International Journal of
Operations Research

International Journal of Operations Research Vol. 12, No. 4, 111−117 (2015)

System reliability of a stochastic project network

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Received September 2015; Revised November 2015; Accepted December 2015

Abstract: Large projects such as the construction of a building, development of a management information system planning and designing of a new product, and implementation of a manufacturing system can be regarded as project networks whose activity durations are independent, finite, and multi-valued random variables. Such a project network is a multistate system with multistate components and its reliability for level $d-1$, i.e., the probability that the duration to complete all activities of the project is less than or equal to $d-1$, can be computed in terms of minimal path vectors to level d (named d-MPs here). The main objective of this paper is to present a simple algorithm to generate all d-MPs of such a stochastic project network for each level d in terms of minimal path sets. Two examples are given to illustrate how all d-MPs are generated by our algorithm and then the reliability of one example is computed.
Keywords — Stochastic project network, reliability analysis, management information system

1. INTRODUCTION

Reliability analysis often assumes that the system under study is represented by a probabilistic graph in a binary state model, and the system operates successfully if there exists at least one path from the source node to the sink node. In such a case, reliability is considered as a matter of connectivity only and so it does not seem to be reasonable as a model for some real-world phenomena. Large projects such as the construction of a building or a factory, development of a management information system, planning and designing of a new product, and implementation of a manufacturing system can be regarded as AOA project networks whose activity durations are independent, finite and multi-valued random variables. For such a project network, it is very practical and desirable to compute its reliability for level $d-1$, i.e., the probability that the duration to complete all activities of the project is less than or equal to $d-1$.

Generally, reliability evaluation can be carried out in terms of either minimal path sets (MPs) or minimal cut sets (MCs) in the binary state model case, and either d-MPs (i.e., minimal path vectors to level *d* (Aven, 1985), lower boundary points of level *d* (Hudson and Kapur. 1983), or upper critical connection vector to level *d* (El-Neweihi et al., 1978) or d-MCs (i.e., minimal cut vectors to level *d* (Aven, 1985)), upper boundary points of level *d* (Hudson and Kapur. 1983), or lower critical connection vector to level *d* (El-Neweihi et al., 1978) for each level *d* in the multistate model case. The AOA project network with random activity durations here can be treated as a multistate system of multistate components and so the need of an efficient algorithm to search for all of its d-MPs arises. The main purpose of this article is to present an algorithm to find all d-MPs of a stochastic project network in terms of minimal path sets. Two examples are given to illustrate how all d-MPs are generated and the reliability of one example is calculated in terms of d-MPs by further applying the state-space decomposition method (Aven, 1985). Finally, the computational complexity and storage requirement of the proposed algorithm are also discussed.

2. BASIC ASSUMPTIONS

A project may be represented by an AOA network in which each activity is represented by an arc and each event is represented by a node. Let $G=(N,A,L,U)$ be such an AOA project network with the unique source node s (i.e., start event) and unique sink node t (i.e., finish event), where N is the set of nodes, $A=\{a_i \mid 1\le i\le n\}$ is the set of arcs, $L=(l_1,l_2,\ldots,l_n)$ and $U=(u_1,u_2,\ldots,u_n)$, where l_i and u_i denote the minimum and maximum

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duration of each arc a_i respectively. Such an AOA project network is assumed to further satisfy the following assumptions:

1. The duration of activity a_i is an integer-valued random variable which takes integer values from l_i to u_i according to a given distribution.
2. The activity duration of a dummy activity is zero.
3. The durations for different activities are statistically independent.

Assumption 3 is made just for convenience. If it fails in practice, the proposed algorithm to search for all d-MPs is still valid except that the reliability computation in terms of such d-MPs should take the joint probability distributions of all activity durations into account.

Let $X = (x_1, x_2, \ldots, x_n)$ be a system-state vector (i.e., the current duration of activity a_i under X is x_i, where x_i takes integer values from l_i to u_i), and $V(X)$, the duration to complete all activities of the project under X. Such a function $V(\cdot)$ plays the role of the so-called structure function of a multistate system with $V(L) = h$ and $V(U) = k$. Under the system-state vector $X = (x_1, x_2, \ldots, x_n)$, the arc set A has the following three important subsets: $N_X = \{a_i \in A \mid x_i > l_i\}$, $Z_X = \{a_i \in A \mid x_i = l_i\}$, and $S_X = \{a_i \in N_X \mid V(X - e_i) < V(X)\}$, where $e_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{im})$, with $\delta_{ij} = 1$ if $j = i$ and $\delta_{ij} = 0$ if $j \neq i$. In fact, $A = S_X \cup (N_X \setminus S_X) \cup Z_X$ is a disjoint union set of A.

