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Unreliable Server Retrial Queue with Optional Service and Multi-Phase Repair

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Abstract: In this paper, the retrial unreliable server queue with batch arrivals is considered. The arrival rates of the units are different and dependent upon the joining probabilities according to the server status. On arrival, if unit finds the busy server, he may retry for the service after a random duration of time. The server facilitates the essential service and optional service, if opted after essential service. Moreover, the server is unreliable and subject to the breakdown while rendering essential/optional service. The failed server may immediately undergo for the compulsory multiphase repair or may wait to start the repair due to any technical reasons. The server can also avail the optional vacation under the Bernoulli schedule after finish the service of each unit or may continue to serve the next unit. The variables corresponding to elapsed times of general distributed service process, retrial process, repair process and vacation duration, as supplementary variables and used to frame the governing equations. By using the probability generating functions of joint distributions of the units at different states of the server, the performance characteristics of the system are derived. To validate the results, the sensitivity analysis has been performed by taking the numerical illustration.
Keyword — Unreliable server, Optional service, Vacation, Retrial queue, Supplementary variable, Queue length.
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1. INTRODUCTION

In all the spheres of day-to-day activities, the formation of the queues can be experienced. In some congestion situations of commercial as well as industrial organizations, the retrial queue or queue with repeated attempts can be noticed. In the situation, when the units are not getting service on joining the system due to busy server, they may decide to join the retrial orbit and retry for the service in other attempts after some random period of time. The concept of retrial queue is important feature and can be observed in manufacturing system, computer, telecommunication systems, network access control process, cellular mobile communication networks, switching systems, etc. To elaborate the situation, we cite the example of cellular mobile network wherein the packet/call attempts join the system to be transmit but due to busy line or due to service interruption, the call attempts may decide to join the orbit and repeat the other attempt to get the service. The queueing systems with bulk arrival/service and essential/optional services are common and meet the practical effectiveness in terms of efficient utilization of resources. During vacation period and repair of breakdown server, the server wishes to perform some internal workout for proper maintenance of the service system to provide the service up to customer's satisfaction.

In performance of the service system, the service interruption is the key factor to observe the efficiency of the system. Many researchers have studied queueing networks with the different variations. Choudhury and Ke (2014) have presented the stochastic model of the single arrival of the customer in unreliable retrial queueing system under Bernoulli vacation schedule for single phase repair/delayed repair. In real time activities, many situations can be experienced that due to service interruption, the service phenomenon can be affected and balking behaviour may be occurred. Motivated with such realistic situation, in the present investigation, we have extended above stated model for the general congestion situation by incorporating some additional realistic features such as (i) arrivals of the units in bulk (ii) provision of second stage optional service (iii) unreliable server with delayed in repair and $m-$ phases of repair (iv) state dependent joining probabilities of the units.

The motivation of this investigation comes from real life situations of manufacturing system encountered in the production of electric appliances industry, wherein the manufacturing process activates the production by providing the raw materials in the bulk and the production of the items has various varieties as per the requirements of the customers with the basic and luxury needs. During manufacturing process, the essential/optional vacation of

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the machines for random time period may be required for efficient outputs of the production, as some modern equipment require the essential vacation for fixed time duration. It is also experienced that the unpredictable failure of the machinery parts may cause of service interruption and affects the input flow of the demands for service (varying arrival rates). The retrial process of the arrived units in the service system is also one of the important key factors of the service environment. In this process, customers may retry for his turn later to fulfil their requirements on finding the service system in idle state. In production process, At the completion of any stage of the service, if the server is not ready to provide the service for next step of the process immediately then ready stock of the items may join the retrial orbit and retry again after some time for the completion of the service process (retrial process).. The service station may breakdowns and failure server requires a phase repair for its recovery to good condition. Furthermore, when immediate repair of the broken down server is not possible due to any technical reason (delayed in repair), some preliminary setting may require before starting the actual repair for the recovery which can be accomplished in many sequential stages of the repair. During the period of repair/delay in repair, the discouragement behaviour of the waiting customers may occur and decide to quit the system without getting the service (balking).

The pioneer works on retrial queue have been done by a number of researchers with different variations (cf. Artalejo (1999), Artalejo and Corral (2008), Shin and choo (2009)). The non-markovian retrial queue with general distributed service time and unreliable server under Bernoulli vacation policy was investigated by Wu and Lian (2013) and studied the behaviour of negative customers by applying the embedded Markov chain approach and supplementary variable technique to obtain the different performance indices of the system. Gao and Wang (2014) have considered the $M/G/1$ retrial queue to discuss the behaviour of the impatient customers and queue size distribution was determined by using the supplementary variable technique and embedded markov chain approach. Recently, Singh *et al.* (2016) have studied the non-markovian retrial queue with two stage service under Bernoulli vacation schedule. In present scenario, the reliable server is a common assumption and can be experienced in many activities. Due to unpredictable failure of the system or due to other cause of service interruption, the discouragement behaviour may occur and can be experienced in computer system, telecommunication network, flexible manufacturing system, banking and transport system, etc. In this direction, some notable works are due to many researchers under different assumptions (cf. Tang (1997), Jain and Agarwal (2009), Choudhury *et al.* (2011)). Jain and Bhagat (2015) investigated the batch arrival retrial queue under Bernoulli vacation schedule and admission control policy to study the behaviour of the system.

In industrial scenario, queue with vacation schedule have great applications in many fields including the communication networks, production management, flexible manufacturing system, etc. Many queue theorists (cf. Doshi (1986), Madan (2000), Zhang and Hou (2010), Singh *et al.* (2012), Dimitriou (2013)) modelled the queueing situations where server can take vacation. The $M/G/1$ unreliable queue with single vacation under (p,N) -policy was presented by Yang and Ke (2014) in order to study the system characteristics of queue size distribution by using the supplementary variable approach. The provision of regular/optional service has also wide applications in the situations of daily and industrial activities. From the literature of queueing theory, it is evident that many researchers presented their work in this direction (cf. Medhi (2002), Wang (2004), Ke (2008)). Recently, Kirupa and Chandrika (2015) investigated a stochastic model for unreliable retrial queue with batch arrival and negative customers having choice of multi-optional service. They have considered the provision of *J*- vacations under feedback policy and analyzed the model by using supplementary variable technique.

To investigate the present model, the remaining paper is explained in the following manner. Section 2 presents the model description by stating the necessary notations and assumptions. The mathematical formulation of the model is done in section 3. The queue length distribution of the model has been provided in Section 4. Section 5 consist the performance measures of the system. Some special cases to validate the results with existing models by setting the parameters are considered in section 6. The numerical illustration has been carried out in Section 7. Finally, conclusion is presented in section 8.

