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A Batch Arrival Unreliable Queue with Two Types of General Heterogeneous Service and Delayed Repair under Repeated Service Policy

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Abstract: This paper deals with a single server $M^X/G/1$ queue under two types of general heterogeneous service with optional repeated service subject to server's breakdowns and delayed repair. We assume that customers arrive at the system according to a compound Poisson process with rate λ. The server provides two types of general heterogeneous service and a customer has the option to choose any type of service. After the completion of either type of service, the customer has the further option to repeat the same type of service. While the server is working with any types of service or repeated service, it may breakdown at any instant and the service channel will fail for a short interval of time. Furthermore, the concept of delay time is also introduced. We carry out an extensive analysis of this model. Finally, we obtain some important performance measure and reliability indices of this model.
Keywords — Two types of service, optional repeated service, elapsed times, remaining time, delay time.

1. INTRODUCTION

The study of queueing system under the special consideration with two phases of service is not new. A considerable attention has been paid by various researchers during last decade. Among them Madan (2000) was the first to consider such a model under the assumption that the first essential service time follows a general distribution law but the second optional service time is exponentially distributed. Medhi (2002) generalized the model by considering the second optional service is also governed by a general distribution. Choudhury (2003a) generalized this model for batch arrival queue (2003b). Choudhury and Paul (2005) investigated such a model under Bernoulli feedback mechanism.

The queueing system with two types of general heterogeneous service which is closely related to two phases of service was first investigated by Madan et, al (2004) for batch arrival queue by introducing the concept of re-service. In such a model the server provides two types of general heterogeneous services and an arriving unit (customer) has the option to choose any one of the two types of service before its service starts. However, if a customer is not satisfied by the service provided by the service channel then it has an option to go for re-service or repeat the service once again. Furthermore, Madan et. al (2005) and Al-khedhairi and Tadj (2007) have investigated similar type of models for batch arrival queue under Bernoulli vacation policy. Recently, Tadj and Ke (2008) have investigated a model with a choice of service and optional re service under a hysteric bulk service policy.

In most of the previous studies, it is assumed that the server is available in the service station on a permanent basis and service station never fails. But in many practical situations, we often meet with the cases where service station may fail and can be repaired. Similar phenomenon always occurs in the area of computer communication networks, in manufacturing systems, etc. Because the performance of such a system may be heavily affected by service station breakdown and delay in repair due to non-availability of the repairman or the apparatus needed for the repairs. This type of queueing system is known as queueing system with unreliable server and delayed repair. These systems with a repairable service station are well worth investigating from the queueing theory point of view as well as from the reliability point of view.

Recently, there have been several contribution considering queueing system with a repairable service station wherein the service channel is subject to breakdown or some other kind of service interruption, which are beyond control of the server or the management. In this context, Ke and Pearn (2004) have discussed an optimal management policy for a Markovian model. Wang (2004) investigated such a model for two phases of service. While Li et. al (1997a,b), Sengupta (1990), Takin and Sengupta (1998), Tang (1997), Madan (1989, 2003) and others have

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investigated some queueing system with service interruption wherein one of the underlying assumption is that as soon as the service channel fails, it instantaneously undergoes repairs. Recently Ke (2004, 2005, 2006a, 2006b, 2007) investigated some control policies for unreliable server, i.e., for breakdown systems. However, in many real life situations it may not be feasible to start the repairs immediately due to non-availability of the repairman in which case the system may also be turned off and there is a delay in repair during which the server stops providing service to the customers. In fact, this type of delay time was introduced by Madan (1994) for an $M/M/1$ queuing system with random breakdowns, general delay time and exponential repair time.

Although several aspects has been discussed for two types of service with optional repeated service and unreliable service system separately, however no works has been done to combine these features together for unreliable server queueing system with batch arrivals. Thus in this present paper we proposed to investigate an $M^X/G/1$ queue provides two types of general heterogeneous service with optional repeated service subject to server breakdown and delayed repair, where the concepts of delay time and repair time for both the types of service are also introduced. Furthermore, we have considered that the service time and repeated service time random variable follows different probability distributions and customers can repeat the same type of service only once. To this end, the methodology used will be based on the inclusion of supplementary variables.

The remaining part of the paper is organized as follows. In section 2 we briefly describe the mathematical model, section 3 deals with derivation of joint distributions of server's state and queue size under elapsed times, in section 4 we derive joint distributions of server's state and queue size under remaining times. Further derivation is also given of steady state queue size distribution at a departure epoch as classical generalization of the well-known *Pollaczek-Khinchin formula* for this model in section 5. Some important particular cases has been included in section 6. The system state probabilities has been derived in section 7. In section 8 we derive Laplace-Stieltjes Transform (LST) of busy period distribution following section 9 derivation of Laplace-Stieltjes Transform (LST) of waiting time distribution. Finally, derivation of some important reliability indices is done in section 10.

2. MATHEMATICAL MODEL

We consider an $M^X/G/1$ queueing system with two types of general heterogeneous service and unreliable server, where the number of individual primary customers arrive at the system according to compound Poisson process with arrival rate λ. The size of successive arriving batches are i.i.d random variables $X_1, X_2, \ldots,$ distributed with probability mass function $c_n = Prob\{X = n\};\, n \geq 1,\, PGFc(z) = E[z^X]$, and finite factorial moments $c_{[k]} = E\left[X\left(X-1\right)\ldots\left(X-k+1\right)\right]$.

The server provide two types of general heterogeneous service to the customers on first come first served (*FCFS*) basis, before its service starts, each customer has the option to select either type of service. i.e., each customer has the option to select either first type of service (*FTS*) denoted by S_1 with probability p_1 or second type of service (*STS*) denoted by S_2 with probability p_2 , where $p_1 + p_2 = 1$. Thus the time required by a customer to complete the service is given by,

$$S = \begin{cases} S_1 \text{ with probability } p_1 \\ S_2 \text{ with probobility } p_2 \end{cases}$$

It is assumed that service time $S_i, i = 1, 2$ (denoting type 1 and type 2 service respectively) (respectively S) of i^{th} type of service (respective total service time) follows general law of probability with distribution function(d.f.) $S_i(x)$ (respectively $S(x)$) , Laplace-Stieltjes Transform (LST) $S_i^*(\theta) = E[e^{(-\theta S_i)}]$ (respectively $S^*(\theta) = E[e^{-\theta S}]$ and finite k^{th} moments $s_i^{(k)}, i = 1, 2$ (respectively s^k) $(k \geq 1)$.

More specifically, the LST of the total service time after the choice of a service is given by

$$S^*(\theta) = p_1 S_1^*(\theta) + p_2 S_2^*(\theta) \tag{2.1}$$

Furthermore, it is assumed that as soon as either type of service completed by a customer, such an customer has further option to repeat the same type of service denoted by B_i once only with probability q_i or leave the system with probability $(1-q_i)$, for $i = 1, 2$. Thus the total service time required to a customer to complete the i^{th} type of service which may be called modified service time $i = 1, 2$ is given by

$$S_i = \begin{cases} S_i + B_i \text{ with probability } q_i \\ S_i \text{ with probability } (1-q_i) \end{cases}$$

Assuming that repeated service time random variable follows general distribution law with probability distribution function $B_i(x)$, LST $B_i^*(\theta) = E[e^{-\theta B_i}]$ and finite k^{th} moments $b_i^{(k)}$ for $i = 1, 2$.

Clearly the LST $S_i^*(\theta)$ of S_i for $i = 1, 2$ is

$$S_i^*(\theta) = (1-q_i)S_i^*(\theta) + q_i S_i^*(\theta)B_i^*(\theta) \tag{2.2}$$

Now utilizing equation (2.2) in (2.1) for $i = 1, 2$, we get the LST of the modified service time is given by

$$S^*(\theta) = \{(1-q_i) + q_1 B_1^*(\theta)\}p_1 S_1^*(\theta) + \{(1-q_2) + q_2 B_2^*(\theta)\}p_2 S_2^*(\theta) \tag{2.3}$$

This type of model is known as $M^X / G / 1$ queue with two types of general heterogeneous service and optional repeated service and has been studied by Madan et, al (2004). Using Kendal's notation the model can be considered an $M^X / \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} / 1$ queue, where $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ stands for two types of parallel general heterogeneous service. Now, for further development for such a type of model we may further introduce the concept of server's breakdown and delay process, where the server is working with any type of service or repeated service, it may breakdown at any time for a short interval of time. The breakdown i.e., server's life times are generated by exogenous Poisson process with rates α_1 for FTS or FTRS (i.e., first type of repeated service) and α_2 for STS or STRS (i.e., second type of repeated service). As soon as breakdown occurs the server is sent for repair during which time it stops providing service to the arriving customers and wait for repair to start, which we may refer to as waiting period of the server. We define this waiting as delay time. The delay time denoted by $D_i, i = 1, 2$ (denoting type 1 and type 2 service or repeated service respectively) of the server for i^{th} type of service or repeated service follows a general law of distribution with distribution function $G_i(y)$, LST $G_i^*(\theta) = E[e^{-\theta D_i}]$ and finite k^{th} moments $g_i^{(\mathrm{k})}$ for $i = 1, 2$. Similarly the repair time R_i of the server for i^{th} type of service or repeated service follows general law of distribution with distribution function $V_i(y)$, LST $V_i^*(\theta) = E[\mathrm{e}^{-\theta R_i}]$ and finite k^{th} moments $v_i^{(k)}$ for $i = 1, 2$. Immediately after the server is fixed (i.e., repaired), the server is ready to start its remaining service to the customers in both types of service and in this case the service times are cumulative, which we may referred to as generalized service time.

Further we assume that arrival process, service time or repeated service time, server's life time, server's delay time and server's repair time random variables are mutually independent of each other. We can explain the situation with the help of the following figure.

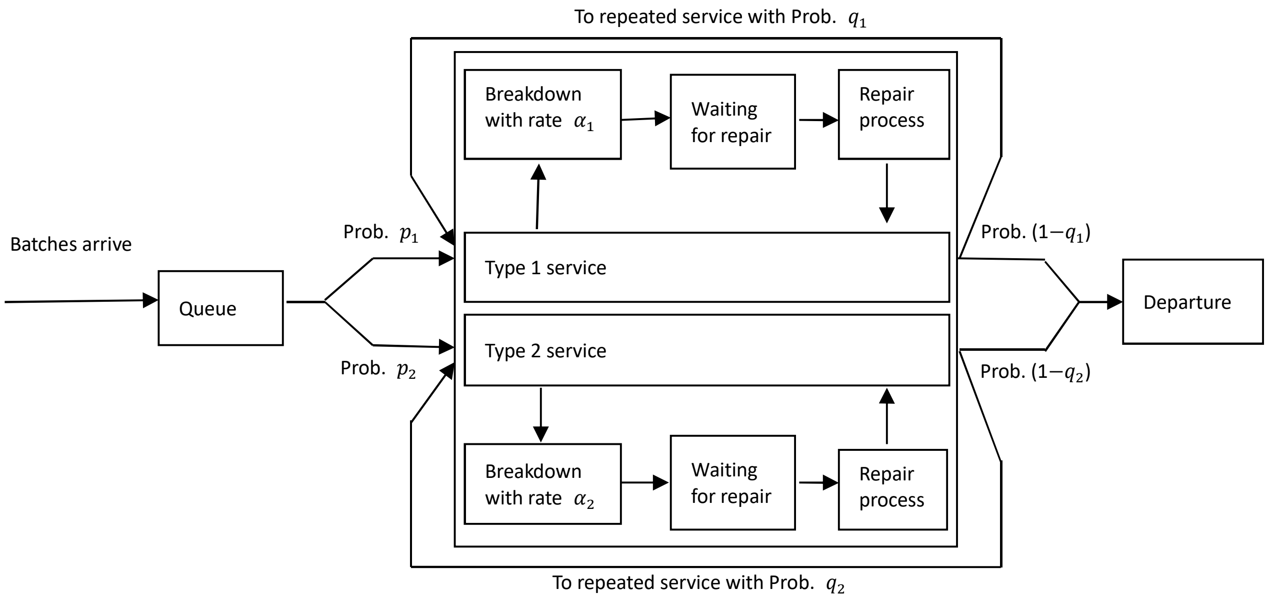

Figure 1. A single server batch arrival queue under two types of general heterogeneous service with optional repeated service subject to server's breakdown and delayed repair

Now if we denote $H_i, i=1,2$, as generalized repair service time for i^{th} type of repair service, $H_i(x)$ and $H_i^*(\theta)=E[e^{-\theta H_i}]$ as its d.f. and LST, respectively and finite k^{th} moments $h_i^{(k)}$ for $i=1,2$, then we have

$$\begin{aligned} H_i^*(\theta) &= \sum_{n=0}^{\infty}\int_0^{\infty} e^{-\theta x}e^{-\alpha_i x}\left[\frac{(\alpha_i x)^n}{n!}\right]\left[G_i^*(\theta)V_i^*(\theta)\right]dS_i(x) \\ &= S_i^*\left(\theta+\alpha_i\left(1-G_i^*(\theta)V_i^*(\theta)\right)\right) \quad \text{for} \quad i=1,2 \end{aligned} \tag{2.4}$$

Similarly, if we denote H_i^R as generalized service time for i^{th} type of repeated service, $H_i^R(x)$ and $H_i^{R^*}(\theta)=E[e^{-\theta H_i^R}]$ as its d.f. and LST, respectively and finite k^{th} moments $h_i^{R^{(k)}}$ for $i=1,2$, then we have

$$\begin{aligned} H_i^{R^*}(\theta) &= \sum_{n=0}^{\infty}\int_0^{\infty} e^{-\theta x}e^{-\alpha_i x}\left[\frac{(\alpha_i x)^n}{n!}\right]\left[G_i^*(\theta)V_i^*(\theta)\right]^n dB_i(x) \\ &= B_i^*\left(\theta+\alpha_i\left(1-G_i^*(\theta)V_i^*(\theta)\right)\right) \quad \text{for} \quad i=1,2 \end{aligned} \tag{2.5}$$

The first two moments for $i=1,2$ are found to be

$$h_i^{(1)} = -\left.\frac{dH_i^*(\theta)}{d\theta}\right|_{\theta=0} = s_i^{(1)}\left\{1+\alpha_i\left(g_i^{(1)}+v_i^{(1)}\right)\right\} \tag{2.6}$$

$$h_i^{R^{(1)}} = -\left.\frac{dH_i^{R^*}(\theta)}{d\theta}\right|_{\theta=0} = b_i^{(1)}\left\{1+\alpha_i\left(g_i^{(1)}+v_i^{(1)}\right)\right\} \tag{2.7}$$

And

$$h_i^{(2)} = (-1)^2 \left.\frac{d^2 H_i^*(\theta)}{d\theta^2}\right|_{\theta=0} = s_i^{(2)} \left\{1 + \alpha_i \left(g_i^{(1)} + v_i^{(1)}\right)\right\}^2 + \alpha_i s_i^{(1)} \left\{g_i^{(2)} + v_i^{(2)} + 2g_i^{(1)} v_i^{(1)}\right\} \tag{2.8}$$

$$h_i^{R^{(2)}} = (-1)^2 \left.\frac{d^2 H_i^{R^*}(\theta)}{d\theta^2}\right|_{\theta=0} = b_i^{(2)} \left\{1 + \alpha_i \left(g_i^{(1)} + v_i^{(1)}\right)\right\}^2 + \alpha_i b_i^{(1)} \left\{g_i^{(2)} + v_i^{(2)} + 2g_i^{(1)} v_i^{(1)}\right\} \tag{2.9}$$

Thus the variances of the generalized service time (denoted by σ^2) and generalized service time for repeated service (denoted by σ_R^2) for $i = 1, 2$ are

$$\sigma^2 = \left(s_i^{(2)} - \left(s_i^{(1)}\right)^2\right) \left\{1 + \alpha_i \left(g_i^{(1)} + v_i^{(1)}\right)\right\}^2 + \alpha_i s_i^{(1)} \left\{g_i^{(2)} + v_i^{(2)} + 2g_i^{(1)} v_i^{(1)}\right\} \tag{2.10}$$

$$\sigma_R^2 = \left(b_i^{(2)} - \left(b_i^{(1)}\right)^2\right) \left\{1 + \alpha_i \left(g_i^{(1)} + v_i^{(1)}\right)\right\}^2 + \alpha_i b_i^{(1)} \left\{g_i^{(2)} + v_i^{(2)} + 2g_i^{(1)} v_i^{(1)}\right\} \tag{2.11}$$

Now the total generalized service time provided by the server to a customer denoted by A and LST of total general service time can be written as

$$\begin{aligned} A^*(\theta) = &\left\{\left(1 - q_1\right) + q_1 B_1^*\left(\theta + \alpha_1\left(1 - G_1^*(\theta) V_1^*(\theta)\right)\right)\right\} p_1 S_1^*\left(\theta + \alpha_1\left(1 - G_1^*(\theta) V_1^*(\theta)\right)\right) \\ &+ \left\{\left(1 - q_2\right) + q_2 B_2^*\left(\theta + \alpha_2\left(1 - G_2^*(\theta) V_2^*(\theta)\right)\right)\right\} p_2 S_2^*\left(\theta + \alpha_2\left(1 - G_2^*(\theta) V_2^*(\theta)\right)\right) \end{aligned} \tag{2.12}$$

where $A^*(\theta)$ denotes the LST of A.

