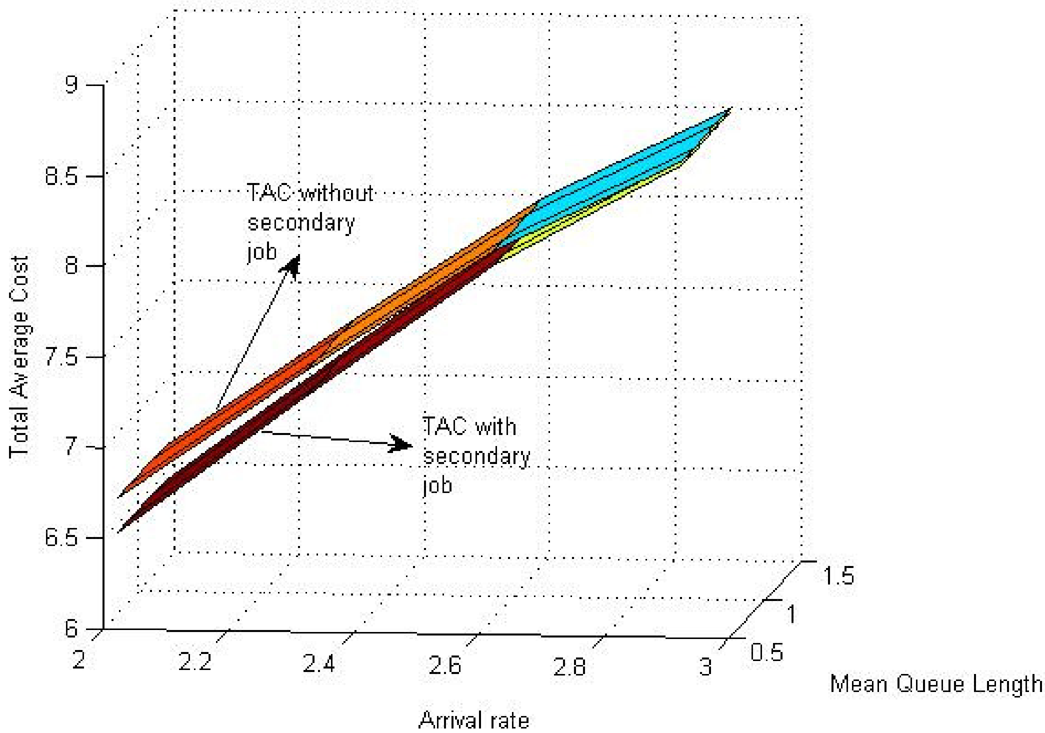

Figure 5: Arrival rate (Vs) Total Average Cost

(For $a = 2,\ b = 5,\ \mu = 2.5,\ \varepsilon = 10,\ \eta = 8,\ \pi = 0.2$)