2. MODEL DESCRIPTION

Consider a retrial queue with heterogeneous arrival rates and single unreliable server. The basic assumptions of the model are described as follows:

2.1 Arrival and Retrial Process

The arrivals of the units occurs in bulk of size X with probability mass function $P(X=j)=c_j,\ j\geq 1$ and follows the Poisson process with mean arrival rates in idle state, busy state, delayed repair, under repair and vacation states with their respective joining probabilities of the units are b, b_1, b_2 and b_3 ; $0 < b, b_1, b_2, b_3 < 1$. The first and second factorial moments of arrival process of the units are denoted as $d^{(1)}$ and $d^{(2)}$, respectively. It is assumed that on arrival, the unit finds the server is idle but available for the service; the unit may either join the queue or wait for their turn in respective orbit. Furthermore, on arrival if the primary unit finds the server is in busy state or on vacation state/ broken down state, the unit may join the group of retrial units (i.e. retrial orbit) of size $N(t)$ at time t. The random variable R denotes the retrial time with distribution function $M(x)$ and its Laplace transform $\tilde{M}(s)$.

2.2 Service Process

The service discipline is FCFS. The regular service time B_1 (optional service time B_2) is general distributed with distribution function $B_1(x)$ ($B_2(x)$), having respective Laplace transform $\tilde{B}_1(s)(\tilde{B}_2(s))$ and finite j^{th} moments $\beta_1^{(j)}(\beta_2^{(j)})$, $j = 1,2$. After completion of essential service of the unit, the optional service may opt with probability r_0 or leave the system with probability $\overline{r}_0 = 1 - r_0$.

2.3 Breakdown State

The server is unreliable and may breakdown in Poisson fashion at any instant of time while rendering regular/optional service with failure rate $\alpha_1(\alpha_2)$. Once the system breaks down, immediately undergoes for the repair. Due to unavailability of the technical staff or some other reason, the waiting time for repair may also occur. The random variable of the delay time to repair is considered as D_i with distribution function $D_i(y)$ and respective Laplace transform $\tilde{D}_i(s)$ and finite moments $\gamma_i^{(j)}, j = 1,2$, when it fails in i^{th} stage of the service. The failed server joins the repair station for repair of m compulsory phases of the repair. The random variable $R_l^{(i)}$ denotes the l^{th} phase repair time of the server with cumulative distribution function $G_{i,l}(y)$, Laplace transform of cumulative distribution function as $\tilde{G}_{i,\,l}(s)$ and first two moments $g_{il}^{(j)}$, $j = 1,2$ when it fails during the i^{th} stage of the service.

2.4 Vacation State

After completion of the service of the unit, the server may go for a vacation of random period V with probability $p(0 < p < 1)$ or may continue to serve the next unit with probability $(1-p)$. The vacation time of the server is general distributed with distribution function $V(y)$ having Laplace transform $\tilde{V}(s)$ and finite moments $E(V^j)$, $j = 1,2$.

Now, we introduce some more notations which will be used for the mathematical formulation of the model. Let $R^0(t), B_i^0(t), D_i^0(t), G_{i,l}(t)$ and $V^0(t)$ denote the elapsed times of retrial units, service of i^{th} stage, delay time to repair, repair of l^{th} phase and vacation state at time t. Consider a markov process $\{N(t), X(t)\}$ as joint distribution of number of units in retrial orbit and queue in the different system states with $N(t) = 0,1,2,....$; $X(t) = 0$ if $\zeta(t) = 0$, $X(t) = R^0(t)$ if $\zeta(t) = 1$, $X(t) = B_1^0(t)$ if $\zeta(t) = 2$,

$X(t) = B_2^0(t)$ if $\zeta(t) = 3$, $X(t) = V^0(t)$ if $\zeta(t) = 4$, $X(t) = D_1^0(t)$ if $\zeta(t) = 5$, $X(t) = D_2^0(t)$ if $\zeta(t) = 6$, $X(t) = G_{1,l}^0(t)$ if $\zeta(t) = 7$, $X(t) = G_{2,\,l}^0(t)$ if $\zeta(t) = 8$, $1 \le l \le m$, where the random variable $\zeta(t)$ denotes the server's status as given by

$$\zeta(t)=\begin{cases}0 & \text{if the server is idle with no unit in the system at time } t,\\ 1 & \text{if the server is idle when only retrial units are present in the system at time } t,\\ 2 & \text{if the server is busy in rendering the first stage regular service at time } t,\\ 3 & \text{if the server is busy in rendering the second stage optional service at time } t,\\ 4 & \text{if the server is on vacation at time } t,\\ 5 & \text{if the server is waiting for repair when failed during regular service at time } t,\\ 6 & \text{if the server is waiting for repair when failed during optional service at time } t,\\ 7 & \text{if the server is under repair when failed during regular service at time } t,\\ 8 & \text{if the server is under repair when failed during optional service at time } t,\end{cases}$$

3. MATHEMATICAL MODEL

In order to obtain the queue length distribution of non-markovian retrial queueing system with general service time, we first develop the mathematical model by using the assumptions and notations discussed in previous section.

3.1 Transient State and Limiting Probabilities

To analyze the mathematical model of queuing system, the transient probabilities are defined as:

$A_n(x,t)$ The probability of n retrial units in the system (including the unit being served or in the orbit, if any) with elapsed service time x at time t.

$P_n^i(x,t)$ The probability of n units in the system (including the unit being served, if any) with elapsed service time 'x' at time t; $i=1,2$.

$V_n(y,t)$ The probability of n units in the vacation state with elapsed service time 'y' at time t.

$D_n^i(x,y,t)$ The probability of n units in the system with elapsed service time x and elapsed delay time y at time t; $i=1,2$.

$R_{l,n}^i(x,y,t)$ The probability of n units in the system with elapsed service time x and elapsed repair time y at time t; $i=1,2; 1\le l\le m$.

The steady state limiting probabilities for different server states are considered as

$$P_0^0=\lim_{t\to\infty}\Pr ob\{N(t)=0,X(t)=0\}$$

$$A_n(x)=\lim_{t\to\infty}\Pr ob\{N(t)=n,X(t)=R^0(t),x<R^0(t)\le x+dx\};x>0;n\ge 1,$$

$$P_n^i(x)dx=\lim_{t\to\infty}\Pr ob\{N(t)=n,X(t)=B_i^0(t),x<B_i^0(t)\le x+dx\};x>0,i=1,2;n\ge 1,$$

$$V_n(y)=\lim_{t\to\infty}\Pr ob\{N(t)=n,X(t)=V^0(t),y<V^0(t)\le y+dy\};y>0;i=1,2;n\ge 1,$$

$$D_n^i(x,y)dx=\lim_{t\to\infty}\Pr ob\{N(t)=n,X(t)=D_i^0(t),y<D_i^0(t)\le y+dy\,|\,B_i^0(t)=x\};(x,y)>0,i=1,2;n\ge 1,$$

$$R_{l,n}^i(x,y)dx=\lim_{t\to\infty}\Pr ob\{N(t)=n,X(t)=G_{i,l}^0(t),y<G_{i,l}^0(t)\le y+dy\,|\,B_i^0(t)=x\};(x,y)>0,i=1,2;\ 1\le l\le m,n\ge 1$$