3. JOINT DISTRIBUTIONS OF SERVER'S STATE AND QUEUE SIZE UNDER ELAPSED TIMES

In this section, we first set up the system equations for its stationary queue size distribution by treating elapsed service time, elapsed delay time and elapsed repair time for both the types of service and repeated service as supplementary variables. Then we solve the equations and derive LST of the probability generating function (PGF)s of the stationary queue size distribution. Let $N_Q(t)$ be the queue size (including the one being served, if any) at time t, $S_i^0(t)$ and $B_i^0(t)$ be the elapsed service time and elapsed repeated service time of the customer for the i^{th} types of service at time t for $i = 1, 2$ denoting type 1 and type 2 service respectively, $D_i^0(t)$ and $R_i^0(t)$ be the elapsed delay and repair time respectively of the server for the i^{th} types of service during which breakdown occurs in the system at time t, where $i = 1, 2$ (denoting type 1 and type 2 service respectively).

Let us now introduce the following random variable:

$$Y(t)=\begin{cases}0, \text{ if the server is idle at time } t\\ 1, \text{ if the server is busy with type 1 service at time } t\\ 2, \text{ if the server is busy with type 2 service at time } t\\ 3, \text{ if the server is busy with repeating type 1 service at time } t\\ 4, \text{ if the server is busy with repeating type 2 service at time } t\\ 5, \text{ if the server is waiting for repair during type 1 service at time } t\\ 6, \text{ if the server is waiting for repair during type 2 service at time } t\\ 7, \text{ if the server is waiting for repair during type 1 repeated service at time } t\\ 8, \text{ if the server is waiting for repair during type 2 repeated service at time } t\\ 9, \text{ if the server is under repair during type 1 service at time } t\\ 10, \text{ if the server is under repair during type 2 service at time } t\\ 11, \text{ if the server is under repair during type 1 repeated service at time } t\\ 12, \text{ if the server is under repair during type 2 repeated service at time } t\end{cases}$$

We now obtain a bivariate Markov process $\{N_Q(t), Y(t)\}$ by introducing the supplementary variables $S_i^0\left(t\right), B_i^0\left(t\right), D_i^0(t)$ and $R_i^0(t)$ for $i=1,2$ and define following limiting probabilities.

$$E_0=\lim_{t\to\infty}P_r\left\{N_Q(t)=0, Y(t)=0\right\}$$

and for $n\geq 1$

$$\begin{aligned}
P_{n,1}(x)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=1; x<S_1^0(t)\leq x+dx\right\}; x>0\\
P_{n,2}(x)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=2; x<S_2^0(t)\leq x+dx\right\}; x>0\\
Q_{n,1}(x)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=3; x<B_1^0(t)\leq x+dx\right\}; x>0\\
Q_{n,2}(x)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=4; x<B_2^0(t)\leq x+dx\right\}; x>0\\
K_{n,1}^P(x,y)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=5; y<D_1^0(t)\leq y+dy \mid S_1^0(t)=x\right\}; (x,y)>0\\
K_{n,2}^P(x,y)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=6; y<D_2^0(t)\leq y+dy \mid S_2^0(t)=x\right\}; (x,y)>0\\
K_{n,1}^Q(x,y)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=7; y<D_1^0(t)\leq y+dy \mid B_1^0(t)=x\right\}; (x,y)>0\\
K_{n,2}^Q(x,y)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=8; y<D_2^0(t)\leq y+dy \mid B_2^0(t)=x\right\}; (x,y)>0\\
R_{n,1}^P(x,y)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=9; y<R_1^0(t)\leq y+dy \mid S_1^0(t)=x\right\}; (x,y)>0\\
R_{n,2}^P(x,y)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=10; y<R_2^0(t)\leq y+dy \mid S_2^0(t)=x\right\}; (x,y)>0\\
R_{n,1}^Q(x,y)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=11; y<R_1^0(t)\leq y+dy \mid B_1^0(t)=x\right\}; (x,y)>0\\
R_{n,2}^Q(x,y)dx&=\lim_{t\to\infty}P_r\left\{N_Q(t)=n, Y(t)=12; y<R_2^0(t)\leq y+dy \mid B_2^0(t)=x\right\}; (x,y)>0
\end{aligned}$$

Further, it is assumed that $S_i(0)=0, \mathrm{S}_i(\infty)=1, B_i(0)=0, \mathrm{B}_i(\infty)=1, G_i(0)=0, G_i(\infty)=1,$ $V_i(0)=0, V_i(\infty)=1$ for $i=1,2$ and for $i=1,2;$ $S_i(x)$ is continuous at $x=0$, $B_i(x)$ is continuous at $x=0$, $G_i(x)$ is continuous at $y=0$ and $V_i(y)$ is continuous at $y=0$, respectively, so that

$$\mu_i(x)dx = \frac{dS_i(x)}{1-S_i(x)},\ \eta_i(x)dx = \frac{dB_i(x)}{1-B_i(x)},\ \xi_i(y)dy = \frac{dG_i(y)}{1-G_i(y)},\ \gamma_i(y)dy = \frac{dV_i(y)}{1-V_i(y)}$$

are the first order differential (hazard rate) function of $S_i,\ B_i,\ D_i$ and R_i respectively for $i = 1,\ 2.$

3.1 THE STEADY STATE EQUATIONS

The Kolmogorov forward equations to analyze the limiting behavior of this system under steady state conditions (e.g. see Cox (1995)) for $i = 1, 2$ can be written as follows:

$$\frac{d}{dx}P_{n,i}(x) + \left[\lambda + \alpha_i + \mu_i(x)\right]P_{n,i}(x) = \lambda\sum_{k=1}^{n} c_k P_{n-k,i}(x) + \int_0^\infty \gamma_i(y)R^P_{n,i}(x,y)\ dy\ ;\ n \geq 1 \tag{3.1}$$

$$\frac{d}{dx}Q_{n,i}(x) + \left[\lambda + \alpha_i + \eta_i(x)\right]Q_{n,i}(x) = \lambda\sum_{k=1}^{n} c_k Q_{n-k,i}(x) + \int_0^\infty \gamma_i(y)R^Q_{n,i}(x,y)\ dy\ ;\ n \geq 1 \tag{3.2}$$

$$\frac{d}{dy}K^P_{n,i}(x,y) + \left[\lambda + \xi_i(y)\right]K^P_{n,i}(x,y) = \lambda\sum_{k=1}^{n} c_k K^P_{n-k,i}(x,y)\ ;\ n \geq 1 \tag{3.3}$$

$$\frac{d}{dy}K^Q_{n,i}(x,y) + \left[\lambda + \xi_i(y)\right]K^Q_{n,i}(x,y) = \lambda\sum_{k=1}^{n} c_k K^Q_{n-k,i}(x,y)\ ;\ n \geq 1 \tag{3.4}$$

$$\frac{d}{dy}R^P_{n,i}(x,y) + \left[\lambda + \gamma_i(y)\right]R^P_{n,i}(x,y) = \lambda\sum_{k=1}^{n} c_k R^P_{n-k,i}(x,y)\ ;\ n \geq 1 \tag{3.5}$$

$$\frac{d}{dy}R^Q_{n,i}(x,y) + \left[\lambda + \gamma_i(y)\right]R^Q_{n,i}(x,y) = \lambda\sum_{k=1}^{n} c_k R^Q_{n-k,i}(x,y)\ ;\ n \geq 1 \tag{3.6}$$

$$\lambda E_0 = \sum_{i=1}^{2}\left[(1-q_i)\int_0^\infty P_{1,1}(x)\mu_i(x)\ dx + \int_0^\infty Q_{1,i}(x)\eta_i(x)\ dx\right] \tag{3.7}$$

Where E_0 is the steady state probability of server being idle, $P_{0,i}(x) = 0$, $Q_{0,i}(x) = 0$, $K^P_{0,i}(x,y) = 0$, $K^Q_{0,i}\left(x,y\right) = 0$, $R^P_{0,i}\left(x,y\right) = 0$ and $R^Q_{0,i}\left(x,y\right) = 0$ for $i = 1,2$ occurring in the equations (3.1) – (3.6). The set of equations are to be solved subject to the following boundary conditions at $x = 0$ for $i = 1,2$;

$$\begin{aligned} P_{n,i}\left(0\right) &= \left(1-q_1\right)p_i\int_0^\infty P_{n+1,1}\left(x\right)\mu_1\left(x\right)dx + \left(1-q_2\right)p_i\int_0^\infty P_{n+1,2}\left(x\right)\mu_2\left(x\right)dx \\ &\quad + p_i\int_0^\infty Q_{n+1,1}\left(x\right)\eta_1\left(x\right)dx + p_i\int_0^\infty Q_{n+1,2}\left(x\right)\eta_2\left(x\right)dx + \lambda c_n E_0 p_i\ ;\ n \geq 1 \end{aligned} \tag{3.8}$$

$$Q_{n,i}\left(0\right) = q_i\int_0^\infty P_{n,i}\left(x\right)\mu_i\left(x\right)dx?\ \ n \geq 1 \tag{3.9}$$

and at $y = 0$ for $i = 1,2$ and fixed value of x;

$$K^P_{n,i}\left(x,0\right) = \alpha_i P_{n,i}\left(x\right);\ n \geq 1 \tag{3.10}$$

$$K^Q_{n,i}\left(x,0\right) = \alpha_i Q_{n,i}\left(x\right);\ n \geq 1 \tag{3.11}$$

$$R^P_{n,i}\left(x,0\right) = \int_0^\infty \xi_i\left(y\right)K^P_{n,1}\left(x,y\right)dy\ ;\ n \geq 1 \tag{3.12}$$

$$R^Q_{n,i}\left(x,0\right) = \int_0^\infty \xi_i\left(y\right)K^Q_{n,1}\left(x,y\right)dy\ ;\ n \geq 1 \tag{3.13}$$

and the normalizing condition

$$E_0+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\Bigg[\int_0^\infty P_{n,i}(x)\,dx+\int_0^\infty Q_{n,i}(x)\,dx+\int_0^\infty\int_0^\infty K_{n,i}^{P}(x,y)\,dx\,dy$$
$$+\int_0^\infty\int_0^\infty K_{n,i}^{Q}(x,y)\,dx\,dy+\int_0^\infty\int_0^\infty R_{n,i}^{P}(x,y)\,dx\,dy+\int_0^\infty\int_0^\infty R_{n,i}^{Q}(x,y)\,dx\,dy\Bigg]=1 \tag{3.14}$$

3.2 THE MODEL SOLUTION

We introduce the following relations for $i=1,2$ and for $n\geq 1$ to solve the above equations

$$\bar{P}_{n,i}(x)=\frac{P_{n,i}(x)}{1-S_i(x)} \tag{3.15}$$

$$\bar{Q}_{n,i}(x)=\frac{Q_{n,i}(x)}{1-B_i(x)} \tag{3.16}$$

$$\bar{K}_{n,i}^{P}(x,y)=\frac{K_{n,i}^{P}(x,y)}{\big(1-S_i(x)\big)\big(1-G_i(y)\big)} \tag{3.17}$$

$$\bar{K}_{n,i}^{Q}(x,y)=\frac{K_{n,i}^{Q}(x,y)}{\big(1-B_i(x)\big)\big(1-G_i(y)\big)} \tag{3.18}$$

$$\bar{R}_{n,i}^{P}(x,y)=\frac{R_{n,i}^{P}(x,y)}{\big(1-S_i(x)\big)\big(1-V_i(y)\big)} \tag{3.19}$$

$$\bar{R}_{n,i}^{Q}(x,y)=\frac{R_{n,i}^{Q}(x,y)}{\big(1-B_i(x)\big)\big(1-V_i(y)\big)} \tag{3.20}$$

Equations (3.1) – (3.13) are then converted into the following equations, for $i=1,2$

$$\frac{d}{dx}\bar{P}_{n,i}(x)+\big[\lambda+\alpha_i\big]\bar{P}_{n,i}(x)=\lambda\sum_{k=1}^{n}c_k\bar{P}_{n-k,i}(x)+\int_0^\infty\bar{R}_{n,i}^{P}(x,y)\,dV_i(y)\ ;\ n\geq 1 \tag{3.21}$$

$$\frac{d}{dx}\bar{Q}_{n,i}(x)+\big[\lambda+\alpha_i\big]\bar{Q}_{n,i}(x)=\lambda\sum_{k=1}^{n}c_k\bar{Q}_{n-k,i}(x)+\int\ \bar{R}_{n,i}^{Q}(x,y)\,dV_i(y)\ ;\ n\geq 1 \tag{3.22}$$

$$\frac{d}{dy}\bar{K}_{n,i}^{P}(x,y)+\lambda\bar{K}_{n,i}^{P}(x,y)=\lambda\sum_{k=1}^{n}c_k\bar{K}_{n-k,i}^{P}(x,y)\ ;\ n\geq 1 \tag{3.23}$$

$$\frac{d}{dy}\bar{K}_{n,i}^{Q}(x,y)+\lambda\bar{K}_{n,i}^{Q}(x,y)=\lambda\sum_{k=1}^{n}c_k\bar{K}_{n-k,i}^{Q}(x,y)\ ;\ n\geq 1 \tag{3.24}$$

$$\frac{d}{dy}\bar{R}_{n,i}^{P}(x,y)+\lambda\bar{R}_{n,i}^{P}(x,y)=\lambda\sum_{k=1}^{n}c_k\bar{R}_{n-k,i}^{P}(x,y)\ ;\ n\geq 1 \tag{3.25}$$

$$\frac{d}{dy}\bar{R}_{n,i}^{Q}(x,y)+\lambda\bar{R}_{n,i}^{Q}(x,y)=\lambda\sum_{k=1}^{n}c_k\bar{R}_{n-k,i}^{Q}(x,y)\ ;\ n\geq 1 \tag{3.26}$$

$$\lambda E_0=\sum_{i=1}^{2}\left[\big(1-q_i\big)\int_0^\infty\bar{P}_{1,i}(x)\,dS_i(x)+\int_0^\infty\bar{Q}_{1,i}(x)\,dB_i(x)\right] \tag{3.27}$$

$$\bar{P}_{n,i}(0)=\big(1-q_1\big)p_i\int_0^\infty\bar{P}_{n+1,1}(x)\,dS_1(x)+\big(1-q_2\big)p_i\int_0^\infty\bar{P}_{n+1,2}(x)\,dS_2(x)$$
$$+p_i\int_0^\infty\bar{Q}_{n+1,1}(x)\,dB_1(x)+p_i\int_0^\infty\bar{Q}_{n+1,2}(x)dB_2(x)+\lambda c_nE_0p_i\ ;\ n\geq 1 \tag{3.28}$$

$$\bar{Q}_{n,i}\left(0\right)=q_i\int_0^{\infty}\bar{P}_{n,i}\left(x\right)dS_i\left(x\right);\ n\geq 1 \tag{3.29}$$

$$\bar{K}^{P}_{n,i}\left(x,0\right)=\alpha_i\bar{P}_{n,i}\left(x\right);\ n\geq 1 \tag{3.30}$$

$$\bar{K}^{Q}_{n,i}\left(x,0\right)=\alpha_i\bar{Q}_{n,i}\left(x\right);\ n\geq 1 \tag{3.31}$$

$$\bar{R}^{P}_{n,i}\left(x,0\right)=\int_0^{\infty}\bar{K}^{P}_{n,i}\left(x,y\right)dG_i\left(y\right);\ n\geq 1 \tag{3.32}$$

$$\bar{R}^{Q}_{n,i}\left(x,0\right)=\int_0^{\infty}\bar{K}^{Q}_{n,i}\left(x,y\right)dG_i\left(y\right);\ n\geq 1 \tag{3.33}$$