3. MODEL CONSTRUCTION

Suppose that $P^1, P^2, \ldots, P^m$ are the collection of all MPs of the AOA project network. For each P^j, the duration to complete all activities along this path is defined as the sum of the durations of all activities in it. This means that the duration to complete all activities of the project is equal to traverse the longest path from the source node s to the sink node t in the AOA network. Hence, the duration to complete all activities of the project under the system-state vector X is defined as

$$V(X) = \max_{1 \le j \le m}\{\textstyle\sum_i \{x_i \mid a_i \in P^j\}\}$$

Since $V(X)$ is non-decreasing in each argument (activity duration) under X, the project network with random activity durations can be treated as a multistate monotone system with the structure function $V(\cdot)$ (Aven, 1985).

A necessary condition for a system-state vector X to be a d-MP is stated in the following lemmas. Our algorithm relies mainly on such a result.

Theorem 1. If X is a d-MP, then $S_X \subseteq \cap_j \{P^j \mid \sum_i \{x_i \mid a_i \in P^j\} = d\}$

Proof: Suppose, on the contrary, that there exists an MP P^r with $\sum_i \{x_i \mid a_i \in P^r\} = d$ such that $S_X \setminus P^r = \{a_i \mid a_i \in S_X \text{ and } a_i \notin P^r\} \neq \varphi$. Choose an $a_i \in S_X - P^r$, and let $Y = X - e_i = (x_1, x_2, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_n) = (y_1, y_2, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n)$. Then $\sum_i \{y_i \mid a_i \in P^r\} = \sum_i \{x_i \mid a_i \in P^r\} = d$ due to the fact that $a_i \notin P^r$ and so $V(Y) = d$, which contradicts to the fact that $a_i \in S_X$. Hence, $S_X \subseteq P^j$ for each P^j with $\sum_i \{x_i \mid a_i \in P^j\} = d$, i.e., $S_X \subseteq \cap_j \{P^j \mid \sum_i \{x_i \mid a_i \in P^j\} = d\}$. **Q.E.D.**

Theorem 2. If X is a d-MP, then there exists at least one MP $P^r = \{a_{r1}, a_{r2}, \ldots, a_{rn_r}\}$, such that the following conditions are satisfied:

$$x_{r1} + x_{r2} + \ldots + x_{rn_r} = d \quad (1)$$

$$l_{ri} \le x_{ri} \le u_{ri} \quad \text{for all} \quad (2)$$

$$x_i = l_i \quad \text{for all } a_i \notin P^r. \quad (3)$$

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Proof: Let I be the non-empty index set of MPs such that:

$\sum_i \{x_i \mid a_i \in P^j\} = d$ for $i \in I$ and $\sum_i \{x_i \mid a_i \in P^j\} < d$ for $i \notin I$.

Choose a P^r with $r \in I$, say $P^r = \{a_{r1}, a_{r2}, \ldots, a_{rn_r}\}$, then

$\sum_i \{x_i \mid a_i \in P^j\} = d,$ i.e., $x_{r1} + x_{r2} + \ldots + x_{rn_r} = d$

$l_{ri} \le x_{ri} \le u_{ri}$ for all $a_i \in P^r$

By Theorem I, $A \setminus P^r \subseteq A \setminus \cap_{j \in I} \{P^j \mid \sum_i \{x_i \mid a_i \in P^j = d\} \subseteq A \setminus S_X = Z_X$,

i.e., $x_i = l_i$ for all $a_i \notin P^r$. **Q.E.D.**

Any system-state vector $X = (x_1, x_2, \ldots, x_n)$ that satisfies constraints (1) - (3) simultaneously will be taken as a d-MP candidate. A d-MP is obviously a d-MP candidate by Lemma 2. By definition, a d-MP candidate X is a d-MP if (1) $V(X) = d$, and (2) $N_X = S_X$. To check whether a d-MP candidate is a d-MP or not, the following lemma is needed.

Theorem 3. If the project network is parallel-series, then each d-MP candidate is a d-MP (where a network is called parallel-series if it can be represented as the parallel of its MPs $P^1, P^2, \ldots, P^m$ and $P^i \cap P^j = \varphi$).