3.2 Hazard Rate Functions

The hazard rate functions for different server states are defined as follows:

$$\mu_i(x)dx = \frac{dB_i(x)}{[1-B_i(x)]}\ ;\ K(x)dx = \frac{dM(x)}{[1-M(x)]}; v(y)dy = \frac{dV(y)}{[1-V(y)]}; \eta_i(y)dy = \frac{dD_i(y)}{[1-D_i(y)]};$$

$$\xi_{i,l}(y)dy = \frac{dG_{i,\,l}(y)}{[1-G_{i,\,l}(y)]}; i = 1,2; 1 \le l \le m.$$

For the analysis purpose, the following probability generating functions (PGF) are used:

$$P^i(x,z) = \sum_{n=1}^{\infty} z^n P_n^i(x), P^i(0,z) = \sum_{n=1}^{\infty} z^n P_n^i(0), A(x,z) = \sum_{n=1}^{\infty} z^n A_n(x), V(y,z) = \sum_{n=1}^{\infty} z^n V_n(y),$$

$$V(0,z) = \sum_{n=1}^{\infty} z^n V_n(0), D^i(x,y,z) = \sum_{n=1}^{\infty} z^n D_n^i(x,y), D^i(x,0,z) = \sum_{n=1}^{\infty} z^n D_n^i(x,0), R_l^i(x,y,z) = \sum_{n=1}^{\infty} z^n R_{l,n}^i(x,y),$$

$$R_l^i(x,0,z) = \sum_{n=1}^{\infty} z^n R_{l,n}^i(x,0).$$

3.3 Governing Equations

To analyze the system, the governing equations of the system in different scenarios of the server states by using the probability reasoning under steady state conditions (cf. Cox (1955)) are as follows:

3.3.1 Idle State (Empty)

The equation for the idle state of the server when neither new unit nor retrial unit is present in the system, is

$$\lambda P_0^0 = q\left[\int_0^{\infty} \mu_2(x) P_1^2(x)dx + \overline{r}_0 \int_0^{\infty} \mu_1(x) P_1^1(x)dx\right] + \int_0^{\infty} v(y) V_1(y)dy \tag{1}$$

3.3.2 Idle State (Non-empty)

This state corresponds to the idle state of the server when no new unit is arrived but retrial unit is present in the system. The equation in this case is

$$\frac{d}{dx} A_n(x) + [\lambda b + K(x)] A_n(x) = 0;\ x > 0;\ \ n \ge 1 \tag{2}$$

3.3.3 Busy State

When the server is busy in providing essential/optional service, we have

$$\frac{d}{dx} P_n^i(x) + [\lambda b + \alpha_i + \mu_i(x)] P_n^i(x) = \lambda b \sum_{j=1}^{n} c_j P_{n-j}^i(x) + \int_0^{\infty} \xi_{i,m}(y) R_{m,n}^i(x,y)dy; x,y > 0,\ n \ge 1, i = 1,2 \tag{3}$$

3.3.4 Delayed Repair State

The server is unreliable and may breakdown in Poisson fashion. Due to some unavoidable reasons, there may be some delay in repair of failed unit occurs. In this case, the equation is framed as

$$\frac{d}{dy} D_n^i(x,y) + [\lambda b_1 + \eta_i(y)] D_n^i(x,y) = \lambda b_1 \sum_{j=1}^{n} c_j D_{n-j}^i(x,y);\ x,\ y > 0, n \ge 1,\ i = 1,2 \tag{4}$$

3.3.5 Repair State

The failed unit completes its repair in m compulsory phases so that, we have

$$\frac{d}{dy}R^i_{l,n}(x,y)+[\lambda b_2+\xi_{i,l}(y)]R^i_{l,n}(x,y)=\lambda b_2\sum_{j=1}^{n}c_jR^i_{l,n-j}(x,y);\ x,\ y>0,\ n\geq 1, i=1,2;1\leq l\leq m \quad (5)$$

3.3.6 Vacation State

After completion of the service of any unit, the server may avail the optional vacation. The equation governs this state is described as

$$\frac{d}{dy}V_n(y)+[\lambda b_3+v(y)]V_n(y)=\lambda b_3\sum_{j=1}^{n}c_jV_{n-j}(y);\quad n\geq 1, y>0 \quad (6)$$

3.4 Boundary Conditions

To analyze the $M^X/G/1$ queue, which is operating under stated assumptions, the following boundary conditions are considered.

To solve the equations (1)-(3), the boundary conditions at $x=0$ are imposed as follows:

$$A_n(0)=q\left[\int_0^\infty \mu_2(x)P^2_{n+1}(x)dx+\overline{r}_0\int_0^\infty \mu_1(x)P^1_{n+1}(x)dx\right]+\int_0^\infty v(y)V_{n+1}(y)dy, n\geq 1 \quad (7)$$

$$P^1_n(0)=\lambda c_nP^0_0+\lambda b\sum_{j=1}^{n}c_j\int_0^\infty A_{n-j}(x)dx+\int_0^\infty k(x)A_n(x)dx, n\geq 1 \quad (8)$$

$$P^2_n(0)=r_0\int_0^\infty \mu_1(x)P^1_n(x)dx, n\geq 1 \quad (9)$$

For fixed value of x, the boundary conditions at $y=0;\ i=1,2$ are considered to solve equations (4)-(6) as follows:

$$D^i_n(x,0)=\alpha_iP^i_n(x);\quad n\geq 1 \quad (10)$$

$$R^i_{1,n}(x,0)=\int_0^\infty \eta_i(y)D^i_n(x,y)dy,\quad n\geq 1;l=1 \quad (11)$$

$$R^i_{l,n}(x,0)=\int_0^\infty \xi_{i,l-1}(y)R^i_{l-1,n}(x,y)dy,\ n\geq 1;2\leq l\leq m \quad (12)$$

$$V_n(0)=p\left[\int_0^\infty \mu_2(x)P^2_n(x)dx+\overline{r}_0\int_0^\infty \mu_1(x)P^1_n(x)dx\right],\ n\geq 1 \quad (13)$$

3.4.1 Normalizing Condition

The normalizing condition can be stated as

$$P_0^0 + \sum_{i=1}^{2}\sum_{n=1}^{\infty}\left[\int_0^{\infty} P_n^i(x)dx + \int_0^{\infty}\int_0^{\infty} D_n^i(x,y)dxdy + \int_0^{\infty}\int_0^{\infty}\sum_{l=1}^{m} R_{l,n}^i(x,y)dxdy\right] + \sum_{n=1}^{\infty}\int_0^{\infty} A_n(x)dx + \sum_{n=1}^{\infty}\int_0^{\infty} V_n(y)dy = 1 \quad (14)$$