We now define the following steady state probability generating function (PGF)s for $|z|<1$ and $i=1,2$ to solve the system of equations from (3.21) to (3.33)

$$\bar{P}_i\left(x;z\right)=\sum_{n=1}^{\infty}\bar{P}_{n,i}\left(x\right)z^n \qquad \bar{Q}_i\left(x;z\right)=\sum_{n=1}^{\infty}\bar{Q}_{n,i}\left(x\right)z^n$$

$$\bar{K}^{P}_i\left(x,y;z\right)=\sum_{n=1}^{\infty}\bar{K}^{P}_{n,i}\left(x,y\right)z^n \qquad \bar{K}^{Q}_i\left(x,y;z\right)=\sum_{n=1}^{\infty}\bar{K}^{Q}_{n,i}\left(x,y\right)z^n$$

$$\bar{R}^{P}_i\left(x,y;z\right)=\sum_{n=1}^{\infty}\bar{R}^{P}_{n,i}\left(x,y\right)z^n \qquad \bar{R}^{Q}_i\left(x,y;z\right)=\sum_{n=1}^{\infty}\bar{R}^{Q}_{n,i}\left(x,y\right)z^n$$

Let $a(z)=\lambda\left(1-c(z)\right)$, then solving equations from (3.23) to (3.26) in usual manner we get the following equations:

$$\bar{K}^{P}_i\left(x,y;z\right)=\bar{K}^{P}_i\left(x,0;z\right)\exp\left\{-a\left(z\right)y\right\};\left(x,y\right)>0\ \text{for}\ i=1,2 \tag{3.34}$$

$$\bar{K}^{Q}_i\left(x,y;z\right)=\bar{K}^{Q}_i\left(x,0;z\right)\exp\left\{-a\left(z\right)y\right\};\left(x,y\right)>0\ \text{for}\ i=1,2 \tag{3.35}$$

$$\bar{R}^{P}_i\left(x,y;z\right)=\bar{R}^{P}_i\left(x,0;z\right)\exp\left\{-a\left(z\right)y\right\};\left(x,y\right)>0\ \text{for}\ i=1,2 \tag{3.36}$$

$$\bar{R}^{Q}_i\left(x,y;z\right)=\bar{R}^{Q}_i\left(x,0;z\right)\exp\left\{-a\left(z\right)y\right\};\left(x,y\right)>0\ \text{for}\ i=1,2 \tag{3.37}$$

Simplifying equations (3.30) and (3.31), we obtain the following results for $i=1,2$

$$\bar{K}^{P}_i\left(x,0;z\right)=\alpha_i\bar{P}_i\left(x;z\right) \tag{3.38}$$

$$\bar{K}^{Q}_i\left(x,0;z\right)=\alpha_i\bar{Q}_i\left(x;z\right) \tag{3.39}$$

Now solving equations (3.32) and (3.33) and using equations (3.34) and (3.35) we obtain the following equations

$$\bar{R}^{P}_i\left(x,0;z\right)=\bar{K}^{P}_i\left(x,0;z\right)G^*_i\left(a\left(z\right)\right),\ \text{for}\ i=1,2 \tag{3.40}$$

$$\bar{R}^{Q}_i\left(x,0;z\right)=\bar{K}^{Q}_i\left(x,0;z\right)G^*_i\left(a\left(z\right)\right),\ \text{for}\ i=1,2 \tag{3.41}$$

Utilizing the equations (3.38) and (3.39) into equations (3.40) and (3.41), we get

$$\bar{R}^{P}_i\left(x,0;z\right)=\alpha_i\bar{P}_i\left(x;z\right)G^*_i\left(a\left(z\right)\right),\ \text{for}\ i=1,2 \tag{3.42}$$

$$\bar{R}^{Q}_i\left(x,0;z\right)=\alpha_i\bar{Q}_i\left(x;z\right)G^*_i\left(a\left(z\right)\right),\ \text{for}\ i=1,2 \tag{3.43}$$

Now, solving differential equations (3.21) and (3.22), we get two differential equations of Lagrangian type whose solutions are given by:

$$\bar{P}_i\left(x;z\right)=\bar{P}_i\left(0;z\right)\exp\left\{-\chi_i\left(z\right)x\right\};x>0,\ \text{for}\ i=1,2 \tag{3.44}$$

$$\bar{Q}_i\left(x;z\right)=\bar{Q}_i\left(0;z\right)\exp\left\{-\chi_i\left(z\right)x\right\};x>0,\ \text{for}\ i=1,2 \tag{3.45}$$

Where $\chi_i\left(z\right)=a\left(z\right)+\alpha_i\left(1-V^*_i\left(a\left(z\right)\right)G^*_i\left(a\left(z\right)\right)\right)$, for $i=1,2$

Multiplying equation (3.28) by z^n and then taking summation over all possible values of $n \geq 1$ and utilizing equations (3.44) and (3.45), we get on simplification

$$z\bar{P}_1(0;z) = (1-q_1)p_1S_1^*(\chi_1(z))\bar{P}_1(0;z) + p_1B_1^*(\chi_1(z))\bar{Q}_1(0;z) + (1-q_2)p_1S_2^*(\chi_2(z))\bar{P}_2(0;z) + p_1B_2^*(\chi_2(z))\bar{Q}_2(0;z) - a(z)zE_0p_1 \tag{3.46}$$

$$z\bar{P}_2(0;z) = (1-q_1)p_2S_1^*(\chi_1(z))\bar{P}_1(0;z) + p_2B_1^*(\chi_1(z))\bar{Q}_1(0;z) + (1-q_2)p_2S_2^*(\chi_2(z))\bar{P}_2(0;z) + p_2B_2^*(\chi_2(z))\bar{Q}_2(0;z) - a(z)zE_0p_2 \tag{3.47}$$

Similarly, multiplying equation (3.29) by z^n and then taking summation over all possible values of $n \geq 1$ and utilizing equation (3.44), we get on simplification

$$\bar{Q}_i(0;z) = q_iS_i^*(\chi_i(z))\bar{P}_i(0;z), \text{ for } i = 1,2 \tag{3.48}$$

Applying equations (3.48) in equations (3.46) and (3.47), we obtain

$$z\bar{P}_1(0;z) = (1-q_1)p_1S_1^*(\chi_1(z))\bar{P}_1(0;z) + p_1B_1^*(\chi_1(z))q_1S_1^*(\chi_1(z))\bar{P}_1(0;z) + (1-q_2)p_1S_2^*(\chi_2(z))\bar{P}_2(0;z) + p_1B_2^*(\chi_2(z))q_2S_2^*(\chi_2(z))\bar{P}_2(0;z) - a(z)zE_0p_1 \tag{3.49}$$

$$z\bar{P}_2(0;z) = (1-q_1)p_2S_1^*(\chi_1(z))\bar{P}_1(0;z) + p_2B_1^*(\chi_1(z))q_1S_1^*(\chi_1(z))\bar{P}_1(0;z) + (1-q_2)p_2S_2^*(\chi_2(z))\bar{P}_2(0;z) + p_2B_2^*(\chi_2(z))q_2S_2^*(\chi_2(z))\bar{P}_2(0;z) - a(z)zE_0p_2 \tag{3.50}$$

Solving equation (3.49) and (3.50), solution yields

$$\bar{P}_i(0;z) = \frac{a(z)zE_0p_i}{\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z)) + \left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z)) - z}, \text{ for } i = 1,2 \tag{3.51}$$

Utilizing (3.51) in equation (3.48), we have

$$\bar{Q}_i(0;z) = \frac{a(z)zE_0p_iq_iS_i^*(\chi_i(z))}{\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z)) + \left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z)) - z}, \text{ for } i = 1,2 \tag{3.52}$$

Applying equations (3.44) and (3.45) in equations (3.38) and (3.39), we obtain

$$\bar{K}_i^P(x,0;z) = \alpha_i\bar{P}_i(0;z)\exp\left\{-\chi_i(z)x\right\}; x > 0, \text{ for } i = 1,2 \tag{3.53}$$

$$\bar{K}_i^Q(x,0;z) = \alpha_i\bar{Q}_i(0;z)\exp\left\{-\chi_i(z)x\right\}; x > 0, \text{ for } i = 1,2 \tag{3.54}$$

From (3.51) and (3.53), for $i = 1,2$ we get

$$\bar{K}_i^P(x,0;z) = \frac{a(z)zE_0p_i\alpha_i\exp\left\{-\chi_i(z)x\right\}}{\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z)) + \left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z)) - z}; x > 0 \tag{3.55}$$

Similarly, from (3.52) and (3.54), we obtain for $i = 1,2$

$$\bar{K}_i^Q(x,0;z) = \frac{a(z)zE_0p_i\alpha_iq_iS_i^*(\chi_i(z))\exp\left\{-\chi_i(z)x\right\}}{\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z)) + \left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z)) - z}; x > 0 \tag{3.56}$$

Utilizing (3.55) in (3.40), we get for $i = 1,2$

$$\bar{R}_i^P\left(x,0;z\right) = \frac{a\left(z\right)zE_0 p_i \alpha_i \exp\left\{-\chi_i\left(z\right)x\right\}G_i^*\left(a\left(z\right)\right)}{\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1 S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2 S_2^*\left(\chi_2\left(z\right)\right)-z}; x>0 \tag{3.57}$$

Similarly, utilizing (3.56) into (3.41), result yields for $i = 1,2$

$$\bar{R}_i^Q\left(x,0;z\right) = \frac{a\left(z\right)zE_0 p_i \alpha_i q_i S_i^*\left(\chi_i\left(z\right)\right)\exp\left\{-\chi_i\left(z\right)x\right\}G_i^*\left(a\left(z\right)\right)}{\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1 S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2 S_2^*\left(\chi_2\left(z\right)\right)-z}; x>0 \tag{3.58}$$

Now letting $z \to 1$ in (3.51), for $i = 1$ we obtain by L' Hospital's rule

$$\bar{P}_1\left(0;1\right) = \frac{\lambda c_{[1]} E_0 p_1}{\left(1-\rho_s\right)} \tag{3.59}$$

Where $\rho_s = p_1\left\{1+\alpha_1(v_1^{(1)}+g_1^{(1)})\right\}\left(\rho_{s_1}+q_1\rho_{b_1}\right)+p_2\left\{1+\alpha_2(v_2^{(1)}+g_2^{(1)})\right\}\left(\rho_{s_2}+q_2\rho_{b_2}\right)$ is the utilizing factor of the system and $\rho_{s_i} = \lambda c_{[1]} s_i^{(1)}$ and $\rho_{b_i} = \lambda c_{[1]} b_i^{(1)}$. From this we obtain the following for $i = 1,2$

$$\bar{P}_i\left(x;1\right) = \frac{\lambda c_{[1]} E_0 p_i}{\left(1-\rho_s\right)} \tag{3.60a}$$

$$\bar{Q}_i\left(x;1\right) = \frac{\lambda c_{[1]} E_0 p_i q_i}{\left(1-\rho_s\right)} \tag{3.60b}$$

$$\bar{K}_i^P\left(x,y;1\right) = \bar{R}_i^P\left(x,y;1\right) = \frac{\alpha_i \lambda c_{[1]} E_0 p_i}{\left(1-\rho_s\right)}$$

$$\bar{K}_i^Q\left(x,y;1\right) = \bar{R}_i^Q\left(x,y;1\right) = \frac{\alpha_i \lambda c_{[1]} E_0 p_i q_i}{\left(1-\rho_s\right)}$$

Now utilizing the normalizing condition (3.14), we get

$$E_0 = \left(1-\rho_s\right) \tag{3.61}$$

Note that the equations (3.60) represent the steady-state probability that the server is idle but available in the system. Also, from equation (3.61), we have $\rho_s < 1$, which is the necessary and sufficient condition of existence of steady-state solution.