Proof: Such a network can be considered as the parallel of its MPs $P^1, P^2, \ldots, P^m$. Let X be a d-MP candidate which is generated with respect to P^r according to Theorem 2. Since the network is parallel-series,

$P^j \cap P^r = \varphi$ for each $j \ne r$. Then

$\sum_i \{x_i \mid a_i \in P^r\} = d$ and $\sum_i \{x_i \mid a_i \in P^j\} = \sum_i \{l_i \mid a_i \in P^i\} \le V(L) = h < d$

In particular, $V(X) = \max_{1 \le j \le m} \{\sum_i \{x_i \mid a_i \in P^j\}\} = \sum_i \{x_i \mid a_i \in P^r\}$

and $N_X \subseteq \cap_j \{P^j \mid \sum_i \{x_i \mid a_i \in P^j\} = d\} = P^r$. Hence, X is a d-MP. **Q.E.D.**

4. ALGORITHM

Suppose that all MPs, $P^1, P^2, \ldots, P^m$, have been stipulated in advance (which can be obtained by assuming that each arc has two states only (Hura, 1983), the family of all d-MPs can be derived by the following steps:

Step 1. For each $P^r = \{a_{r1}, a_{r2}, \ldots, a_{rn_r}\}$, find all integer-valued solutions of the following constraints by applying an implicit enumeration method:

$x_{r1} + x_{r2} + \ldots + x_{rn_r} = d$

$l_{rj} \le x_{rj} \le u_{rj}$ for $j = 1, 2, \ldots, n_r$

Step 2. Set $x_i = l_i$ for all $a_i \notin P^r$.

Step 3. Obtain the family of d-MP candidates $X = (x_1, x_2, \ldots, x_n)$ by steps 1 and 2.

Step 4. Check each candidate X one at a time to determine whether it is a d-MP:

(a) If the network is parallel-series, then each candidate is a d-MP.

(b) If the network is non parallel-series, then test each candidate whether $V(X) = d$ and $N_X \subseteq \cap_j \{P^j \mid \sum_i \{x_i \mid a_i \in P^j\} = d\}$, i.e.,

(4.1) If there exists an $i \ne r$ such that $\sum_i \{x_i \mid a_i \in P^r\} > d$, then X is not a d-MP and go to step (4.4).

(4.2) Let index set $I = \{j \mid \sum_i \{x_i \mid a_i \in P^j\} = d\}$

(4.3) If there exists an $a_i \in A \setminus \cap_j \{P^j \mid \sum_i \{x_i \mid a_i \in P^j\} = d\}$ such that $x_i \ne l_i$, then X is not a d-MP, else X is a d-MP.

(4.4) Next candidate.

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5. EXAMPLES

The following two examples are used to illustrate the proposed algorithm:

Example 1.

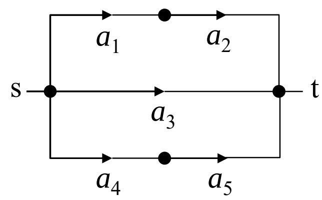

Figure 1: A parallel-series project network

It is known in this example that $L = (l_1, l_2, l_3, l_4, l_5) = (1,3,3,3,2)$ with $V(L) = 5$, $U = (u_1, u_2, u_3, u_4, u_5) = (2,5,5,4,3)$ with $V(U) = 7$, and there exists 3 MPs; $P^1 = \{a_1, a_2\}$, $P^2 = \{a_3\}$, $P^3 = \{a_4, a_5\}$.

Hence,

(1) $n = 5, m = 3,$ and

(2) The system has 3 levels: 5, 6, 7.

Given $d = 6$, the family of 6-Mps is derived as follows:

Step 1. For $P^1 = \{a_1, a_2\}$, find all integer-valued solutions of the following constraints by applying the implicit enumeration ,method:

$$x_1 + x_2 = 6$$
$$1 \le x_1 \le 2$$
$$3 \le x_2 \le 5$$

Step 2. Set $x_3 = 3, x_4 = 3,$ and $x_5 = 2.$

Step 3. Two 6-MP candidates $X^1 = (1,5,3,3,2)$ and $X^2 = (2,4,3,3,2)$ are obtained.

Step 4. Since the network is parallel-series, $(1,5,3,3,2)$ and $(2,4,3,3,2)$ are 6-MPs.