4. QUEUE LENGTH DISTRIBUTION

To establish the queue length distribution, the mathematical analysis by using the probability generating function approach is presented as follows (cf. Singh *et al.* (2016)):

$$P^i(x,z) = P^i(0,z)[1 - B_i(x)]\exp\{-\tau_i(z)x\};\ \ i = 1,2 \quad (15)$$

$$P^2(0,z) = r_0 P^1(0,z)\tilde{B}_1(\tau_1(z)) \quad (16)$$

$$V(0,z) = p[\overline{r}_0 + r_0\tilde{B}_2(\tau_2(z))]P^1(0,z)\tilde{B}_1(\tau_1(z)) \quad (17)$$

$$A(0,z) = (z^{-1})P^1(0,z)\tilde{B}_1(\tau_1(z))\{\overline{r}_0 + r_0\tilde{B}_2(\tau_2(z))\}\{q + p\tilde{V}(\varphi_5(z))\} - \lambda P_0^0 \quad (18)$$

$$P^1(0,z) = \lambda P_0^0 X(z) + A(0,z)[\tilde{M}(\lambda b) + X(z)(1 - \tilde{M}(\lambda b))] \quad (19)$$

$$D^i(x,y,z) = \alpha_i P^i(0,z)[1 - B_i(x)]\exp\{-\tau_i(z)x\}[1 - D_i(y)]\exp\{-\varphi_3(z)y\}; i = 1,2 \quad (20)$$

$$R_1^i(x,y,z) = \alpha_i P^i(0,z)[1 - B_i(x)]\exp(-\tau_i(z)x)[1 - G_{i,1}(y)]\exp(-\varphi_4(z)y)\tilde{D}_i(\varphi_3(z));\ i = 1.2 \quad (21)$$

$$R_l^i(x,y,z) = \alpha_i P^i(0,z)[1 - B_i(x)]\exp(-\tau_i(z)x)[1 - G_{i,l}(y)]\exp(-\varphi_4(z)y)\tilde{D}_i(\varphi_3(z)) \times \prod_{j=1}^{l-1}\tilde{G}_{i,j}(\varphi_4(z)); 2 \le l \le m \quad (22)$$

$$V(y,z) = [zP_0^0 p\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[\overline{r}_0 + r_0\tilde{B}_2(\tau_2(z))](1 - V(y))\exp\{-\varphi_5(z)y][S(z)]^{-1} \quad (23)$$

$$A(x,z) = [\lambda P_0^0(1 - M(x))\exp\{-\lambda bx\}[z - \tilde{B}_1(\tau_1(z))\{\overline{r}_0 + r_0\tilde{B}_2(\tau_2(z))\}\{q + p\tilde{V}(\varphi_5(z))\}X(z)]][S(z)]^{-1} \quad (24)$$

$$P_0^0 = b[1 - r_1 - d^{(1)}(1 - \tilde{M}(\lambda b))][b + \chi_1 + (1 - b)r_1 + \tilde{M}(\lambda b)r_2]^{-1} \quad (25)$$

where

$\varphi_1(z) = \lambda(1 - X(z)), \varphi_2(z) = \lambda b(1 - X(z)), \varphi_3(z) = \lambda b_1(1 - X(z)), \varphi_4(z) = \lambda b_2(1 - X(z)),$

$\varphi_5(z) = \lambda b_3(1 - X(z)),$

$\tau_i(z) = \varphi_2(z) + \alpha_i(1 - \tilde{D}_i(\varphi_3(z))\prod_{j=1}^{m}\tilde{G}_{i,j}(\varphi_4(z))); i = 1,2; \chi_1 = [1 - \tilde{M}(\lambda b)][(1 - b)d^{(1)} - 1];$

$S(z) = \tilde{B}_1(\tau_1(z))\{\overline{r}_0 + r_0\tilde{B}_2(\tau_2(z))\}\{q + p\tilde{V}(\varphi_5(z))\}[\tilde{M}(\lambda b) + X(z)(1 - \tilde{M}(\lambda b))] - z$

$r_1 = \lambda d^{(1)}[\beta_1^{(1)}(b + \alpha_1(\gamma_1^{(1)}b_1 + b_2\sum_{j=1}^{m} g_{1j}^{(1)})) + r_0\beta_2^{(1)}(b + \alpha_2(\gamma_2^{(1)}b_1 + b_2\sum_{j=1}^{m} g_{2j}^{(1)})) + pE(V)b_3]$

$r_2 = \lambda d^{(1)}[\beta_1^{(1)}\alpha_1(\gamma_1^{(1)}(b - b_1) + (b - b_2)\sum_{j=1}^{m} g_{1j}^{(1)}) + r_0\beta_2^{(1)}\alpha_2(\gamma_2^{(1)}(b - b_1) + (b - b_2)\sum_{j=1}^{m} g_{2j}^{(1)}) + (b - b_3)pE(V)]$

Lemma1: The necessary and sufficient stability condition for the system is given by inequality

$$r_1 + d^{(1)}(1 - \tilde{M}(\lambda b)) < 1 \quad (26)$$

Proof: By following the Foster's criterion on the mean drift in irreducible and periodic Markov chain $\{X_n, n \in Z^+\}$ (cf. Singh *et al.*, (2016)), we can establish the required result of stability condition as given in equation (26).

Theorem 1: Under the stability condition, the partial PGF of joint probability distributions of the server state and orbit size are given by

$$A(x,z)=[\lambda b\ \varepsilon_1(1-M(x))\exp\{-\lambda bx\}\ [z-\tilde{B}_1(\tau_1(z))\ \{\overline{r_0}+r_0\tilde{B}_2(\tau_2(z))\}\ \{q+p\tilde{V}(\varphi_5(z))\}\ X(z)]]\ [\varepsilon_2]^{-1} \quad (27)$$

$$P^1(x,z)=[zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)[1-B_1(x)]\exp\{-\tau_1(z)x\}][\varepsilon_2]^{-1} \quad (28)$$

$$P^2(x,z)=[r_0zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[1-B_2(x)]\exp\{-\tau_2(z)x\}][\varepsilon_2]^{-1} \quad (29)$$

$$V(y,z)=[zpb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[\overline{r_0}+r_0\tilde{B}_2(\tau_2(z))](1-V(y))\exp\{-\varphi_5(z)y\}][\varepsilon_2]^{-1} \quad (30)$$

$$D^1(x,y,z)=[\alpha_1zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)[1-B_1(x)]\exp\{-\tau_1(z)x\}[1-D_1(y)]\exp\{-\varphi_3(z)y\}][\varepsilon_2]^{-1} \quad (31)$$

$$D^2(x,y,z)=[\alpha_2r_0zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[1-B_2(x)]\exp\{-\tau_2(z)x\}[1-D_2(y)]\exp\{-\varphi_3(z)y\}][\varepsilon_2]^{-1} \quad (32)$$

$$R_1^1(x,y,z)=[\alpha_1zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)[1-B_1(x)]\exp\{-\tau_1(z)x\}\tilde{D}_1(\varphi_3(z))[1-G_{1,1}(y)]\exp\{-\varphi_4(z)y\}][\varepsilon_2]^{-1} \quad (33)$$

$$R_1^2(x,y,z)=[\alpha_2r_0zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[1-B_2(x)]\exp\{-\tau_2(z)x\}\tilde{D}_2(\varphi_3(z))[1-G_{2,1}(y)] \times\exp\{-\varphi_4(z)y\}][\varepsilon_2]^{-1} \quad (34)$$