Thus, summarizing the above results we have, for $i = 1,2$

$$\bar{P}_i\left(x;z\right) = \frac{a\left(z\right)z\left(1-\rho_s\right)p_i \exp\left\{-\chi_i\left(z\right)x\right\}}{\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1 S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2 S_2^*\left(\chi_2\left(z\right)\right)-z} \tag{3.62}$$

$$\bar{Q}_i\left(x;z\right) = \frac{a\left(z\right)z\left(1-\rho_s\right)p_i q_i S_i^*\left(\chi_i\left(z\right)\right)\exp\left\{-\chi_i\left(z\right)x\right\}}{\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1 S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2 S_2^*\left(\chi_2\left(z\right)\right)-z} \tag{3.63}$$

$$\bar{K}_i^P\left(x,y;z\right) = \frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_i \exp\left\{-\chi_i\left(z\right)x\right\}\exp\left\{-a\left(z\right)y\right\}}{\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1 S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2 S_2^*\left(\chi_2\left(z\right)\right)-z} \tag{3.64}$$

$$\bar{K}_i^Q\left(x,y;z\right) = \frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_i q_i S_i^*\left(\chi_i\left(z\right)\right)\exp\left\{-\chi_i\left(z\right)x\right\}\exp\left\{-a\left(z\right)y\right\}}{\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1 S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2 S_2^*\left(\chi_2\left(z\right)\right)-z} \tag{3.65}$$

$$\bar{R}_i^P\left(x,y;z\right)=\frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_i\exp\left\{-\chi_i\left(z\right)x\right\}\exp\left\{-a\left(z\right)y\right\}G_i^*\left(a\left(z\right)\right)}{\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z} \tag{3.66}$$

$$\bar{R}_i^Q\left(x,y;z\right)=\frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_iq_iS_i^*\left(\chi_i\left(z\right)\right)\exp\left\{-\chi_i\left(z\right)x\right\}\exp\left\{-a\left(z\right)y\right\}G_i^*\left(a\left(z\right)\right)}{\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z} \tag{3.67}$$

Now from equations (3.62) – (3.67), we get the following double transforms for $i=1,2$

$$\begin{aligned}
P_i^*\left(\theta;z\right)&=\sum_{n=1}^{\infty}z^n\int_0^{\infty}e^{-\theta x}P_{n,i}\left(x\right)dx\\
&=\sum_{n=1}^{\infty}z^n\int_0^{\infty}e^{-\theta x}\bar{P}_{n,i}\left(x\right)\left(1-S_i\left(x\right)\right)dx\\
&=\int_0^{\infty}e^{-\theta x}\bar{P}_{n,i}\left(x;z\right)\left(1-S_i\left(x\right)\right)dx\\
&=\frac{a\left(z\right)z\left(1-\rho_s\right)p_i}{\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z}\times\int_0^{\infty}e^{-\left\{\theta+\chi_i\left(z\right)\right\}x}\left(1-S_i\left(x\right)\right)dx\\
&=\frac{a\left(z\right)z\left(1-\rho_s\right)p_i\left(1-S_i^*\left(\theta+\chi_i\left(z\right)\right)\right)}{\left[\theta+\chi_i\left(z\right)\right]\left[\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z\right]}
\end{aligned} \tag{3.68}$$

$$\begin{aligned}
Q_i^*\left(\theta;z\right)&=\sum_{n=1}^{\infty}z^n\int_0^{\infty}e^{-\theta x}Q_{n,i}\left(x\right)dx\\
&=\sum_{n=1}^{\infty}z^n\int_0^{\infty}e^{-\theta x}\bar{Q}_{n,i}\left(x\right)\left(1-B_i\left(x\right)\right)dx\\
&=\int_0^{\infty}e^{-\theta x}\bar{P}_{n,i}\left(x;z\right)\left(1-B_i\left(x\right)\right)dx\\
&=\frac{a\left(z\right)z\left(1-\rho_s\right)p_iq_iS_i^*\left(\chi_i\left(z\right)\right)}{\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z}\times\int_0^{\infty}e^{-\left\{\theta+\chi_i\left(z\right)\right\}x}\left(1-B_i\left(x\right)\right)dx\\
&=\frac{a\left(z\right)z\left(1-\rho_s\right)p_iq_iS_i^*\left(\chi_i\left(z\right)\right)\left(1-B_i^*\left(\theta+\chi_i\left(z\right)\right)\right)}{\left[\theta+\chi_i\left(z\right)\right]\left[\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z\right]}
\end{aligned} \tag{3.69}$$

$$\begin{aligned}
K_i^{P^*}\left(\theta,\omega;z\right)&=\sum_{n=1}^{\infty}z^n\int_0^{\infty}\int_0^{\infty}e^{-\theta x}e^{-\omega y}K_{n,i}^P\left(x,y\right)dx\,dy\\
&=\sum_{n=1}^{\infty}z^n\int_0^{\infty}\int_0^{\infty}e^{-\theta x}e^{-\omega y}\bar{K}_{n,i}^P\left(x,y\right)\left(1-S_i\left(x\right)\right)\left(1-G_i\left(y\right)\right)dx\,dy\\
&=\frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_i}{\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z}\\
&\qquad\times\int_0^{\infty}e^{-\left\{\theta+\chi_i\left(z\right)\right\}x}\left(1-S_i\left(x\right)\right)dx\times\int_0^{\infty}e^{-\left\{\omega+a\left(z\right)\right\}x}\left(1-G_i\left(y\right)\right)dy\\
&=\frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_i\left(1-S_i^*\left(\theta+\chi_i\left(z\right)\right)\right)\left(1-G_i^*\left(\omega+a\left(z\right)\right)\right)}{\left[\theta+\chi_i\left(z\right)\right]\left[\omega+a\left(z\right)\right]\left[\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z\right]}
\end{aligned} \tag{3.70}$$

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$$K_i^{Q^*}\left(\theta,\omega;z\right)=\sum_{n=1}^{\infty}z^n\int_0^{\infty}\int_0^{\infty}e^{-\theta x}e^{-\omega y}K_{n,i}^{Q}\left(x,y\right)\,dx\;dy$$

$$=\sum_{n=1}^{\infty}z^n\int_0^{\infty}\int_0^{\infty}e^{-\theta x}e^{-\omega y}\overline{K}_{n,i}^{Q}\left(x,y\right)\left(1-B_i\left(x\right)\right)\left(1-G_i\left(y\right)\right)\,dx\;dy$$

$$=\frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_iq_iS_i^*\left(\chi_i\left(z\right)\right)}{\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z}$$

$$\times\int_0^{\infty}e^{-\left\{\theta+\chi_i(z)\right\}x}\left(1-B_i\left(x\right)\right)dx\times\int_0^{\infty}e^{-\left\{\omega+a(z)\right\}x}\left(1-G_i\left(y\right)\right)dy$$

$$=\frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_iq_iS_i^*\left(\chi_i\left(z\right)\right)\left(1-B_i^*\left(\theta+\chi_i\left(z\right)\right)\right)\left(1-G_i^*\left(\omega+a\left(z\right)\right)\right)}{\left[\theta+\chi_i(z)\right]\left[\omega+a(z)\right]\left[\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z\right]}\tag{3.71}$$

$$R_i^{P^*}\left(\theta,\omega;z\right)=\sum_{n=1}^{\infty}z^n\int_0^{\infty}\int_0^{\infty}e^{-\theta x}e^{-\omega y}R_{n,i}^{P}\left(x,y\right)\,dx\;dy$$

$$=\sum_{n=1}^{\infty}z^n\int_0^{\infty}\int_0^{\infty}e^{-\theta x}e^{-\omega y}\overline{R}_{n,i}^{P}\left(x,y\right)\left(1-S_i\left(x\right)\right)\left(1-G_i\left(y\right)\right)\,dx\;dy$$

$$=\frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_iG_i^*\left(a\left(z\right)\right)}{\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z}$$

$$\times\int_0^{\infty}e^{-\left\{\theta+\chi_i(z)\right\}x}\left(1-S_i\left(x\right)\right)dx\times\int_0^{\infty}e^{-\left\{\omega+a(z)\right\}x}\left(1-G_i\left(y\right)\right)dy$$

$$=\frac{\alpha_i a\left(z\right)z\left(1-\rho_s\right)p_iG_i^*\left(a\left(z\right)\right)\left(1-S_i^*\left(\theta+\chi_i\left(z\right)\right)\right)\left(1-V_i^*\left(\omega+a\left(z\right)\right)\right)}{\left[\theta+\chi_i\left(z\right)\right]\left[\omega+a\left(z\right)\right]\left[\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z\right]}\tag{3.72}$$

$$R_i^{Q^*}(\theta,\omega;z)=\sum_{n=1}^{\infty}z^n\int_0^{\infty}\int_0^{\infty}e^{-\theta x}e^{-\omega y}R_{n,i}^{Q}(x,y)\,dx\;dy$$

$$=\sum_{n=1}^{\infty}z^n\int_0^{\infty}\int_0^{\infty}e^{-\theta x}e^{-\omega y}\overline{R}_{n,i}^{Q}(x,y)(1-B_i(x))(1-G_i(y))\,dx\;dy$$

$$=\frac{\alpha_i a(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))G_i^*(a(z))}{\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z}$$

$$\times\int_0^{\infty}e^{-\{\theta+\chi_i(z)\}x}(1-B_i(x))\,dx\times\int_0^{\infty}e^{-\{\omega+a(z)\}x}(1-G_i(y))\,dy$$

$$=\frac{\alpha_i a(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))G_i^*(a(z))\left(1-B_i^*(\theta+\chi_i(z))\right)\left(1-V_i^*(\omega+a(z))\right)}{[\theta+\chi_i(z)][\omega+a(z)]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}\tag{3.73}$$

Now, we may summarize the main results of this section in the following theorem.

Theorem 3.1: Under the stability condition $\rho_s<1$, the LST of $PGFs$ of the joint distribution of server's state and queue size under elapsed times for $i=1,2$ are given by:

$$P_i^*(\theta;z)=\frac{a(z)z(1-\rho_s)p_i\left(1-S_i^*(\theta+\chi_i(z))\right)}{[\theta+\chi_i(z)]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$Q_i^*(\theta;z)=\frac{a(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))\left(1-B_i^*(\theta+\chi_i(z))\right)}{[\theta+\chi_i(z)]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$K_i^{P^*}(\theta,\omega;z)=\frac{\alpha_i a(z)z(1-\rho_s)p_i\left(1-S_i^*(\theta+\chi_i(z))\right)\left(1-G_i^*(\omega+a(z))\right)}{[\theta+\chi_i(z)][\omega+a(z)]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$K_i^{Q^*}(\theta,\omega;z)=\frac{\alpha_i a(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))\left(1-B_i^*(\theta+\chi_i(z))\right)\left(1-G_i^*(\omega+a(z))\right)}{[\theta+\chi_i(z)][\omega+a(z)]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$R_i^{P^*}(\theta,\omega;z)=\frac{\alpha_i a(z)z(1-\rho_s)p_i G_i^*(a(z))\left(1-S_i^*(\theta+\chi_i(z))\right)\left(1-V_i^*(\omega+a(z))\right)}{[\theta+\chi_i(z)][\omega+a(z)]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$R_i^{Q^*}(\theta,\omega;z)=\frac{\alpha_i a(z)z(1-\rho_s)p_i q_i S_i^*(\chi_i(z))G_i^*(a(z))\left(1-B_i^*(\theta+\chi_i(z))\right)\left(1-V_i^*(\omega+a(z))\right)}{[\theta+\chi_i(z)][\omega+a(z)]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

4. JOINT DISTRIBUTIONS OF SERVER'S STATE AND QUEUE SIZE UNDER REMAINING TIMES

This section provides the joint distribution of queue size $N_Q(t)$ and remaining times. Let $S_i^+(t)$, $B_i^+(t)$, $D_i^+(t)$ and $R_i^+(t)$ be the remaining service, repeated service, delay and repair times respectively for $i=1,2$, which is the time needed to complete the service, repeated service, delay and repair under way at time t. We now define the following limiting probabilities.
For $n\geq 1$

$$\Pi_{n,1}(u)du=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=1;u<S_1^+(t)\leq u+du\};u>0$$
$$\Pi_{n,2}(u)du=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=2;u<S_2^+(t)\leq u+du\};u>0$$
$$\Omega_{n,1}(u)du=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=3;u<B_1^+(t)\leq u+du\};u>0$$
$$\Omega_{n,2}(u)du=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=4;u<B_2^+(t)\leq u+du\};u>0$$
$$\Phi_{n,1}^{\Pi}(u,w)dw=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=5;w<D_1^+(t)\leq w+dw\mid S_1^+(t)=u\};(u,w)>0$$
$$\Phi_{n,2}^{\Pi}(u,w)dw=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=6;w<D_2^+(t)\leq w+dw\mid S_2^+(t)=u\};(u,w)>0$$
$$\Phi_{n,1}^{\Omega}(u,w)dw=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=7;w<D_1^+(t)\leq w+dw\mid B_1^+(t)=u\};(u,w)>0$$
$$\Phi_{n,2}^{\Omega}(u,w)dw=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=8;w<D_2^+(t)\leq w+dw\mid B_2^+(t)=u\};(u,w)>0$$
$$\Psi_{n,1}^{\Pi}(u,w)dw=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=9;w<R_1^+(t)\leq w+dw\mid S_1^+(t)=u\};(u,w)>0$$
$$\Psi_{n,2}^{\Pi}(u,w)dw=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=10;w<R_2^+(t)\leq w+dw\mid S_2^+(t)=u\};(u,w)>0$$
$$\Psi_{n,1}^{\Omega}(u,w)dw=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=11;w<R_1^+(t)\leq w+dw\mid B_1^+(t)=u\};(u,w)>0$$
$$\Psi_{n,2}^{\Omega}(u,w)dw=\lim_{t\to\infty}P_r\{N_Q(t)=n,Y(t)=12;w<R_2^+(t)\leq w+dw\mid B_2^+(t)=u\};(u,w)>0$$

The double transforms of these joint distributions for $i=1,2$ can be obtained as follows:

$$\Pi_i^*(\theta;z)=\sum_{n=1}^{\infty}z^n\int_0^{\infty}e^{-\theta u}\Pi_{n,i}(u)\,du$$

However considering the condition that the service time $S_i(i=1,2)$ has already exceeded x, the distribution for the remaining service time is given by

$$P_r\left[u<S_i^+(t)\leq u+du\middle|S_i>x\right]=\frac{S_i(x+u)du}{1-S_i(x)}$$

Therefore, we have

$$\Pi_i^*(\theta;z)=\sum_{n=1}^{\infty}z^n\int_0^{\infty}P_{n,i}(x)\,dx\int_0^{\infty}e^{-\theta x}\frac{S_i(x+u)}{1-S_i(x)}\,du$$
$$=\int_0^{\infty}\overline{P}_{n,i}(x;z)\,dx\int_0^{\infty}e^{-\theta x}S_i(x+u)\,du$$

$$=\frac{a(z)z(1-\rho_s)p_i\left[S_i^*(\chi_i(z))-S_i^*(\theta)\right]}{\left[\theta-\chi_i(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]} \tag{4.1}$$

Similarly,

$$\Omega_i^*(\theta;z)=\frac{a(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))\left[B_i^*(\chi_i(z))-B_i^*(\theta)\right]}{\left[\theta-\chi_i(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]} \tag{4.2}$$

Now

$$\Phi_i^{\Pi^*}(\theta,\omega;z)=\sum_{n=1}^{\infty}z^n\int_0^\infty\int_0^\infty e^{-\theta u}e^{-\omega w}\Phi_{n,i}^{\Pi}(u,w)\,du\,dw$$

Again considering the condition that the service time $S_i(i=1,2)$ and delay time $D_i(i=1,2)$ has already exceeded x and y, the distribution for the remaining delay time is given by

$$P_r\left[\left(u<S_i^+(t)\le u+du\middle|S_i>x\right)\cap\left(w<D_i^+(t)\le w+dw\middle|D_i>y\right)\right]=\frac{S_i(x+u)G_i(y+w)dudw}{(1-S_i(x))(1-G_i(y))}$$

Therefore, we have

$$\Phi_i^{\Pi^*}(\theta,\omega;z)=\sum_{n=1}^{\infty}z^n\int_0^\infty\int_0^\infty K_{n,i}^P(x,y)\,dx\,dy\int_0^\infty\int_0^\infty e^{-\theta u}e^{-\omega w}\frac{S_i(x+u)G_i(y+w)}{(1-S_i(x))(1-G_i(y))}\,du\,dw$$

$$=\int_0^\infty\int_0^\infty \bar{K}_i^P(x,y;z)\,dx\,dy\int_0^\infty\int_0^\infty e^{-\theta u}e^{-\omega w}S_i(x+u)G_i(y+w)\,du\,dw$$

$$=\frac{\alpha_ia(z)z(1-\rho_s)p_i\left[S_i^*(\chi_i(z))-S_i^*(\theta)\right]\left[G_i^*(a(z))-G_i^*(\omega)\right]}{\left[\theta-\chi_i(z)\right]\left[\omega-a(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]} \tag{4.3}$$

Similarly,

$$\Phi_i^{\Omega^*}(\theta,\omega;z)=\frac{\alpha_ia(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))\left[B_i^*(\chi_i(z))-B(\theta)\right]\left[G_i^*(a(z))-G_i^*(\omega)\right]}{\left[\theta-\chi_i(z)\right]\left[\omega-a(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]} \tag{4.4}$$

Similarly, the distribution of the remaining repair time is obtained by considering the condition that that the service time $S_i(i=1,2)$ and repair time $R_i(i=1,2)$ has already exceeded x and y, which is given by

$$P_r\left[\left(u<S_i^+(t)\le u+du\middle|S_i>x\right)\cap\left(w<R_i^+(t)\le w+dw\middle|R_i>y\right)\right]=\frac{S_i(x+u)V_i(y+w)dudw}{(1-S_i(x))(1-V_i(y))}$$

Therefore,

$$\Psi_i^{\Pi^*}(\theta,\omega;z)=\sum_{n=1}^{\infty}z^n\int_0^\infty\int_0^\infty R_{n,i}^P(x,y)\,dx\,dy\int_0^\infty\int_0^\infty e^{-\theta u}e^{-\omega w}\frac{S_i(x+u)V_i(y+w)}{(1-S_i(x))(1-V_i(y))}\,du\,dw$$

$$=\int_0^\infty\int_0^\infty \bar{R}_i^P(x,y;z)\,dx\,dy\int_0^\infty\int_0^\infty e^{-\theta u}e^{-\omega w}S_i(x+u)V_i(y+w)\,du\,dw$$

$$=\frac{\alpha_ia(z)z(1-\rho_s)p_iG_i^*(a(z))\left[S_i^*(\chi_i(z))-S_i^*(\theta)\right]\left[V_i^*(a(z))-V_i^*(\omega)\right]}{\left[\theta-\chi_i(z)\right]\left[\omega-a(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]} \tag{4.5}$$

Similarly, we have

$$\Psi_i^{\Omega^*}(\theta,\omega;z)=\frac{\alpha_ia(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))G_i^*(a(z))\left[B_i^*(\chi_i(z))-B(\theta)\right]\left[G_i^*(a(z))-G_i^*(\omega)\right]}{\left[\theta-\chi_i(z)\right]\left[\omega-a(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]} \tag{4.6}$$

The results obtained in this section may be summarized in the following theorem.