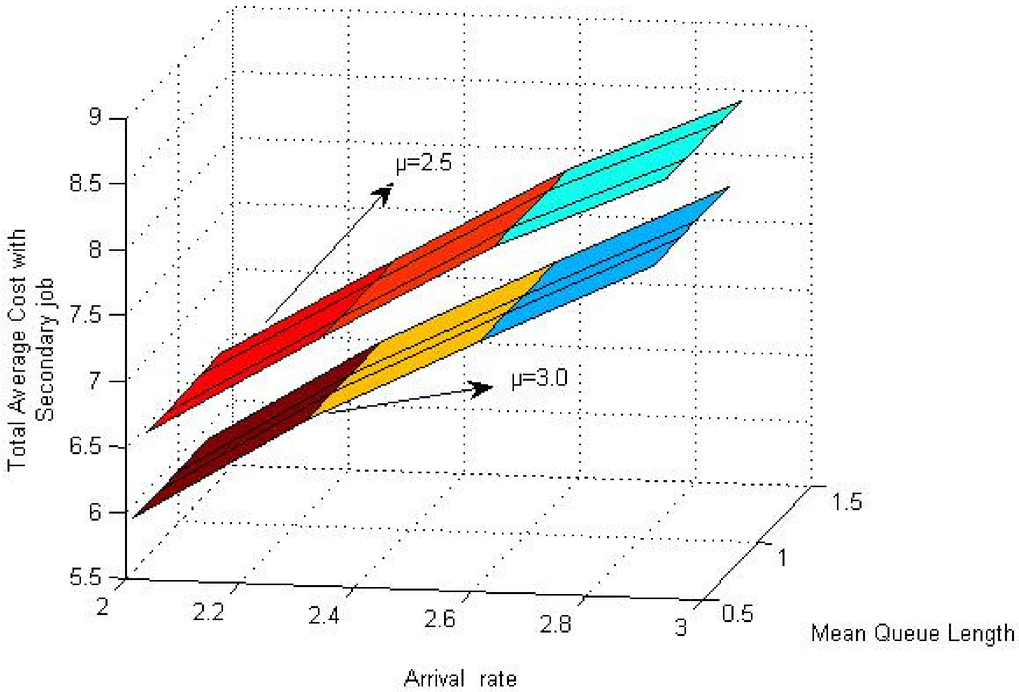

Figure 6: Arrival rate (Vs) Total Average Cost

(For $a = 2,\ b = 5,\ \varepsilon = 10,\ \eta = 8,\ \pi = 0.2$)

9. CONCLUSION

In this paper, the server breakdown with vacation interruption in a $M^X / G(a,b)/1$ queueing system is analysed. In the literature, all vacation models with bulk service considered that the server can start the service only when he completes the secondary job. But in emergency, the server has to terminate the secondary job and must give priority for primary job. Once the required level is reached to start the primary service, there is no point in continuing the secondary job. This model is proposed to overcome this difficulty and to make the system operate more efficiently. *The model so considered is unique in the sense that breakdown is introduced in a bulk service vacation interrupted queueing model, which is the significant contribution.* Probability generating function of the steady state queue size distribution at an arbitrary time epoch is obtained. Expressions for various performances are obtained.

A real time example from industry is discussed for the proposed model. *The results so obtained in this paper can be used for managerial decision to minimize the overall cost and search for the best operating policy in a waiting line system. The theoretical development of the model is justified with the numerical results which are consistent with the fact that when the server is allotted to secondary job, the idle time is properly utilized and hence the total average cost is minimized.* The model considers general bulk service with minimum of 'a' customers and maximum of 'b' customers. Hence, the probability generating function involves 'b' unknown constants. So we have the limitations of expressing the probability generating function in a closed form.

In the direction of future research, the model can be extended with close down and set up time concepts. An attempt may be made to derive the busy period distributions and idle period distributions. A discrete time model can also be developed.

Acknowledgement

The authors would be grateful to thank the referees and the editor for their insightful comments and suggestions, which helped in bringing the paper to its present form. *The first author of this article was supported by University Grants Commissions, India, Grant No. F MRP - 4983/14 (SERO/UGC).*

REFERENCES

1. Arumuganathan, R. and Jeyakumar, S. (2005). Steady state analysis of a bulk queue with multiple Vacations, setup times with N-policy and closedown times. Applied Mathematical Modeling, Vol. 29, No. 10: 972-986.