$$R_l^1(x,y,z)=[\alpha_1zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)[1-B_1(x)]\exp(-\tau_1(z)x)[1-G_{1,l}(y)]\exp(-\varphi_4(z)y)\tilde{D}_1(\varphi_3(z)) \times\prod_{j=1}^{l-1}\tilde{G}_{1,j}(\varphi_4(z))][\varepsilon_2]^{-1};\ 2\le l\le m \quad (35)$$

$$R_l^2(x,y,z)=[\alpha_2r_0\ zb\,\varepsilon_1\,\varphi_1(z)\,\tilde{M}(\lambda b)\,\tilde{B}_1(\tau_1(z))[1-B_2(x)]\exp(-\tau_2(z)x)\ [1-G_{2,l}(y)] \times\exp(-\varphi_4(z)y)\ \tilde{D}_2(\varphi_3(z))\prod_{j=1}^{l-1}\tilde{G}_{2,j}(\varphi_4(z))]\ [\varepsilon_2]^{-1},\ 2\le l\le m \quad (36)$$

where $\varepsilon_1=[1-r_1-d^{(1)}(1-\tilde{M}(\lambda b))];\varepsilon_2=[b+\chi_1+(1-b)r_1+\tilde{M}(\lambda b)r_2]S(z)$

Proof: By using (25) in equation (24), we get the equation (27). For detailed proof see (cf. Singh *et al.* (2016)).

Theorem 2: Under the stability condition, the marginal PGF of the system states are

$$A(z)=[\varepsilon_1(1-\tilde{M}(\lambda b))[z-\tilde{B}_1(\tau_1(z))\{\overline{r_0}+r_0\tilde{B}_2(\tau_2(z))\}\{q+p\tilde{V}(\varphi_5(z))\}X(z)]][\varepsilon_2]^{-1} \quad (37)$$

$$P^1(z)=[zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)[1-\tilde{B}_1(\tau_1(z))]][\varepsilon_2\tau_1(z)]^{-1} \quad (38)$$

$$P^2(z)=[r_0zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[1-\tilde{B}_2(\tau_2(z))]][\varepsilon_2\tau_2(z)]^{-1} \quad (39)$$

$$V(z)=[zpb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[\overline{r_0}+r_0\tilde{B}_2(\tau_2(z))](1-\tilde{V}(\varphi_5(z)))][\varepsilon_2\varphi_5(z)]^{-1} \quad (40)$$

$$D^1(z)=[\alpha_1zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)[1-\tilde{B}_1(\tau_1(z))][1-\tilde{D}_1(\varphi_3(z))]][\varepsilon_2\tau_1(z)\varphi_3(z)]^{-1} \quad (41)$$

$$D^2(z)=[\alpha_2r_0zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[1-\tilde{B}_2(\tau_2(z))][1-\tilde{D}_2(\varphi_3(z))]][\varepsilon_2\tau_2(z)\varphi_3(z)]^{-1} \quad (42)$$

$$R_1^1(z)=[\alpha_1zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)[1-\tilde{B}_1(\tau_1(z))]\tilde{D}_1(\varphi_3(z))[1-\tilde{G}_{1,1}(\varphi_4(z))]][\varepsilon_2\tau_1(z)\varphi_4(z)]^{-1} \quad (43)$$

$$R_1^2(z)=[\alpha_2r_0zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[1-\tilde{B}_2(\tau_2(z))]\tilde{D}_2(\varphi_3(z))[1-\tilde{G}_{2,1}(\varphi_4(z))]][\varepsilon_2\tau_2(z)\varphi_4(z)]^{-1} \quad (44)$$

$$R_l^1(z)=[\alpha_1zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)[1-\tilde{B}_1(\tau_1(z))][1-\tilde{G}_{1,l}(\varphi_4(z))]\tilde{D}_1(\varphi_3(z)) \times\prod_{j=1}^{l-1}\tilde{G}_{1,j}(\varphi_4(z))][\varepsilon_2\tau_1(z)\varphi_4(z)]^{-1};\ 2\le l\le m \quad (45)$$

$$R_l^2(z)=[\alpha_2r_0zb\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b)\tilde{B}_1(\tau_1(z))[1-\tilde{B}_2(\tau_2(z))][\ 1-\tilde{G}_{2,l}(\varphi_4(z))]\ \tilde{D}_2(\varphi_3(z)) \times\prod_{j=1}^{l-1}\tilde{G}_{2,j}(\varphi_4(z))][\ \varepsilon_2\tau_2(z)\varphi_4(z)]^{-1};2\le l\le m \quad (46)$$

Proof: Taking integration of equations (27)-(36) with respect to appropriate variable and using the result $\int_0^\infty e^{-su}(1-M(u))du=[1-\tilde{M}(s)][s]^{-1}$, we get the required results.

Further, the effective arrival rates (λ_e) of the units in different server's status, are obtained by putting $z=1$ in equations (37)-(46) and using

$\lambda_e=\lambda P_0^0+\lambda b\left(\sum_{i=1}^2 P^i(1)+A(1)\right)+\lambda b_1\left(\sum_{i=1}^2 D^i(1)\right)+\lambda b_2\sum_{l=1}^m\left(\sum_{i=1}^2 R_l^i(1)\right)+\lambda b_3V(1),$ we get

$$\lambda_e = [\lambda b\tilde{M}(\lambda b))][b + \chi_1 + (1-b)r_1 + \tilde{M}(\lambda b)r_2]^{-1} \quad (47)$$

Also, $\rho = \lambda_e d^{(1)}\ [\beta_1^{(1)}(1+\alpha_1(\gamma_1^{(1)} + \sum_{j=1}^{m} g_{1j}^{(1)})) + r_0\ \beta_2^{(1)}(1+\alpha_2(\gamma_2^{(1)} + \sum_{j=1}^{m} g_{2j}^{(1)})) + pE(V)].$ (48)

Theorem 3: Under the stability condition, at the departure epoch PGF of the stationary queue size is

$$\omega(z) = [\varepsilon_1[1-X(z)]\tilde{B}_1(\tau_1(z))\{\overline{r}_0 + r_0\tilde{B}_2(\tau_2(z))\}\{q + p\tilde{V}(\varphi_5(z))\}][d^{(1)}S(z)]^{-1} \quad (49)$$

Proof: The probability (w_j) with j units in the queue at departure epoch with normalizing constant (K_0) is considered and queue size distribution at the departure epoch is obtained as follows: (cf. Wolf (1982))

$$w_j = K_0\left[\overline{r}_0\int_0^\infty \mu_1(x)P_{j+1}^1(x)dx + \int_0^\infty \mu_2(x)P_{j+1}^2(x)dx + \int_0^\infty v(y)V_{j+1}(y)dy\right] \quad (50)$$

By using $\omega(z) = \sum_{j=0}^{\infty} w_j z^j$, equation (50) yields

$$\omega(z) = [K_0 b\varepsilon_1\varphi_1(z)\tilde{M}(\lambda b))\tilde{B}_1(\tau_1(z))\{\overline{r}_0 + r_0\tilde{B}_2(\tau_2(z))\}\{q + p\tilde{V}(\varphi_5(z))\}][\varepsilon_2]^{-1} \quad (51)$$

The normalizing condition $\omega(1) = 1$ gives

$$K_0 = [b + \chi_1 + (1-b)r_1 + \tilde{M}(\lambda b)r_2][\lambda d^{(1)}b\tilde{M}(\lambda b)]^{-1} \quad (52)$$

Using equations (51)-(52), we obtain the required result as given in equation (49).