Theorem 4.1: Under the stability condition $\rho_s<1$, the *LST* of *PGFs* of the joint distribution of server's state and queue size under remaining times for $i=1,2$ are given by:

$$\Pi_i^*(\theta;z)=\frac{a(z)z(1-\rho_s)p_i\left[S_i^*(\chi_i(z))-S_i^*(\theta)\right]}{\left[\theta-\chi_i(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$\Omega_i^*(\theta;z)=\frac{a(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))\left[B_i^*(\chi_i(z))-B_i^*(\theta)\right]}{\left[\theta-\chi_i(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$\Phi_i^{\Pi^*}(\theta,\omega;z)=\frac{\alpha_ia(z)z(1-\rho_s)p_i\left[S_i^*(\chi_i(z))-S_i^*(\theta)\right]\left[G_i^*(a(z))-G_i^*(\omega)\right]}{\left[\theta-\chi_i(z)\right]\left[\omega-a(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$\Phi_i^{\Omega^*}(\theta,\omega;z)=\frac{\alpha_ia(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))\left[B_i^*(\chi_i(z))-B(\theta)\right]\left[G_i^*(a(z))-G_i^*(\omega)\right]}{\left[\theta-\chi_i(z)\right]\left[\omega-a(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$\Psi_i^{\Pi^*}(\theta,\omega;z)=\frac{\alpha_ia(z)z(1-\rho_s)p_iG_i^*(a(z))\left[S_i^*(\chi_i(z))-S_i^*(\theta)\right]\left[V_i^*(a(z))-V_i^*(\omega)\right]}{\left[\theta-\chi_i(z)\right]\left[\omega-a(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

$$\Psi_i^{\Omega^*}(\theta,\omega;z)=\frac{\alpha_ia(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))G_i^*(a(z))\left[B_i^*(\chi_i(z))-B(\theta)\right]\left[G_i^*(a(z))-G_i^*(\omega)\right]}{\left[\theta-\chi_i(z)\right]\left[\omega-a(z)\right]\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}$$

Remark 4.1: It is important to note that marginal server's state queue size distribution at stationary point of time can be obtained either considering joint distribution of server' state and queue size under elapsed times condition or joint distribution of server' state and queue size under remaining times condition. Hence, we obtain these distributions in corollary 4.1.

Corollary 4.1: The marginal *PGFs* of the server's state queue size distribution at steady state condition for $i=1,2$ are given by:

$$P_i(z)=\frac{a(z)z(1-\rho_s)p_i\left(1-S_i^*(\chi_i(z))\right)}{\chi_i(z)\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}\tag{4.7}$$

$$Q_i(z)=\frac{a(z)z(1-\rho_s)p_iq_iS_i^*(\chi_i(z))\left(1-B_i^*(\chi_i(z))\right)}{\chi_i(z)\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}\tag{4.8}$$

$$K_i^P(z)=\frac{z\alpha_i(1-\rho_s)p_i\left(1-S_i^*(\chi_i(z))\right)\left(1-G_i^*(a(z))\right)}{\chi_i(z)\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}\tag{4.9}$$

$$K_i^Q(z)=\frac{z\alpha_i(1-\rho_s)p_iq_iS_i^*(\chi_i(z))\left(1-B_i^*(\chi_i(z))\right)\left(1-G_i^*(a(z))\right)}{\chi_i(z)\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}\tag{4.10}$$

$$R_i^P(z)=\frac{z\alpha_i(1-\rho_s)p_iG_i^*(a(z))\left(1-S_i^*(\chi_i(z))\right)\left(1-V_i^*(a(z))\right)}{\chi_i(z)\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}\tag{4.11}$$

$$R_i^Q(z)=\frac{z\alpha_i(1-\rho_s)p_iq_iS_i^*(\chi_i(z))\left(1-B_i^*(\chi_i(z))\right)G_i^*(a(z))\left(1-V_i^*(a(z))\right)}{\chi_i(z)\left[\left\{(1-q_1)+q_1B_1^*(\chi_1(z))\right\}p_1S_1^*(\chi_1(z))+\left\{(1-q_2)+q_2B_2^*(\chi_2(z))\right\}p_2S_2^*(\chi_2(z))-z\right]}\tag{4.12}$$

Proof: By setting $\theta=0$ and $\omega=0$ in equations (3.68)-(3.73) or in equations (4.1) - (4.6), we obtain the above results.

Remark 4.2: Similarly, LST of the marginal distributions of service period and repeated service period, LST of the joint distributions of service period and delay period, repeated service period and delay period, service period and repair period, repeated service period and repair period can also be derived either considering joint distribution of server' state and queue size under elapsed times condition or joint distribution of server' state and queue size under remaining times condition. Consequently we have the following corollary 4.2.

Corollary 4.2: The LST of the marginal *PDF*(probability density function)*s* of service period and repeated service period distributions, LST of the joint *PDFs* of service period and delay period, repeated service period and delay period, service period and repair period, repeated service period and repair period distributions for $i = 1,2$ has been given below:

(i) The *LST* of the marginal *PDF* of service period distribution of type i service is given by:

$$P_i^*\left(\theta;1\right) = \frac{\lambda c_{[1]} p_i \left(1 - S_i^*\left(\theta\right)\right)}{\theta}; \text{for } i = 1,2$$

(ii) The *LST* of the marginal *PDF* of repeated service period distribution of type i repeated service is given by:

$$Q_i^*\left(\theta;1\right) = \frac{\lambda c_{[1]} p_i q_i \left(1 - B_i^*\left(\theta\right)\right)}{\theta}; \text{for } i = 1,2$$

(iii) The *LST* of the joint *PDF* of service period and delay period distribution during type i service is given by:

$$K_i^{P^*}\left(\theta,\omega;1\right) = \frac{\lambda c_{[1]} \alpha_i p_i \left(1 - S_i^*\left(\theta\right)\right)\left(1 - G_i^*\left(\omega\right)\right)}{\theta\omega}; \text{for } i = 1,2$$

(iv) The *LST* of the joint *PDF* of repeated service period and delay period distribution during type i repeated service is given by:

$$K_i^{Q^*}\left(\theta,\omega;1\right) = \frac{\lambda c_{[1]} p_i \alpha_i q_i \left(1 - B_i^*\left(\theta\right)\right)\left(1 - G_i^*\left(\omega\right)\right)}{\theta\omega}; \text{for } i = 1,2$$

(v) The *LST* of the joint *PDF* of service period and repair period distribution during type i service is given by:

$$R_i^{P^*}\left(\theta,\omega;1\right) = \frac{\lambda c_{[1]} \alpha_i p_i \left(1 - S_i^*\left(\theta\right)\right)\left(1 - V_i^*\left(\omega\right)\right)}{\theta\omega}; for\ i = 1,2$$

(vi) The *LST* of the *PDF* of joint distribution of repeated service period and repair period during type i repeated service is given by:

$$R_i^{Q^*}\left(\theta,\omega;1\right) = \frac{\lambda c_{[1]} p_i \alpha_i q_i \left(1 - B_i^*\left(\theta\right)\right)\left(1 - V_i^*\left(\omega\right)\right)}{\theta\omega}; for\ i = 1,2$$

Proof: Taking limit $z \to 1$ by using the L' Hospital's rule in equations (3.68) – (3.73) or in equations (4.1) – (4.6) for $i = 1,2$, we obtain the above results.

Remark 4.3: From above expressions of the LST of the joint distributions of service period, repeated service period, repair period and delay period, it is clear that service, repeated service, delay and repair time random variables are mutually independent of each other.

Theorem 4.2: Let $P\left(z\right)$ be the server's state queue size distribution at random epoch, then we have

$$P\left(z\right) = \frac{\left(1-\rho_s\right)\left(1-z\right)\left[\left\{\left(\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right)p_1S_1^*\left(\chi_1\left(z\right)\right)+\left(\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right)p_2S_2^*\left(\chi_2\left(z\right)\right)\right\}\right]}{\left[\left\{\left(1-q_1\right)+q_1B_1^*\left(\chi_1\left(z\right)\right)\right\}p_1S_1^*\left(\chi_1\left(z\right)\right)+\left\{\left(1-q_2\right)+q_2B_2^*\left(\chi_2\left(z\right)\right)\right\}p_2S_2^*\left(\chi_2\left(z\right)\right)-z\right]} \quad (4.13)$$

Proof: $P\left(z\right) = E_0 + \sum_{i=1}^{2}\left(P_i\left(z\right) + Q_i\left(z\right) + K_i^P\left(z\right) + K_i^Q\left(z\right) + R_i^P\left(z\right) + R_i^Q\left(z\right)\right)$

5. DEPARTURE POINT QUEUE SIZE DISTRIBUTION

In this section we derive PGF of the queue size distribution at a departure epoch, i.e. system size distribution as a classical generalization of the well known Pollaczek-Khinchin formula to our $M^X / \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} / 1$ queueing system. Following the argument of PASTA (see Wolf(1982)) and state that a departing customer will see $'j'$ customer in the queue just after a departure if and only if there were $\left(j+1\right)$ customer in the type 1 and type 2 service or repeated service just before the departure. Denoting $\{\phi_j ; j \geq 0\}$ as the probability that there are $'j'$ customer in the queue at the departure point of time, then we may write for $j \geq 0$

$$\phi_j = K\left[\left(1-q_1\right)\int_0^\infty \bar{P}_{j+1,1}\left(x\right) dS_1\left(x\right) + \left(1-q_2\right)\int_0^\infty \bar{P}_{j+1,2}\left(x\right) dS_2\left(x\right) + \int_0^\infty \bar{Q}_{j+1,1}\left(x\right) dB_1\left(x\right) \right.$$
$$\left. + \int_0^\infty \bar{Q}_{j+1,2}\left(x\right) dB_2(x)\right] \tag{5.1}$$

Where K is normalizing constant.

Let $\phi\left(z\right) = \sum_{j=0}^{\infty} \phi_j z^j$

Now multiplying both sides of equation (5.1) by z^j and taking summation over $j \geq 0$ and utilizing equations (3.44) and (3.45) for $i = 1,2$, we get on simplification

$$\phi\left(z\right) = \frac{K\ E_0 a(z)\left[\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\} p_1 S_1^*\left(\chi_1\left(z\right)\right) + \left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\} p_2 S_2^*\left(\chi_2\left(z\right)\right)\right]}{\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\} p_1 S_1^*\left(\chi_1\left(z\right)\right) + \left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\} p_2 S_2^*\left(\chi_2\left(z\right)\right) - z} \tag{5.2}$$

Then utilizing normalizing condition $\phi\left(1\right) = 1$, by L' Hospital's rule we get on simplification

$$K = \frac{\left(1-\rho_s\right)}{\lambda c_{[1]} E_0} \tag{5.3}$$

By inserting (5.3) in formula (5.2) we obtain the following result.

$$\phi\left(z\right) = \frac{\left(1-\rho_s\right)\left(1-c\left(z\right)\right)\left[\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\} p_1 S_1^*\left(\chi_1\left(z\right)\right) + \left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\} p_2 S_2^*\left(\chi_2\left(z\right)\right)\right]}{c_{[1]}\left[\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\} p_1 S_1^*\left(\chi_1\left(z\right)\right) + \left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\} p_2 S_2^*\left(\chi_2\left(z\right)\right) - z\right]} \tag{5.4}$$

Summarizing the results obtained in this section, we can form a theorem which is given by:

Theorem 5.1: Under stability condition $\rho_s < 1$, the *PGF* of the stationary queue size at departure point of time of this model is given by:

$$\phi\left(z\right) = \frac{\left(1-\rho_s\right)\left(1-c\left(z\right)\right)\left[\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\} p_1 S_1^*\left(\chi_1\left(z\right)\right) + \left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\} p_2 S_2^*\left(\chi_2\left(z\right)\right)\right]}{c_{[1]}\left[\left\{\left(1-q_1\right)+q_1 B_1^*\left(\chi_1\left(z\right)\right)\right\} p_1 S_1^*\left(\chi_1\left(z\right)\right) + \left\{\left(1-q_2\right)+q_2 B_2^*\left(\chi_2\left(z\right)\right)\right\} p_2 S_2^*\left(\chi_2\left(z\right)\right) - z\right]}$$

Remark 5.1: The relationship between stationary queue size distribution at random epoch $P(z)$ and at departure epoch $\phi(z)$ can be obtained by comparing equation (4.13) with equation (5.4) and is given by:

$$\phi(z) = \frac{[1 - c(z)]}{c_{[1]}(1 - z)} P(z) = A(z) P(z)$$

Where $A(z)$ is the *PGF* of the number of units placed before an arbitrary test customer (tagged customer) in a batch in which the tagged customer arrives. This number is given as backward recurrence time in the discrete time renewal process where renewal points are generated by the arrival size random variable. This is due to randomness nature of the arrival size random variable.

Remark 5.2: Now by setting $z = 0$ in equation (5.4), we have

$$\begin{aligned}\phi_0 &= \text{Prob.}[\text{No unit is waiting in the system at the departure epoch}] \\ &= \phi(0) = (1 - \rho_s) / c_{[1]}\end{aligned}$$

Thus the relationship between $\phi(0)$ and E_0 is given by

$$E_0 = c_{[1]}\phi_0$$

The relationship exhibits an interesting phenomenon. It states that an observer is more likely to find the system empty than a departing customer leaving the system.