We repeat our algorithm to find candidates.

Step 1. For $P^2 = \{a_3\}$, find all integer-valued solutions of the following constraints by applying the implicit enumeration ,method:

$$x_3 = 6$$
$$3 \le x_3 \le 5$$

Step 2. Set $x_1 = 1, x_2 = 3,\ x_4 = 3,$ and $x_5 = 2.$

Step 3. No 6-MP candidate is obtained.

We repeat our algorithm to find candidates.

Step 1. For $P^3 = \{a_4, a_5\}$, find all integer-valued solutions of the following constraints by applying the implicit enumeration ,method:

$$x_4 + x_5 = 6$$
$$3 \le x_4 \le 4$$
$$2 \le x_5 \le 3$$

Step 2. Set $x_1 = 1, x_2 = 3,$ and $x_3 = 3.$

Step 3. Two 6-MP candidates $X^3 = (1,3,3,3,3)$ and $X^4 = (1,3,3,4,2)$ are obtained.

Step 4. Since the network is parallel-series, $(1,3,3,3,3)$ and $(1,3,3,4,2)$ are 6-MPs.

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Example 2.

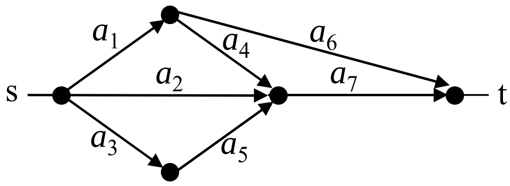

Figure 2: A small project network

It is known in this example that $L=(l_1,l_2,l_3,l_4,l_5,l_6,l_7)=(5,6,2,3,4,4,1)$ with $V(L)=9$, $U=(u_1,u_2,u_3,u_4,u_5,u_6,u_7)=(6,8,4,5,5,6,2)$ with $V(U)=12$, and there exists 4 MPs; $P^1=\{a_1,a_6\}$, $P^2=\{a_1,a_4,a_7\}$, $P^3=\{a_3,a_5,a_7\}$, and $P^4=\{a_2,a_7\}$.

Hence,

(1) $n=7, m=4,$ and

(2) The system has 4 levels: 9, 10, 11, 12.

Given $d=10$, the family of 10-Mps is derived as follows:

Step 1. For $P^1=\{a_1,a_6\}$, find all integer-valued solutions of the following constraints by applying the implicit enumeration ,method:

$$x_1+x_6=10$$
$$5\le x_1\le 6$$
$$4\le x_6\le 6$$

Step 2. Set $x_2=6, x_3=2, x_4=3, x_5=4$ and $x_7=1$.

Step 3. Two 10-MP candidates $X^1=(5,6,2,3,4,5,1)$ and $X^2=(6,6,2,3,4,4,1)$ are obtained.

Step 4. Check $X^1=(5,6,2,3,4,5,1)$ whether it is a 10-MP.

(4.1) $\sum_i\{x_i \mid a_i\in P^i\}\le 10$ for each P^i.

(4.2) $I=\{j \mid \sum_i\{x_i \mid a_i\in P^j\}=10\}=\{1\}$

(4.3) $X^1=(5,6,2,3,4,5,1)$ is a 10-MP.

(4.4) Next candidate (i.e., check $X^2=(6,6,2,3,4,4,1)$ whether it is a 10-MP)

(4.1) $\sum_i\{x_i \mid a_i\in P^i\}\le 10$ for each P^i.

(4.2) $I=\{j \mid \sum_i\{x_i \mid a_i\in P^j\}=10\}=\{1,2\}$

(4.3) $X^2=(6,6,2,3,4,4,1)$ is a 10-MP.

Table 1. Probability distributions of activity durations in Example 2

Activity	Duration	Probability	Activity	Duration	Probability
a_2	6	0.30	a_1	5	0.40
	7	0.50		6	0.60
	8	0.20	a_5	4	0.80
a_3	2	0.35		5	0.20
	3	0.60	a_6	4	0.20
	4	0.05		5	0.70
a_4	3	0.10		6	0.10
	4	0.60	a_7	1	0.70
	5	0.30		2	0.30

We repeat our algorithm to find candidates.