Theorem 4: The PGF of stationary queue size in the system and orbit at arbitrary epoch are given as

$$\begin{aligned} P(z) = {} & \varepsilon_1[bS(z) + (1-\tilde{M}(\lambda b))[z - \chi_6\{q + p\tilde{V}(\varphi_5(z))\}X(z)] + zb\varphi_1(z)\tilde{M}(\lambda b)\big(\chi_2 + r_0\tilde{B}_1(\tau_1(z))\chi_3\big) \\ & + S(z)\big(\chi_4 + r_0\tilde{B}_1(\tau_1(z))\chi_5\big) + zbp\chi_6\varphi_1(z)\tilde{M}(\lambda b)(1-\tilde{V}(\varphi_5(z))[\varphi_5(z)]^{-1}][\varepsilon_2]^{-1} \end{aligned} \quad (53)$$

$$\begin{aligned} O(z) = {} & \varepsilon_1[bS(z) + (1-\tilde{M}(\lambda b))[z - \chi_6\{q + p\tilde{V}(\varphi_5(z))\}X(z)] + zb\varphi_1(z)\tilde{M}(\lambda b)\big(\chi_2 + r_0\tilde{B}_1(\tau_1(z))\chi_3\big) \\ & + S(z)\big(\chi_4 + r_0\tilde{B}_1(\tau_1(z))\chi_5\big) + [zbp\chi_6\varphi_1(z)\tilde{M}(\lambda b)(1-\tilde{V}(\varphi_5(z))][\varphi_5(z)]^{-1}][\chi_6\{q + p\tilde{V}(\varphi_5(z))\}\varepsilon_2]^{-1} \end{aligned} \quad (54)$$

with

$\chi_2 = (1-\tilde{B}_1(\tau_1(z)))[\tau_1(z)]^{-1}\{1 + \alpha_1(1-\tilde{D}_1(\varphi_3(z)))(b_2 - b_1)(b_1b_2)^{-1} - bb_2^{-1}; \chi_4 = (1-\tilde{B}_1(\tau_1(z)))(\varphi_4(z))^{-1};$

$\chi_3 = (1-\tilde{B}_2(\tau_2(z)))(\tau_2(z))^{-1}\{1 + \alpha_2(1-\tilde{D}_2(\varphi_3(z)))(b_2 - b_1)(b_1b_2)^{-1} - bb_2^{-1}\}; \chi_5 = (1-\tilde{B}_2(\tau_2(z)))(\varphi_4(z))^{-1};$

$\chi_6 = \tilde{B}_1(\tau_1(z))\{\overline{r}_0 + r_0\tilde{B}_2(\tau_2(z))$.

Proof: Using the relations

$$P(z) = P_0^0 + A(z) + \sum_{i=1}^{2} P^i(z) + V(z) + \sum_{i=1}^{2} D^i(z) + \sum_{i=1}^{2}\sum_{l=1}^{m} R_l^i(z);$$

$$P(z) = O(z)\tilde{B}_1(\tau_1(z))\{\overline{r}_0 + r_0\tilde{B}_2(\tau_2(z))\}\{q + p\tilde{V}(\varphi_5(z))\}$$

We get the required results given in equations (53) - (54).

5. PERFORMANCE MEASURES

The main objective of studying the queueing system is to derive the performance measures which can be interpreted in terms of its queue length distribution.

5.1 Queueing Indices

The server's states in the system are considered on the basis of different time periods; namely idle period I (N) as the length of time per cycle when the server is idle when no unit (only retrial unit) present in the system. The length of time per cycle when the server is busy in rendering essential service (optional service) is defined as busy period (B_i). The time duration for which the server is waiting for the repair (under repair) when broken down during essential service (optional service) is considered as D_i (R_i), respectively. The duration for which the server is under vacation is defined as vacation period (V).

The expected length of cycle is given by $E(C) = E(I) + E(H)$ (55)

where

$$E(I) = [\lambda_e d^{(1)}]^{-1} \tag{56}$$

$$E(H) = \sum_{i=1}^{2} [E(B_i) + E(D_i) + E(R_i)] + E(V) \tag{57}$$

5.1.1 Long Run Probabilities

To evaluate the queue length and other metrics of the system, the probabilities of the system at different server states are important characteristics. The long run probabilities for different server states of the system are obtained as

$$P(I) = [\lambda b\tilde{M}(\lambda b)(1-\rho) - \lambda_e[r_1 + d^{(1)} - 1][1 - \tilde{M}(\lambda b)]][\lambda b\tilde{M}(\lambda b)]^{-1} = E(I)[E(C)]^{-1} \tag{58}$$

$$P(B_1) = \lambda_e d^{(1)}\beta_1^{(1)} = E(B_1)[E(C)]^{-1} \tag{59}$$

$$P(B_2) = \lambda_e r_0 d^{(1)}\beta_2^{(1)} = E(B_2)[E(C)]^{-1} \tag{60}$$

$$P(D_1) = \alpha_1\lambda_e d^{(1)}\gamma_1^{(1)}\beta_1^{(1)} = E(D_1)[E(C)]^{-1} \tag{61}$$

$$P(D_2) = \alpha_2\lambda_e r_0 d^{(1)}\gamma_2^{(1)}\beta_2^{(1)} = E(D_2)[E(C)]^{-1} \tag{62}$$

$$P(R_l^1) = \alpha_1\lambda_e d^{(1)} g_{1l}^{(1)}\beta_1^{(1)} = E(R_1)[E(C)]^{-1} \tag{63}$$

$$P(R_l^2) = \alpha_2\lambda_e r_0 d^{(1)} g_{2l}^{(1)}\beta_2^{(1)} = E(R_2)[E(C)]^{-1} \tag{64}$$

$$P(V) = p\lambda_e d^{(1)} E(V) \tag{65}$$

$$P(N) = [\lambda_e[r_1 + d^{(1)} - 1][1 - \tilde{M}(\lambda b)]][\lambda b\tilde{M}(\lambda b)]^{-1} \tag{66}$$

where

$$P(R_1) = \sum_{l=1}^{m} P(R_l^1); P(R_2) = \sum_{l=1}^{m} P(R_l^2);$$

$$[E(C)]^{-1} = [\lambda_e d^{(1)}\{\lambda b\tilde{M}(\lambda b)(1-\rho) - \lambda_e[r_1 + d^{(1)} - 1][1 - \tilde{M}(\lambda b)]\}][\lambda b\tilde{M}(\lambda b)]^{-1} \tag{67}$$

Proof: The long run probabilities can be obtained using $P(B_i) = \lim_{z\to 1} P^i(z)$, $P(D_i) = \lim_{z\to 1} D^i(z)$,