Corollary 5.1: Let L_Q and L_D be the mean queue sizes at a random and departure epoch respectively, then we have

$$\begin{aligned}L_Q = \rho_s &+ \frac{(\lambda c_{[1]})^2 \left[p_1 q_1 s_1^{(1)} b_1^{(1)} \left\{ 1 + \alpha_1 \left(g_1^{(1)} + v_1^{(1)} \right) \right\}^2 \right]}{(1 - \rho_s)} + \frac{(\lambda c_{[1]})^2 \left[p_2 q_2 s_2^{(1)} b_2^{(1)} \left\{ 1 + \alpha_2 \left(g_2^{(1)} + v_2^{(1)} \right) \right\}^2 \right]}{(1 - \rho_s)} \\ &+ \frac{(\lambda c_{[1]})^2 \left[p_1 \left(s_1^{(2)} + q_1 b_1^{(2)} \right) \left\{ 1 + \alpha_1 \left(g_1^{(1)} + v_1^{(1)} \right) \right\}^2 + p_2 \left(s_2^{(2)} + q_2 b_2^{(2)} \right) \left\{ 1 + \alpha_2 \left(g_2^{(1)} + v_2^{(1)} \right) \right\}^2 \right]}{2(1 - \rho_s)} \\ &+ \frac{(\lambda c_{[1]})^2 \left[p_1 \alpha_1 \left(s_1^{(1)} + q_1 b_1^{(1)} \right) \left(g_1^{(2)} + v_1^{(2)} + 2 g_1^{(1)} v_1^{(1)} \right) + p_2 \alpha_2 \left(s_2^{(1)} + q_2 b_2^{(1)} \right) \left(g_2^{(2)} + v_2^{(2)} + 2 g_2^{(1)} v_2^{(1)} \right) \right]}{2(1 - \rho_s)} + \frac{\rho_s c_{[2]}}{2 c_{[1]} (1 - \rho_s)}\end{aligned}$$

and $L_D = L_Q + \dfrac{c_{[2]}}{2c_{[1]}}$; respectively.

Proof: The result follows directly by differentiating expression (4.13) and (5.4) once with respect to z and then setting to unity.

6. SYSTEM STATE PROBABILITIES

In this section we derive system state probabilities under steady state conditions. To derive it we take limit $\theta \to 0$ and $\omega \to 0$ by using the L'Hospital's rule in LSTs of the PDFs (i)-(vi) of corollary 4.2) for $i = 1, 2$ hence we get the following system state probabilities

(i) the probability that the server is busy with type 1 service is $P_{S_1} = p_1\rho_{s_1}$;

(ii) the probability that the server is busy with type 2 service is $P_{S_2} = p_2\rho_{s_2}$;

(iii) the probability that the server is busy with type 1 repeated service is $Q_{b_1} = p_1 q_1 \rho_{b_1}$;

(iv) the probability that the server is busy with type 2 repeated service is $Q_{b_2} = p_2 q_2 \rho_{b_2}$;

(v) the probability that the server is waiting for repair during type 1 service is $K_{S_1}^{P} = \alpha_1 p_1 \rho_{s_1} g_1^{(1)}$;

(vi) the probability that the server is waiting for repair during type 2 service is $K_{S_2}^{P} = \alpha_2 p_2 \rho_{s_2} g_2^{(1)}$;

(vii) the probability that the server is waiting for repair during type 1 repeated service is

$$K_{b_1}^{Q} = \alpha_1 p_1 q_1 \rho_{b_1} g_1^{(1)};$$

(viii) the probability that the server is waiting for repair during type 2 repeated service is

$$K_{b_2}^{Q} = \alpha_2 p_2 q_2 \rho_{b_2} g_2^{(1)}.$$

(ix) the probability that the server is under repair during type 1 service is $R_{S_1}^{P} = \alpha_1 p_1 \rho_{s_1} v_1^{(1)}$;

(x) the probability that the server is under repair during type 2 service is $R_{S_2}^{P} = \alpha_2 p_2 \rho_{s_2} v_2^{(1)}$;

(xi) the probability that the server is under repair during type 1 repeated service is

$$R_{b_1}^{Q} = \alpha_1 p_1 q_1 \rho_{b_1} v_1^{(1)};$$

(xii) the probability that the server is under repair during type 2 repeated service is

$$R_{b_2}^{Q} = \alpha_2 p_2 q_2 \rho_{b_2} v_2^{(1)}.$$

and

(xiii) the probability that the system is idle is

$$\begin{aligned} E_0 &= 1 - \sum_{i=1}^{2}\left(P_{S_i} + Q_{b_i} + K_{S_i}^{P} + K_{b_i}^{Q} + R_{S_i}^{P} + R_{b_i}^{Q}\right) \\ &= 1 - p_1\left\{1 + \alpha_1(v_1^{(1)} + g_1^{(1)})\right\}\left(\rho_{s_1} + q_1\rho_{b_1}\right) - p_2\left\{1 + \alpha_2(v_2^{(1)} + g_2^{(1)})\right\}\left(\rho_{s_2} + q_2\rho_{b_2}\right) \end{aligned}$$

7. PARTICULAR CASE

If we take $S_i = B_i$, for $i = 1,2$ (i.e., service time and repeated service time random variables follows same distribution) in the above expression (5.4), then we obtain

$$\phi(z) = \frac{\left(1-\rho_s\right)\left(1-c(z)\right)\left[\left\{\left(1-q_1\right)+q_1S_1^*\left(\chi_1(z)\right)\right\}p_1S_1^*\left(\chi_1(z)\right)+\left\{\left(1-q_2\right)+q_2S_2^*\left(\chi_2(z)\right)\right\}p_2S_2^*\left(\chi_2(z)\right)\right]}{c_{[1]}\left[\left\{\left(1-q_1\right)+q_1S_1^*\left(\chi_1(z)\right)\right\}p_1S_1^*\left(\chi_1(z)\right)+\left\{\left(1-q_2\right)+q_2S_2^*\left(\chi_2(z)\right)\right\}p_2S_2^*\left(\chi_2(z)\right)-z\right]} \quad (7.1)$$

Where $\rho_s = \rho_{s_1}\left(1+q_1\right)p_1\left\{1+\alpha_1(v_1^{(1)}+g_1^{(1)})\right\} + \rho_{s_2}\left(1+q_2\right)p_2\left\{1+\alpha_2(v_2^{(1)}+g_2^{(1)})\right\}$ is

the utilizing factor of the system and $\rho_{s_i} = \lambda c_{[1]} s_i^{(1)}$

Further, if we take $\alpha_1 = \alpha_2 = 0$ (i.e. there is no breakdown in the system) in the expression (6.1), then we obtain

$$\phi(z) = \frac{(1-\rho_s)(1-c(z))\left[\left\{(1-q_1)+q_1S_1^*(a(z))\right\}p_1S_1^*(a(z))+\left\{(1-q_2)+q_2S_2^*(a(z))\right\}p_2S_2^*(a(z))\right]}{c_{[1]}\left[\left\{(1-q_1)+q_1S_1^*(a(z))\right\}p_1S_1^*(a(z))+\left\{(1-q_2)+q_2S_2^*(a(z))\right\}p_2S_2^*(a(z))-z\right]} \tag{7.2}$$

Where $\rho_s = \rho_{s_1}(1+q_1)p_1 + \rho_{s_2}(1+q_2)p_2$ is the utilization factor of such a system;

which is consistent with the result obtained by Madan, et al [6] where they have considered queue size as excluding one in service. This shows that the result found in this section agree with the existing literature. The only difference between two results is from reliability point of view.

Similarly, if we take $q_1 = q_2 = 0$ (i.e. there is no repeated service in the system) in the expression (6.3), then we have

$$\phi(z) = \frac{(1-\rho_s)[1-c(z)]\left[p_1S_1^*(a(z))+p_2S_2^*(a(\mathrm{z}))-z\right]}{c_{[1]}\left[p_1S_1^*(a(z))+p_2S_2^*(a(\mathrm{z}))-z\right]} \tag{7.3}$$

Where $\rho_s = \rho_{s_1}p_1 + \rho_{s_2}p_2$ is the utilization factor of such a system;

which is the expression for PGF of stationary queue size distribution at departure epoch for classical $M^X/\mathrm{G}/1$ queue with two types of service facility. Also, if we put either $p_1 = 1$ and $p_2 = 0$ or $p_1 = 0$ and $p_2 = 1$ in the expression (6.3) then it reduces to *Pollaczek-Khinchin* formula for classical $M^X/\mathrm{G}/1$ queueing system. Thus we may consider this result as classical generalization of *Pollaczek-Khinchin* formula for $M^X/\mathrm{G}/1$ queue with two types of general heterogeneous service and optional repeated service subject to server's breakdown and delayed repair.

8. BUSY PERIOD DISTRIBUTION

In the present section, we derive busy period distribution for this model. The busy period is defined as the length of the time interval during which the server remains busy and this continues till the epoch when the server becomes free again. The LST of the busy period distribution of this $M^X/G/1$ unreliable queue under two types of service with optional repeated service and delayed repair can be obtained as follows:

Let

T_{Sb} = length of the busy period

T_c = length of the busy cycle

and T_0 = length of the idle period

Let $T_{Sb}^*(\theta) = E\left[e^{-\theta T_{Sb}}\right]$ be the LST of T_{Sb}, then Takac's functional equation under the steady state condition is given by

$$T_{Sb}^*(\theta) = A^*\left(\theta + \lambda - \lambda c\left(T_{Sb}^*(\theta)\right)\right)$$

Where

$$\begin{aligned} A^*(\theta) = &\left\{(1-q_1)+q_1B_1^*\left(\theta+\alpha_1\left(1-V_1^*(\theta)G_1^*(\theta)\right)\right)\right\}p_1S_1^*\left(1-V_1^*(\theta)G_1^*(\theta)\right) \\ &+\left\{(1-q_1)+q_2B_2^*\left(\theta+\alpha_2\left(1-V_2^*(\theta)G_2^*(\theta)\right)\right)\right\}p_2S_2^*\left(1-V_2^*(\theta)G_2^*(\theta)\right) \end{aligned}$$

is LST of total generalized service time is obtained from (2.12).

The mean busy period is found to be

$$E\left[T_{Sb}\right] = -\left.\frac{dT_{Sb}^{*}\left(\theta\right)}{d\theta}\right|_{\theta=0}$$

$$= \frac{p_1\left\{1+\alpha_1\left(v_1^{(1)}+g_1^{(1)}\right)\right\}\left(s_1^{(1)}+q_1b_1^{(1)}\right)+p_2\left\{1+\alpha_2\left(v_2^{(1)}+g_2^{(1)}\right)\right\}\left(s_2^{(1)}+q_2b_2^{(1)}\right)}{\left(1-\rho_s\right)}$$

$$= \frac{\rho_s}{\left(1-\rho_s\right)} \times \frac{1}{\lambda c_{[1]}}$$

Now, since $E\left(T_0\right) = \frac{1}{\lambda c_{[1]}}$, therefore utilizing the relationship $E\left(T_c\right) = E\left(T_{Sb}\right) + E\left(T_0\right)$, we have

$$E\left(T_c\right) = \frac{1}{\lambda c_{[1]}\left(1-\rho_s\right)}$$

9. WAITING TIME DISTRIBUTION

To obtain the waiting time distribution in the queue, we consider a randomly chosen arriving batch and derive the waiting time distribution of the first customer of this batch, W_1 (say) and $W_1^{*}\left(\theta\right)$ be the LST of W_1.
Now if we identify a batch with a single customer, then its service time is just the modified service time of customers constituting the batch. In this case, the batch will have as its batch size $c\left(z\right) = z$. The mean arrival rate will λ and LST of the total generalized service time of the batch will replace

$$A^{*}\left(\theta\right) = \left\{\left(1-q_1\right)+q_1H_1^{R^{*}}\left(\theta\right)\right\}p_1H_1^{*}\left(\theta\right)+\left\{\left(1-q_2\right)+q_2H_2^{R^{*}}\right\}p_2H_2^{*}\left(\theta\right),$$

Where $H_i^{R^{*}}\left(\theta\right) = B_i^{*}\left(\theta+\alpha_i\left(1-G_1^{*}\left(\theta\right)V_1^{*}\left(\theta\right)\right)\right)$ and $H_i^{*}\left(\theta\right) = S_i^{*}\left(\theta+\alpha_i\left(1-G_1^{*}\left(\theta\right)V_1^{*}\left(\theta\right)\right)\right)$ for $i = 1,2$
by $c\left(A^{*}\left(\theta\right)\right)$

Using the transformation and results by Chaudhury and Templeton (1983) (See Chapter 3), from equation (5.4) we have

$$\phi\left(z\right) = \frac{\left(1-\rho_s\right)\lambda\left[1-z\right]c\left[A^{*}\left(\lambda-\lambda z\right)\right]}{\lambda\left[c\left[A^{*}\left(\lambda-\lambda z\right)\right]-z\right]} \tag{9.1}$$

If the waiting time of each batch is independent of the part of arrival process following the arrival time of the batches left behind a departing batch and those that arrive during the time it spends in the queue and in service, it follows that

$$\phi\left(z\right) = W_1^{*}\left(\lambda-\lambda z\right)c\left[A^{*}\left(\lambda-\lambda z\right)\right] \tag{9.2}$$

Now putting $\theta = \lambda - \lambda z$ in (9.2) and utilizing (9.1) in (9.2), we have on simplification

$$W_1^{*}\left(\theta\right) = \frac{\left(1-\rho_s\right)\theta}{\left[\theta-\lambda+\lambda c[A^{*}\left(\theta\right)]\right]} \tag{9.3}$$

Next, let W_Q be the waiting time of an arbitrary customer in a batch and $W_Q^*(\theta)$ be the LST of W_Q. If $j \geq 1$ is the position of the customer within arrival batch, then

$$W_Q = W_1 + \sum_{i=1}^{j-1} A_i^{'}; j \geq 1 \tag{9.4}$$

Where $A_i^{'}$ denotes the difference between total generalized service time and inter arrival time of the i customer in the batch.

If ψ_j is the probability of an arbitrary customer being the j^{th} position of an arriving batch, then applying the results of Chaudhury and Templeton (1983) (see chapter 3), we may write

$$\Pr[\sum_{i=1}^{j-1} A_i^{'} \leq t] = \sum_{j=1}^{\infty} \psi_j A(t)^{(j-1)^*};$$

Where $A(t) = \Pr\left[A_i^{'} \leq t\right]$ and $\psi_j = (1 - \sum_{i=1}^{j-1} c_i) / c_{[1]}$.

Consequently taking LST of (9.4), we get on simplification

$$\begin{aligned} W_Q^*(\theta) &= E\left[e^{-\theta W_Q}\right] \\ &= E\left[e^{-\theta W_1}\right].E\left[e^{-\theta \sum_{i=1}^{j-1} A_i^{'}}\right] \\ &= \frac{W_1^*(\theta)}{c_{[1]}}.\frac{\left[1 - c\left[A^*(\theta)\right]\right]}{\left[1 - A^*(\theta)\right]} \end{aligned}$$

and therefore LST of the waiting time distribution in the queue for this model is given by

$$W_Q^*(\theta) = \frac{(1-\rho_s)\theta\left[1 - c\left[A^*(\theta)\right]\right]}{c_{[1]}\left[\theta - \lambda + \lambda c\left[A^*(\theta)\right]\right]\left[1 - A^*(\theta)\right]} \tag{9.5}$$

Mean waiting time of a test customer found to be

$$\begin{aligned} E[W] &= p_1\left\{1 + \alpha_1(v_1^{(1)} + g_1^{(1)})\right\}\left(s_1^{(1)} + q_1 b_1^{(1)}\right) + p_2\left\{1 + \alpha_2(v_2^{(1)} + g_2^{(1)})\right\}\left(s_2^{(1)} + q_2 b_2^{(1)}\right) \\ &+ \frac{\left(\lambda c_{[1]}\right)\left[p_1 q_1 s_1^{(1)} b_1^{(1)}\left\{1 + \alpha_1\left(g_1^{(1)} + v_1^{(1)}\right)\right\}^2\right]}{(1-\rho_s)} + \frac{\left(\lambda c_{[1]}\right)\left[p_2 q_2 s_2^{(1)} b_2^{(1)}\left\{1 + \alpha_2\left(g_2^{(1)} + v_2^{(1)}\right)\right\}^2\right]}{(1-\rho_s)} \\ &+ \frac{\left(\lambda c_{[1]}\right)\left[p_1\left(s_1^{(2)} + q_1 b_1^{(2)}\right)\left\{1 + \alpha_1\left(g_1^{(1)} + v_1^{(1)}\right)\right\}^2 + p_2\left(s_2^{(2)} + q_2 b_2^{(2)}\right)\left\{1 + \alpha_2\left(g_2^{(1)} + v_2^{(1)}\right)\right\}^2\right]}{2(1-\rho_s)} \\ &+ \frac{\left(\lambda c_{[1]}\right)\left[p_1\alpha_1\left(s_1^{(1)} + q_1 b_1^{(1)}\right)\left(g_1^{(2)} + v_1^{(2)} + 2g_1^{(1)}v_1^{(1)}\right) + p_2\alpha_2\left(s_2^{(1)} + q_2 b_2^{(1)}\right)\left(g_2^{(2)} + v_2^{(2)} + 2g_2^{(1)}v_2^{(1)}\right)\right]}{2(1-\rho_s)} + \frac{\rho_s c_{[2]}}{2\lambda\left(c_{[1]}\right)^2(1-\rho_s)}; \end{aligned}$$

which verifies Little's formula.