2. Balasubramanian, M. and Arumuganathan, R. (2011). Steady state analysis of a bulk arrival general bulk service queueing system with modififed M-vacation policy and variant arrival rate. International Journal of Operational Research, Vol. 11, No. 4: 383-407.
3. Borthakur, A. and Medhi, J. (1974) . A queueing system with arrival and services in batches of variable size. Cahiers du. C.E.R.O. Vol. 16: 117-126.
4. Chaudhry, M.L. and Templeton, J.G.C. (1983). A first course in bulk queues. New York, John Wiley and Sons.
5. Doshi, B.T. (1986). Queueing systems with vacations: a survey. Queueing Systems, Vol. 1: 29-66.
6. Gautam Choudhury and Mitali Deka (2013). A Batch Arrival Unreliable Server Bernoulli Vacation Queue with Two Phases of Service and Delayed Repair. International Journal of Operations Research, Vol. 10, No. 3: 134-152.
7. Haridass, M. and Arumuganathan, R. (2012). Optimum Cost Analysis of a Bulk Queueing system with Multiple Vacations and Restricted Admissibility of Arriving Batches. International Journal of Operations Research, Vol. 9, No. 1: 27 - 43.
8. Haridass, M. and Arumuganathan, R. (2011). Analysis of a batch arrival general bulk service queueing system with variant threshold policy for secondary jobs. International Journal of Mathematics in Operational Research, Vol. 3, No. 1: 56-77.
9. Haridass, M. and Arumuganathan, R. (2012). Analysis of a M^X/G (a, b)/1 queueing system with vacation interruption. RAIRO-Operations Research, Vol. 46: 305-334.
10. Jain, M. Agrawal, P K. (2009). Optimal policy for bulk queue with multiple types of server breakdown. International Journal of Operational Research, Vol. 4, No. 1: 35-54.
11. Jain, M. and Charu Bhargava. (2008). Bulk Arrival Retrial Queue with Unreliable Server and Priority Subscribers. International Journal of Operations Research, Vol. 5, No. 4: 242-259.
12. Jau-Chuan Ke and Kuo-Hsiung Wang. (2006). A General Input Queue with N Policy and Service Rate Depending on Bulk Size. International Journal of Operations Research, Vol. 3, No. 1: 7-15.
13. Jau-Chuan Ke, Chia-Huang Wu and Zhe George Zhang (2010). Recent Developments in Vacation Queueing Models: A Short Survey. International Journal of Operations Research, Vol. 7, No. 4: 3-8.
14. Jeyakumar, S. and Senthilnathan, B. (2012). A study on the behaviour of the server breakdown without interruption in a M^X/G (a, b)/1 queueing system with multiple vacations and closedown time. Applied Mathematics and Computation, Vol. 219: 2618-2633.
15. Ji-Hong, Li and Nai-Shuo Tian. (2007). The M/M/1 queue with working vacations and vacation interruptions. J. Syst. Sci. Eng., Vol. 16: 121-127.
16. Krishna Reddy, GV. and Natarajan, R. and Arumuganathan, R. (1998). Analysis of a bulk queue with N-policy, multiple vacations and setup times. Comput. Oper. Res. Vol. 25, No. 11: 957-967.
17. Li, W., Shi, D. and Chao, X. (1997). Reliability analysis of M/G/1 queueing system with server breakdowns and vacations. Journal of Applied Probability, Vol. 34: 546-555.
18. Madan, KC., Abu-Dayyeh, W. and Gharaibeh, M. (2003). Steady state analysis of two M^X/M (a, b)/1 queue models with random breakdowns. International Journal of Information and Management Sciences, Vol. 14, No. 3: 37-51.
19. Madhu Jain and Anamika Jain. (2014). Batch Arrival Priority Queueing Model with Second Optional Service and Server Breakdown. International Journal of Operations Research, Vol. 11, No. 4: 112-130.
20. Medhi J. (1984). Recent Developments in Bulk Queueing Models. Wiley Eastern Ltd. New Delhi.
21. Mian Zhang and Zhenting Hou. (2010). Performance analysis of M/G/1 queue with working vacations and vacation interruption. Journal of Computational and Applied Mathematics. Vol. 234, No. 10: 2977-2985.
22. Rakesh Kumar and Sumeet K. Sharma. (2012). A Multi-server Markovian Queueing System with Discouraged Arrivals and Retention of Reneged Customers. International Journal of Operations Research, Vol. 9 No. 4: 173-184.
23. Suganya, S. (2014). $M^X/G(a,b)/1$ with vacation interrupted, optional reservice and balking. International Journal of Scientific and Research Publications, Vol. 4, No. 11: 1-9.
24. Tian, N. and Zhang, S G. (2006). Vacation Queueing Models: Theory and Applications. Springer, New York.
25. Zhang, H. and Shi, D. (2009). The M/M/1 queue with Bernoulli-Schedule-Controlled vacation and vacation interruption. International Journal of Information and Management Sciences, Vol. 20, No. 4: 579-587.
26. Zhe George Zhang, Naishuo Tian, Ernie Love, and Zhanyou Ma. (2011). Analysis of Queues with an Imperfectly Repairable Server. International Journal of Operations Research, Vol. 8, No. 2: 1-7.