$P(R_l^i) = \lim_{z\to 1} R_l^i(z)\,; i = 1,2,\quad 1 \le l \le m$

On putting $z = 1$ in equations (37)-(46) and using the equation (47) we get the equations (59)-(66) and

$$P(I) = 1 - \left[\sum_{i=1}^{2} [P(B_i) + P(D_i) + \sum_{l=1}^{m} P(R_l^i)] + P(V) + P(N)\right]. \tag{68}$$

5.1.2 Mean Queue Length at Departure Epoch

The mean queue length at departure epoch (L_D) is determined by using

$$L_D = \left(\frac{d\omega(z)}{dz}\right)_{z=1} = r_1 + \frac{d^{(2)}}{2d^{(1)}} + \frac{S''(1)}{2(1 - r_1 - d^{(1)}(1 - \tilde{M}(\lambda b)))} \tag{69}$$

where

$$\begin{aligned} S''(1) = [&d^{(2)}(1 - \tilde{M}(\lambda b)) + 2r_0\beta_1^{(1)}\beta_2^{(1)}(\lambda d^{(1)})^2\psi_1\psi_2 - r_0\beta_2^{(1)}\tau_2''(1) + 2\,r_0\beta_2^{(1)}\lambda(d^{(1)})^2\psi_2(1 - \tilde{M}(\lambda b)) \\ &+ 2\,r_0 pE(V)b_3\beta_2^{(1)}(\lambda d^{(1)})^2\psi_2 - \beta_1^{(1)}\tau_1''(1) + 2\beta_1^{(1)}\lambda(d^{(1)})^2\psi_1(1 - \tilde{M}(\lambda b)) + pE(V)\lambda b_3\{d^{(2)} \\ &+ 2pE(V)b_3\beta_1^{(1)}(\lambda d^{(1)})^2\psi_1 + 2(d^{(1)})^2(1 - \tilde{M}(\lambda b))\} + pE(V^2)(\lambda b_3 d^{(1)})^2 \\ &+ r_0\beta_2^{(2)} \times \{\psi_2\lambda d^{(1)}\}^2 + \beta_1^{(2)}\{\psi_1\lambda d^{(1)}\}^2] \end{aligned}$$

$$\psi_1 = (b + \alpha_1(b_1\gamma_1^{(1)} + b_2\sum_{j=1}^{m} g_{1j}^{(1)}));\psi_2 = (b + \alpha_2(b_1\gamma_2^{(1)} + b_2\sum_{j=1}^{m} g_{2j}^{(1)}))$$

$$\tau_i''(1) = -\{\lambda b d^{(2)} + \alpha_i (2b_1 b_2 (\lambda\, d^{(1)})^2 \gamma_i^{(1)} \sum_{j=1}^{m} (g_{ij}^{(1)}) + 2(\lambda b_2 d^{(1)})^2 \sum_{l=2}^{m} \sum_{j=1}^{l-1} (g_{ij}^{(1)} g_{il}^{(1)})$$
$$+ [\lambda b_1 d^{(2)})\gamma_i^{(1)} + (\lambda b_1 d^{(1)})^2 \gamma_i^{(2)}] + \sum_{j=1}^{m} [\lambda b_2 d^{(2)}) g_{ij}^{(1)} + (\lambda b_2 d^{(1)})^2 g_{ij}^{(2)}])\}; i = 1, 2.$$

5.1.3 Mean Queue Length at Arbitrary Epoch

The mean queue length at arbitrary epoch (L_q) is determined as

$$L_q = \left(\frac{dP(z)}{dz}\right)_{z=1}$$
$$= \varepsilon_1 [b + \chi_1 + (1-b) r_1 + \tilde{M}(\lambda b) r_2]^{-1} [(1 - \tilde{M}(\lambda b))(r_1 + d^{(1)}(1 - \tilde{M}(\lambda b)) - 1)\psi_3 - (1 - r_1 - d^{(1)}) S''(1)[2\varepsilon_1^2]^{-1}]$$
$$+ b\,\tilde{M}(\lambda b)\{\lambda d^{(1)} \psi_4 [\varepsilon_1]^{-1} + ((\beta_1^{(1)} + r_0 \beta_2^{(1)})(1 - b b_2^{-1}) + b_2^{-1}(\beta_1^{(1)} \psi_1 + r_0 \beta_2^{(1)} \psi_2)) \times$$
$$([1 - r_1 - d^{(1)}(1 - \tilde{M}(\lambda b))][\lambda d^{(2)} + 2\lambda d^{(1)}] + \lambda d^{(1)} S''(1)[2\varepsilon_1^2]^{-1})\}$$
$$+ b\, p \tilde{M}(\lambda b) b_3^{-1} \{[1 - r_1 - d^{(1)}(1 - \tilde{M}(\lambda b))]\psi_5 + \lambda b_3 E(V) d^{(1)} S''(1)\}[2\varepsilon_1^2]^{-1}\} \tag{70}$$

where

$$\psi_3 = -\left(S''(1) + \tilde{M}(\lambda b)\left\{d^{(2)}) + 2\, r_0 \beta_2^{(1)} \lambda (d^{(1)})^2 \psi_2 + 2\beta_1^{(1)} \lambda (d^{(1)})^2 \psi_1 + 2pE(V)\lambda b_3 (d^{(1)})^2\right\}\right)$$
$$\psi_4 = -\beta_1^{(1)} \alpha_1 \lambda b_1 d^{(1)} \gamma_1^{(1)} (b_2 - b_1)[b_1 b_2]^{-1} + (b_2 - b)(2b_2)^{-1} \beta_1^{(2)} \lambda d^{(1)} \psi_1$$
$$+ 2r_0 \left[-[\beta_2^{(1)} \alpha_2 \lambda b_1 d^{(1)} \gamma_2^{(1)} (b_2 - b_1)][b_1 b_2]^{-1} + (b_2 - b)(2b_2)^{-1} \{\beta_2^{(2)} \lambda d^{(1)} \psi_2 + 2\beta_1^{(1)} \beta_2^{(1)} \lambda d^{(1)} \psi_1\}\right]$$
$$+ [\lambda b_2 d^{(1)} \{\beta_1^{(2)} (\lambda d^{(1)} \psi_1)^2 - \beta_1^{(1)} \tau_1''(1)\} - \lambda^2 b_2 \beta_1^{(1)} d^{(1)} E(X^{(2)}) \psi_1][2(\lambda b_2 d^{(1)})^2]^{-1}$$
$$+ r_0 [[\lambda b_2 d^{(1)} \{\beta_2^{(2)} (\lambda d^{(1)} \psi_2)^2 - \beta_2^{(1)} \tau_2''(1)\} - \lambda^2 b_2 \beta_2^{(1)} d^{(1)} d^{(2)} \psi_2][2(\lambda b_2 d^{(1)})^2]^{-1}$$
$$+ [\beta_1^{(1)} \beta_2^{(1)} (\lambda d^{(1)})^2 \psi_1 \psi_2][\lambda b_2 d^{(1)}]^{-1}]$$
$$\psi_5 = -[2\lambda b_3 d^{(1)} E(V) + 2b_3 E(V)(\lambda d^{(1)})^2 (\beta_1^{(1)} \psi_1 + r_0 \beta_2^{(1)} \psi_2) + E(V^2)(\lambda b_3 d^{(1)})^2 + E(V)\lambda b_3 d^{(2)}]$$