10. RELIABILITY INDICES

Finally, we derive two important reliability indices viz. - the system availability and failure frequency under stability condition. Suppose that the system is empty in the beginning. Let $A_S(t)$ be the pointwise availability of the server at time t, i.e., the probability that the system is working at time t (the server is either working on a customer or in an idle period), such that the steady state availability of the server will be

$$A_S = \lim_{t\to\infty} A_S(t).$$

10.1 Availability of the server

The steady state availability of the server can be obtain utilizing the equations (4.7) and (4.8) for $i = 1,2$ and by considering the equation

$$A_S = E_0 + \lim_{z\to 1}\left[P_1(z) + P_2(z) + Q_1(z) + Q_2(z)\right]$$

after simplification we obtain the following result

$$A_S = 1 - p_1\alpha_1\left(v_1^{(1)} + g_1^{(1)}\right)\left(\rho_{s_1} + q_1\rho_{b_1}\right) - p_2\alpha_2\left(v_2^{(1)} + g_2^{(1)}\right)\left(\rho_{s_2} + q_2\rho_{b_2}\right) \tag{10.1}$$

10.2 Failure frequency of the server

By utilizing the argument of Li et. al. [13] in equations (3.60a) and (3.60b), the failure frequency of the server under stability condition is obtained as follows

$$\begin{aligned} M_{fs} = {} & \alpha_1\left[\bar{P}_1(x;1)\times\int_0^\infty\left(1 - S_1(x)\right)dx + \bar{Q}_1(x;1)\times\int_0^\infty\left(1 - B_1(x)\right)dx\right] \\ & +\alpha_2\left[\bar{P}_2(x;1)\times\int_0^\infty\left(1 - S_2(x)\right)dx + \bar{Q}_2(x;1)\times\int_0^\infty\left(1 - B_2(x)\right)dx\right] \end{aligned}$$

Since $\int_0^\infty\left(1 - S_i(x)\right)dx = \int_0^\infty x\, dS_i(x) = s_i^{(1)}$ and $\int_0^\infty\left(1 - B_i(x)\right)dx = \int_0^\infty x\, dB_i(x) = b_i^{(1)}$ for $i = 1,2$; therefore we have

$$M_{fs} = p_1\alpha_1\left(\rho_{s_1} + q_1\rho_{b_1}\right) + p_2\alpha_2\left(\rho_{s_2} + q_2\rho_{b_2}\right) \tag{10.2}$$

11. A REAL WORLD APPLICATION WITH THE COST EFFECTIVENESS MAXIMIZATION MODEL

In this section, we present a possible application and some numerical examples in some situations to explain that present model can represent the possible application reasonably well. For example, in company's customer relationship management system, a contact centre is a location for centralized handling of individual communications. Contact centers run support or help desks which provide two types of supports viz., answer technical questions from customers and assist them using their equipment or software. Support desks are used by companies in the computing, telecommunications and consumer electronics industries. A contact centre supports interaction with customers over a variety of media, including telephony, e-mail, social media, and live chat (online chat). Live chat is mainly used for text-based communication systems where customers communicate with website's live chat agents through live support applications. The live support applications open a window that connects the customers to an agent. Typically, text massage arrives at the agent's system following Poisson stream requesting any

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one of the two types of support. In practice, the live chat may be interrupted due to some unpredictable events (breakdowns) in the system and the system can be repaired. Moreover, it may so happen that customers can ask for re-support due to interruption in the previous chat. In this scenario, text massage, the agent's system, two types of support or help by agents and re- support correspond to batch arrival, the server, two types of service and repeated service in the queueing terminology.

For the system of the agent, we define cost effectiveness as (Availability)/(Expected out of pocket cost rate) to be an alternative cost criterion, which reflects the efficiency per dollar outlay. This criterion is useful for the effective use of available money. This criterion is helpful in the situation that benefits obtained from the investment are not reducible in monetary terms.

Let C_b be the expected cost rate per busy cycle, then

$C_b = \dfrac{C_S}{E(T_c)}$, where C_S is the out of pocket cost per cycle.

The cost effectiveness is defined as

$$C_e = \frac{A_S}{C_b}$$

The following graphs show how the cost rate and cost effectiveness listed vary with system parameter (such as arrival rate λ of massages, breakdown rates (α_i, for $i = 1,2$) of the agent's system and repeated service probabilities). For computational convenience, the settings of system's parameter are as follows:

- $C_S = 100$
- Geometric batch size with mean $c_{[1]} = 1.0$ for the request arrival.
- Exponential service time with a mean $s_1^{(1)} = 0.2$ and $s_2^{(1)} = 0.25$ for type 1 and type 2 services respectively.

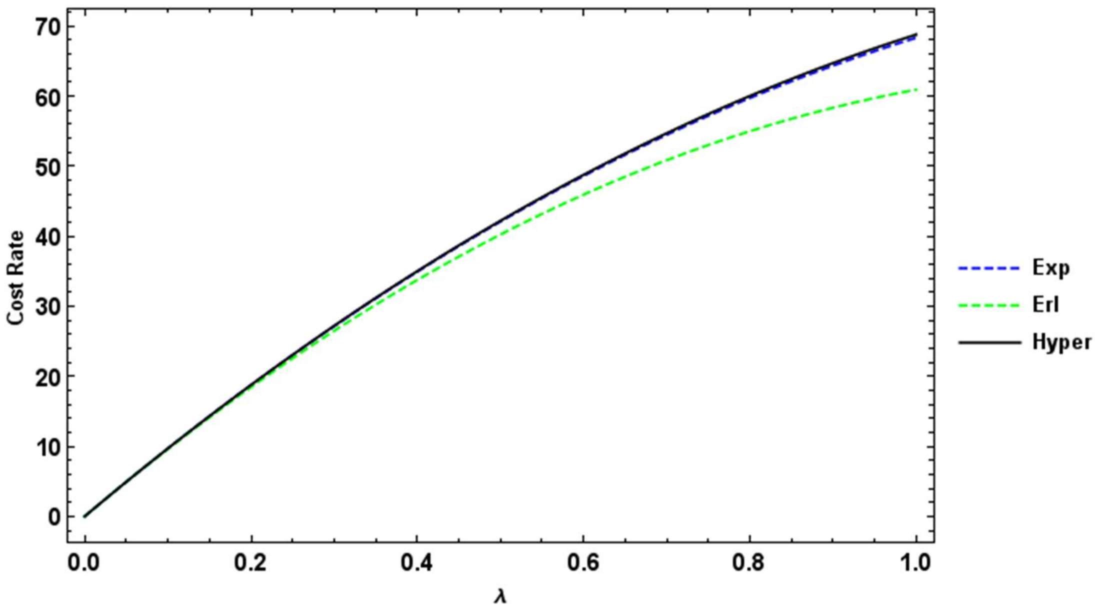

Figure 2. Cost rate for various values of λ for different repeated service time

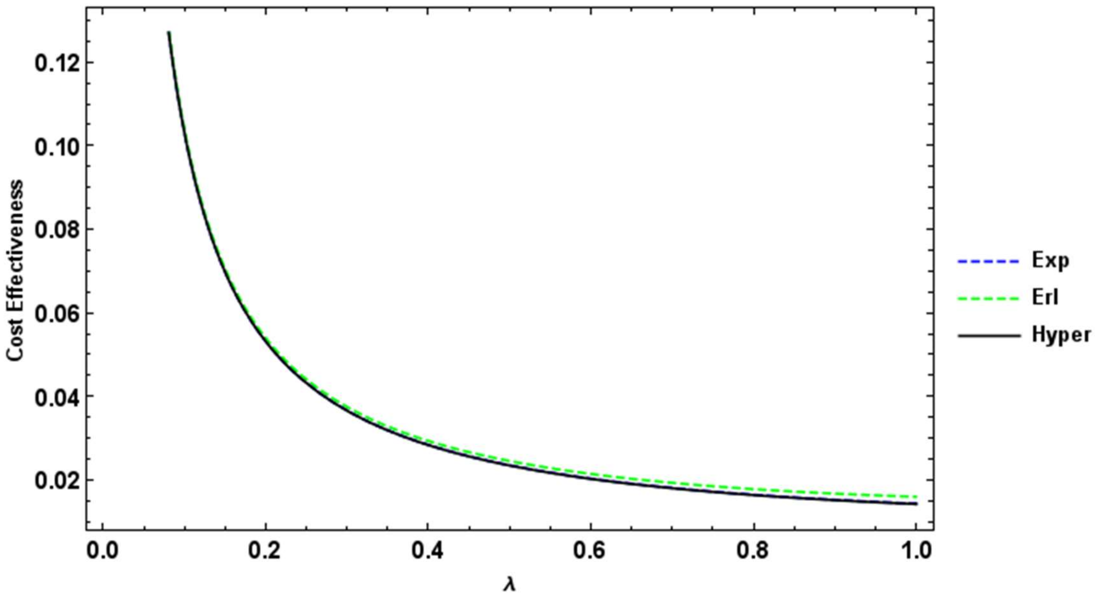

Figure 3.Cost effectiveness for various values of λ for different repeated service time

- Exponential delay time with a mean $g_1^{(1)} = 1.0$ and $g_2^{(1)} = 0.75$ for type 1 and type 2 services respectively.
- Exponential repair time with mean $v_1^{(1)} = 1.0$ and $v_2^{(1)} = 0.8$ for type 1 and type 2 services respectively.

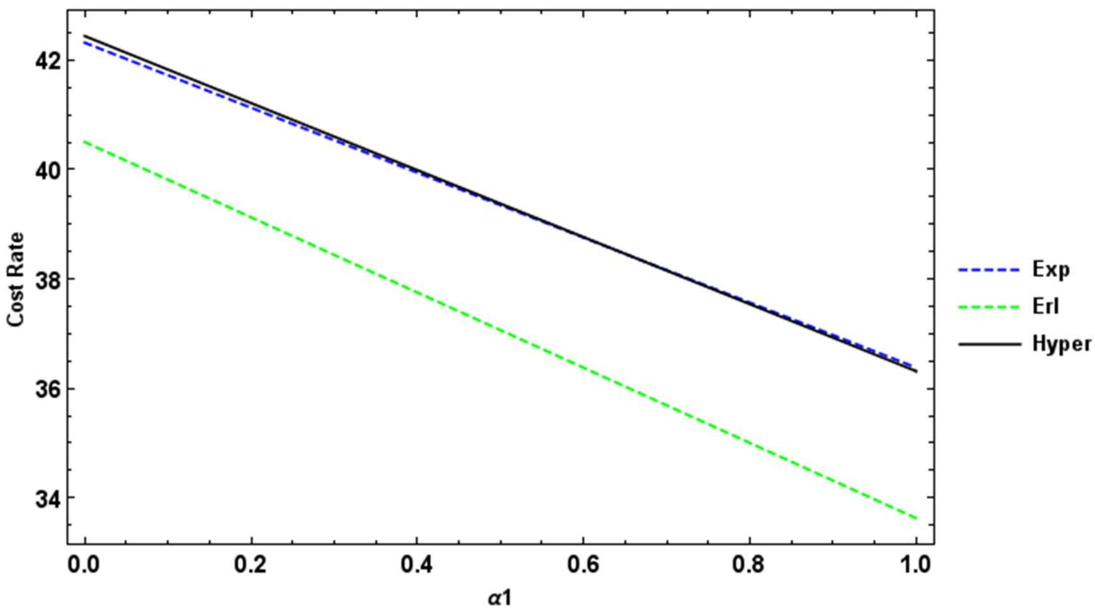

Figure 4. Cost rate for various values of α_1 for different repeated service time

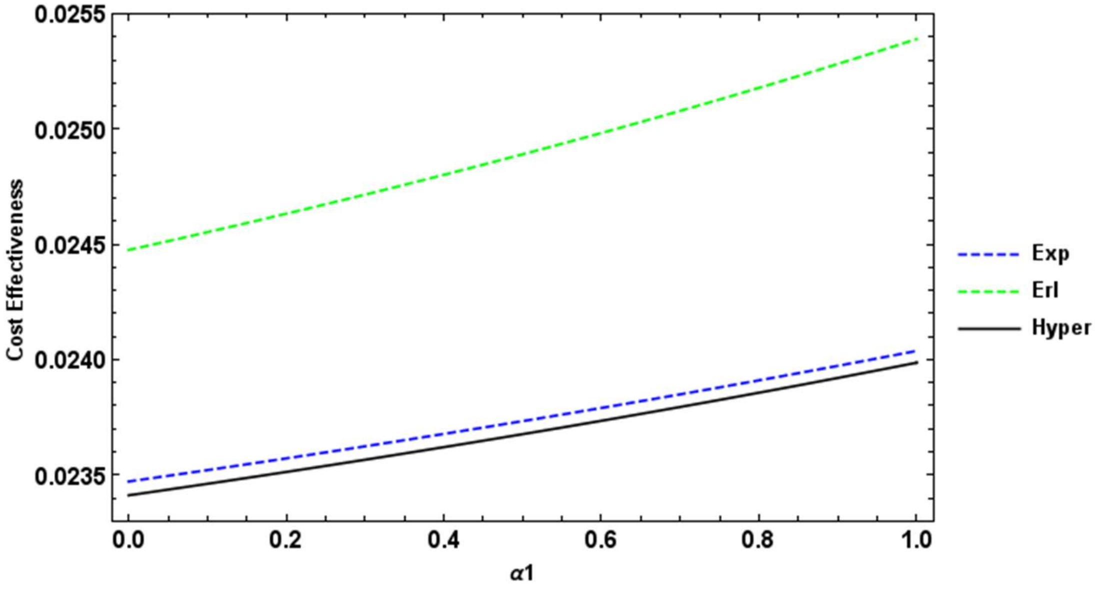

Figure 5. Cost effectiveness for various values of α_1 for different repeated service time

For illustrative purpose, repeated service time is considered as Exponential (denoted by Exp), 2- stage Erlang (denoted by Erl), 2- stage Hyper exponential (denoted by Hyper) respectively with service rate 4 and 2 for type 1 and type 2 services respectively.

The results of cost rate and cost effectiveness are shown respectively, in figures 2-6 for the following five cases.

Case 1: We choose $\alpha_1 = 0.04$, $\alpha_2 = 0.05$, $p_1 = p_2 = 0.5$, $q_1 = 0.15$ and $q_2 = 0.20$ and vary the values of λ from 0 to 1.0 for different service time.