Step 1. For $P^2=\{a_1,a_4,a_7\}$, find all integer-valued solutions of the following constraints by applying the implicit enumeration ,method:

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$x_1 + x_4 + x_7 = 10$

$5 \le x_1 \le 6$

$3 \le x_4 \le 5$

$1 \le x_7 \le 2$

Step 2. Set $x_2 = 6, x_3 = 2, x_5 = 4,$ and $x_6 = 4.$

Step 3. Three 10-MP candidates $X^3 = (5,6,2,3,4,4,2)$, $X^4 = (5,6,2,4,4,4,1)$, and $X^5 = (6,6,2,3,4,4,1)$ are obtained.

Step 4. Check $X^3 = (5,6,2,3,4,4,2)$ whether it is a 10-MP.

(4.1) $\sum_i \{x_i \mid a_i \in P^i\} \le 10$ for each P^i.

(4.2) $I = \{j \mid \sum_i \{x_i \mid a_i \in P^j\} = 10\} = \{2\}$

(4.3) $X^3 = (5,6,2,3,4,4,2)$ is a 10-MP.

The result is listed in Table 2.

Table 2. List of all 5-MPs in Example 2

10-MP candidates	10-MP?
$X^1 = (5,6,2,3,4,5,1)$	Yes
$X^2 = (6,6,2,3,4,4,1)$	Yes
$X^3 = (5,6,2,3,4,4,2)$	Yes
$X^4 = (5,6,2,4,4,4,1)$	Yes
$X^5 = (6,6,2,3,4,4,1)$	Yes
$X^6 = (5,6,3,3,5,4,2)$	No
$X^7 = (5,6,4,3,4,4,2)$	No
$X^8 = (5,6,4,3,5,4,1)$	Yes
$X^9 = (5,8,2,3,4,4,2)$	No

6. RELIABILITY EVALUATION

If $Y^1, Y^2, \ldots, Y^{m_d}$ are the collection of all d-MPs of the project network, then the system reliability for level $d-1$ is defined as

$$R_{d-1} = \Pr\{X \mid V(X) \le d-1\} = 1 - \Pr\{X \mid V(X) \ge d\} = 1 - \Pr\{\cup_{i=1}^{m_d} \{X \mid X \ge Y^i\}\}.$$

To compute $\Pr\{\cup_{i=1}^{m_d} \{X \mid X \ge Y^i\}\}$ in terms of d-MPs, several methods such as inclusion-exclusion (El-Neweihi et al., 1978; Hudson and Kapur, 1983), disjoint subset (Hudson and Kapur, 1985), and state-space decomposition (Aven, 1985) are available. Here we apply the state-space decomposition method to Example 2 and obtain that $\Pr\{\cup_{i=1}^{m_d} \{X \mid X \ge Y^i\}\} = 0.9955$. Hence, $R_9 = P\{X \mid V(X) \le 9\} = 1 - 0.9955 = 0.0055$. Similarly, we have $R_{10} = 0.1955,$ $R_{11} = 0.6448,$ and $R_{12} = 1.000$. If level d of Example 2 is a random variable and its probability distribution is known as $\pi_9 = 0.2, \pi_{10} = 0.3,$ $\pi_{11} = 0.4,$, and $\pi_8 = 0.1$, then the system reliability is $R = \sum_{d=9}^{12} R_d \pi_d = 0.4177.$

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7. DISCUSSION

The numbers of non-negative integer solutions that satisfies (1) $x_{j1} + x_{j2} + \ldots + x_{jn_j} = d$ and (2) $l_{ji} \le x_{ji} \le u_{ji}$ for $i = 1, 2, \ldots, n_j$ are bounded by $\binom{n_j + d - 1}{d}$ and $\prod_{i=1}^{n_j}(u_{ji} - l_{ji} + 1)$, respectively. Hence, the total number of d-MP candidates of the proposed algorithm is bounded by $\sum_{j=1}^{m} \min\{\binom{n_j + d - 1}{d}, \prod_{i=1}^{n_j}(u_{ji} - l_{ji} + 1)\}$.

As each d-MP candidate is an n-tuple, it thus requires at most $O(n \cdot \sum_{j=1}^{m} \min\{\binom{n_j + d - 1}{d}, \prod_{i=1}^{n_j}(u_{ji} - l_{ji} + 1)\})$ units of storage space to save all d-MP candidates.

8. CONCLUSION

Given all MPs that are stipulated in advance, the proposed method can generate all d-MPs of a stochastic project network for each level d. The system reliability for level $d-1$, i.e., the probability that the duration to complete all activities of the project is less than or equal to $d-1$, can then be computed in terms of these d-MPs.

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