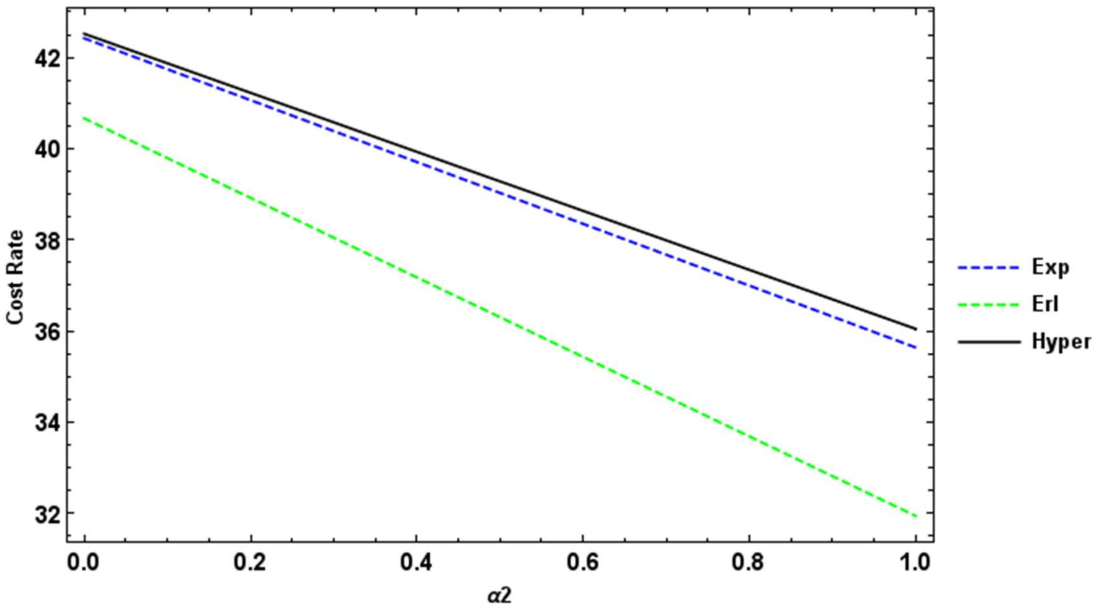

Figure 6. Cost rate for various values of α_2 for different repeated service time

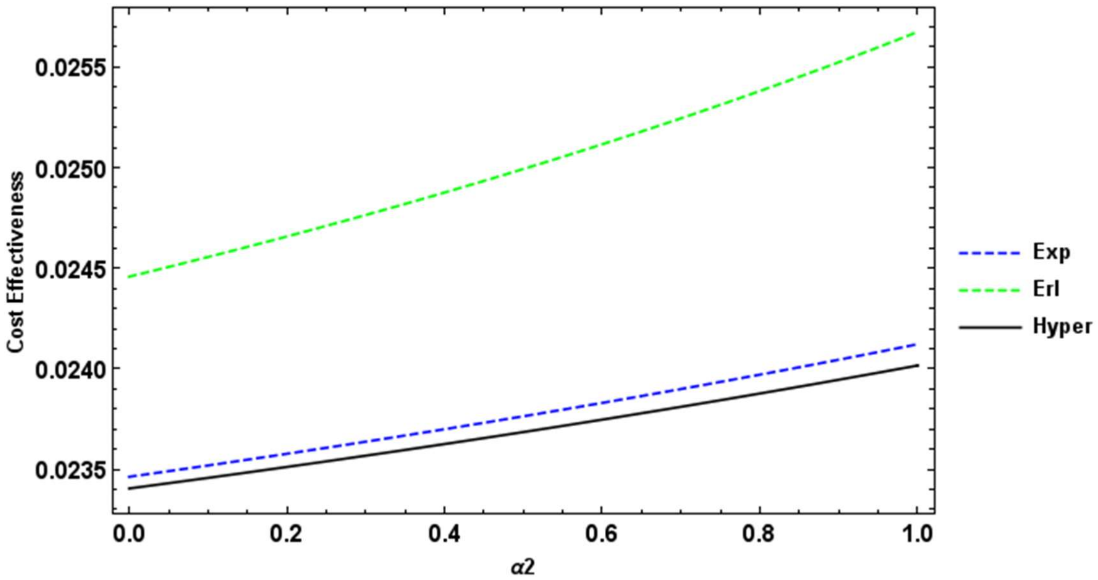

Figure 7. Cost effectiveness for various values of α_2 for different repeated service time

Case 2: We choose $\lambda = 0.5$, $\alpha_2 = 0.05$, $p_1 = p_2 = 0.5$, $q_1 = 0.15$ and $q_2 = 0.20$ and vary the values of α_1 from 0 to 1.0 for different service time.

Case 3: We choose $\lambda = 0.5$, $\alpha_1 = 0.04$, $p_1 = p_2 = 0.5$, $q_1 = 0.15$ and $q_2 = 0.20$ and vary the values of α_2 from 0 to 1.0 for different service time.

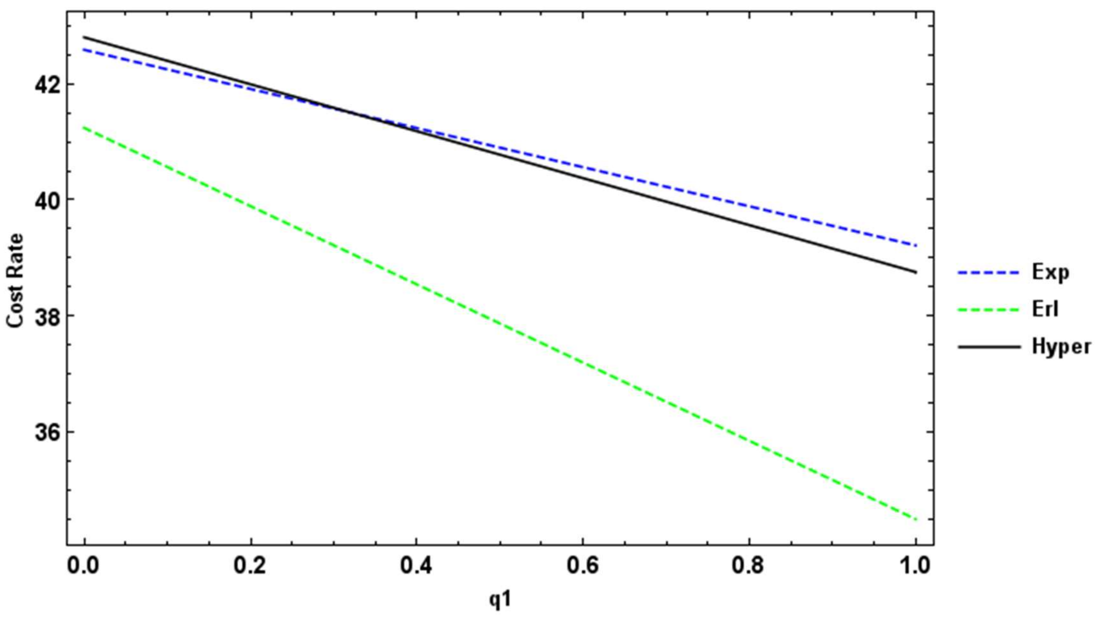

Figure 8. Cost rate for various values of q_1 for different repeated service time

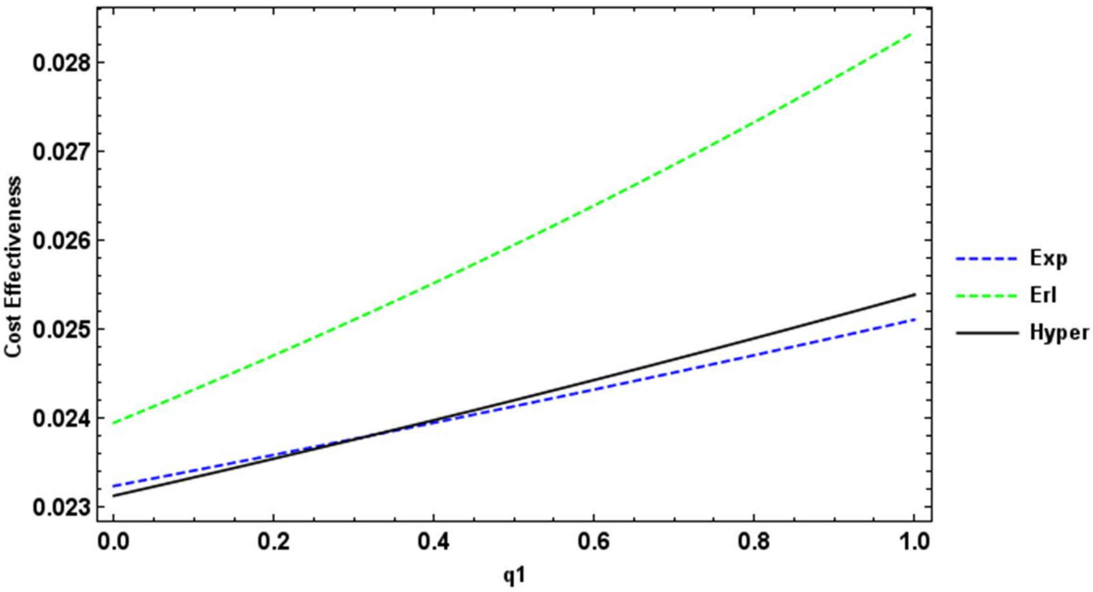

Figure 9. Cost effectiveness for various values of q_1 for different repeated service time

Case 4: We choose $\lambda = 0.5$, $\alpha_1 = 0.04$, $\alpha_2 = 0.05$, $p_1 = p_2 = 0.5$ and $q_2 = 0.20$ and vary the values of q_1 from 0 to 1.0 for different service time.

Case 5: We choose $\lambda = 0.5$, $\alpha_1 = 0.04$, $\alpha_2 = 0.05$, $p_1 = p_2 = 0.5$ and $q_1 = 0.15$ and vary the values of q_2 from 0 to 1.0 for different service time.

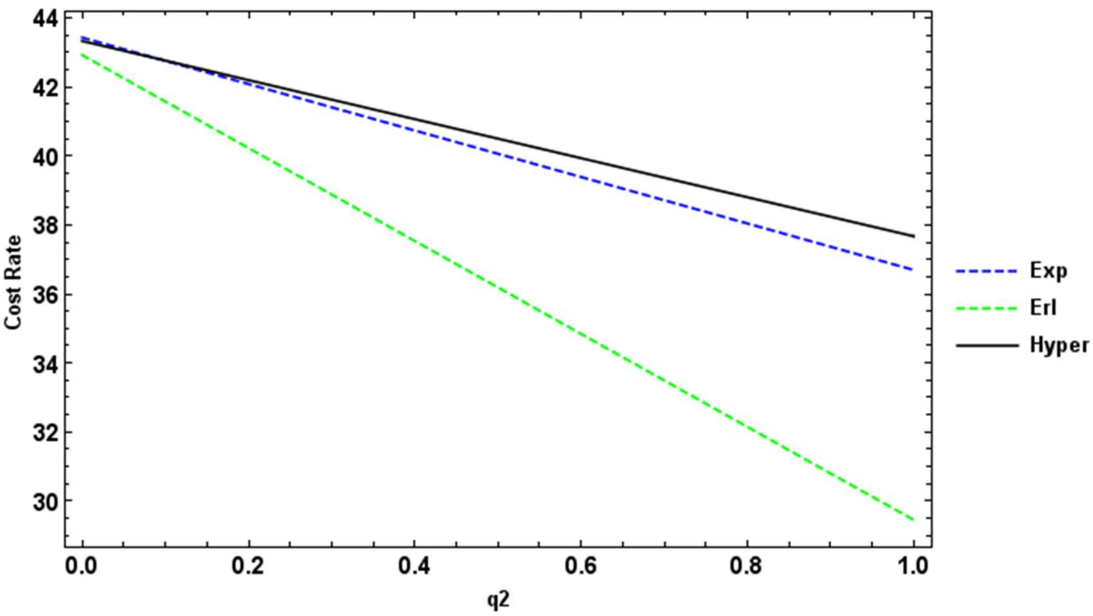

Figure 10. Cost rate for various values of q_2 for different repeated service time

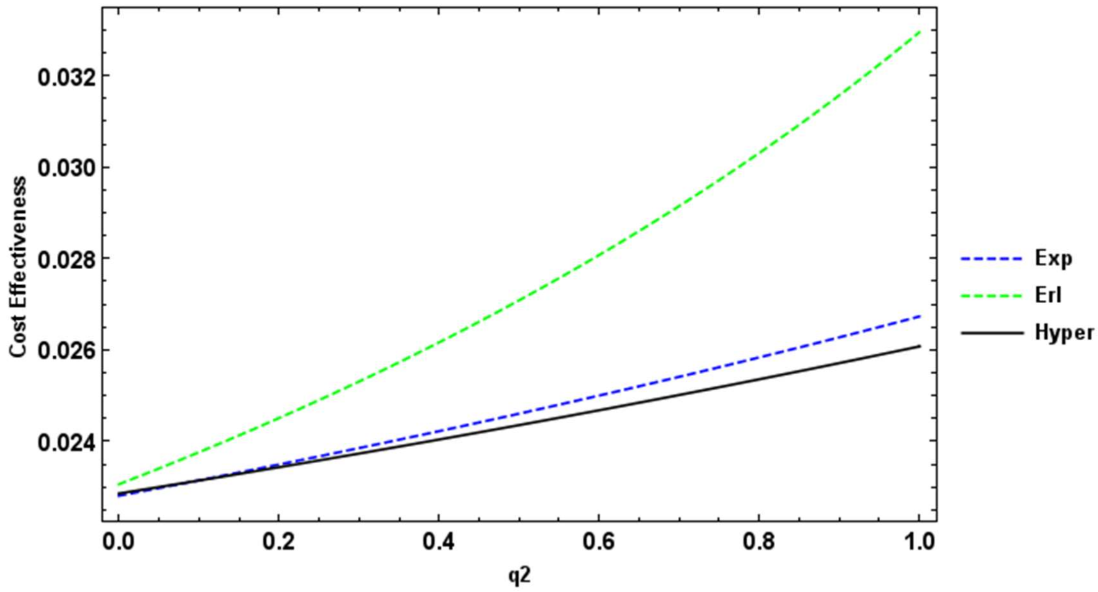

Figure 11. Cost effectiveness for various values of q_2 for different repeated service time

From fig 2 and 3 we observe that cost rate increases first and then decreases, while the cost effectiveness decreases first and then decreases and tends to stability, when the arrival rate of text massage increases from 0.0 to 1.0. The reason is that the agent's system has to take more supporting power to deal with the arriving massage in the initial period. Along with more and more massage being served; new arriving massage with the same distribution can be served quickly.

Fig 4, 5, 6, 7, 8, 9, 10 and 11 reveal the effect of various breakdown rates and repeated service probability on the cost rate and cost effectiveness. From the figures we observe that cost rate decreases, while the cost effectiveness increases, when the breakdown rates or probability of repeated service demand increases from 0.0 to 1.0. This is because that the increasing of breakdown rate or repeated service demand represents more massages are waiting in the queue. From the view point of the agent's system, more service or repeated service demand can be treated using the same servicing capacity no matter the service or repeated service demand being served or waiting in the queue.

12. CONCLUSION

We have analyzed in this paper an $M^X/G/1$ queue under types of general heterogeneous service with optional repeated service subject to server's breakdowns and delayed repair. By using the most popular classical supplementary variable technique, we obtain the stationary measures of the joint distributions of the state of the server and number of customers present in the system, i.e. queue size, stationary queue size distribution at a departure epoch, busy period distribution, waiting time distribution, the system availability, the failure frequency. Finally, the numerical analysis clearly demonstrates the meaningful impact of the server breakdowns and repeated service on the system's performance.

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