5.1.4 Mean Waiting Time at Departure Epoch

The mean waiting time of the units in the queue (W_q) at departure epoch is obtained using Little's formula and given by

$$W_q = L_q [\lambda_e d^{(1)}]^{-1} \tag{71}$$

5.1.5 Mean Queue Length of Orbit at Arbitrary Epoch

The mean queue length of orbit at arbitrary epoch (L_0) is

$$L_0 = \left(\frac{dO(z)}{dz}\right)_{z=1} = L_q - r_1 [b\varepsilon_1 + [r_1 + d^{(1)} - 1][1 - \tilde{M}(\lambda b)]] S(z)[\varepsilon_2]^{-1} + \rho) \tag{72}$$

5.2 Reliability Indices

The efficiency of the unreliable server system can be predicted in terms of the reliability indices. During the design and development stage of the system, these indices can play important role. Under the steady state conditions, we get the following reliability indices:

The availability of the server which gives the probability of the server being available in the system is

$$A_v = P_0^0 + \sum_{i=1}^{2}\int_0^\infty P^i(x,1)dx = P_0^0 + \lim_{z\to 1}[\sum_{i=1}^{2} P^i(z)]$$
$$= 1 - [[1-\tilde{M}(\lambda b)][r_1 + d^{(1)} - 1] + \lambda b\tilde{M}(\lambda b)d^{(1)}\,[\beta_1^{(1)}\alpha_1(\gamma_1^{(1)} + \sum_{j=1}^{m} g_{1j}^{(1)}) + r_0\,\beta_2^{(1)}\alpha_2(\gamma_2^{(1)} + \sum_{j=1}^{m} g_{2j}^{(1)})$$
$$+ pE(V)]]S(z)[\varepsilon_2]^{-1} \tag{73}$$

The steady state failure frequency which measure the rate of failure of the server is

$$F_f = \alpha_1\int_0^\infty P^1(x,1) + \alpha_2\int_0^\infty P^2(x,1) = \lim_{z\to 1}[\alpha_1 P^1(z) + \alpha_2 P^2(z)] = \lambda_e d^{(1)}[\alpha_1\beta_1^{(1)} + r_0\alpha_2\beta_2^{(1)}].$$

5.3 Cost Analysis

To find the decision variables corresponding to optimum cost of the system, we construct the cost function in terms of cost elements associated with different activities such as holding cost (C_h) per unit time for each unit present in the system, start up cost (C_S) per unit time, (C_{B_i}) cost per unit time when the server is busy with essential/optional service, delayed repair cost (C_{D_i}) per unit time when the server is broken down during essential/optional service, repair cost (C_{R_i}) per unit time incurred on the server failed during essential/optional service, (C_v) cost per unit time incurred on the vacationing server.

The total expected cost function per unit time (TC) is

$$TC = C_h L_q + [E(C)]^{-1}\left[C_S + \sum_{i=1}^{2}\{C_{B_i}E(B_i) + C_{D_1}E(D_i) + C_{R_1}E(R_i)\} + C_v E(V)\right] \tag{74}$$

6. SPECIAL CASES

In this section, we discuss some special cases of our model which are deduced by setting the appropriate parameters and results match with the existing models.

Case (i): $M^X/G/1$ ***unreliable retrial queue with uniform arrival, no delayed repair, single phase repair, optional service, vacation.***

By setting $b = b_1 = b_2 = b_3 = 1, m = 1$ and $\tilde{D}_i(\varphi_3(z)) = 1$ in equations (49) and (53), we have

$$\omega(z) = [[1-\rho-d^{(1)}(1-\tilde{M}(\lambda))][1-X(z)]\tilde{B}_1(\tau_1(z))\{\overline{r_0} + r_0\tilde{B}_2(\tau_2(z))\}\{q + p\tilde{V}(\varphi(z))\}][d^{(1)}S(z)]^{-1} \tag{75}$$
$$P(z) = [[1-\rho-d^{(1)}(1-\tilde{M}(\lambda))](1-z)\tilde{B}_1(\tau_1(z))\{\overline{r_0} + r_0\tilde{B}_2(\tau_2(z))\}\{q + p\tilde{V}(\varphi(z))\}][S(z)]^{-1}; \tag{76}$$
$$\varphi(z) = \lambda(1-X(z)),\ \tau_i(z) = \varphi(z) + \alpha_i(1-\tilde{G}_{i,1}(\varphi_4(z)));\ i = 1,2$$

The results obtained above are same as formulated recently in Singh *et al.* (2016).

Case (ii): $M/G/1$ ***unreliable retrial queue with uniform arrival, delayed repair, single phase repair, vacation, no optional service.***

By setting $P(X=1) = 1, b = b_1 = b_2 = b_3 = 1,\ r_0 = 0,\ m = 1$, the equation (49) yields

$$\omega(z) = (1-\rho)(1-z)\tilde{B}_1(\tau_1(z))\{q + p\tilde{V}(\lambda(1-z))\}[\tilde{M}(\lambda)(1-z)\tilde{B}_1(\tau_1(z)\{q + p\tilde{V}(\lambda(1-z))\}$$
$$- z[1 - \{q + p\tilde{V}(\lambda(1-z))\}\tilde{B}_1(\tau_1(z)]]^{-1}; \tag{77}$$
$$\tau_1(z) = \lambda(1-z) + \alpha_1(1-\tilde{D}_1(\lambda(1-z))\tilde{G}_{1,1}(\lambda(1-z)))$$

This result coincides with the model of Choudhury and Ke (2014).

Case (iii): $M/G/1$ ***unreliable queue uniform arrival, delayed repair, single phase repair, optional service, no retrial unit, no vacation.***

By setting $P(X=1)=1$, $b=b_1=b_2=b_3=1$, $m=1, p=0$, $\tilde{M}(\lambda b)=1$ equation (49), coverts to

$$\omega(z)=[(1-\rho)(1-z)\tilde{B}_1(\tau_1(z))\{\overline{r}_0+r_0\tilde{B}_2(\tau_2(z))\}][(\tilde{B}_1(\tau_1(z))\{\overline{r}_0+r_0\tilde{B}_2(\tau_2(z))\}-z)]^{-1}; \quad (78)$$

$$\tau_i(z)=\lambda(1-z)+\alpha_i(1-\tilde{D}_i(\lambda(1-z))\tilde{G}_{i,1}(\lambda(1-z))); i=1,2.$$

The result obtained tallies with the result of Choudhury and Tadj (2009).

Case (iv): $M/G/1$ ***model with reliable server, uniform arrival, optional service, no retrial, no vacation.***

By setting $P(X=1)=1$, $\alpha_1=\alpha_2=0$, $b=b_1=b_2=b_3=1, \tilde{D}_i(\varphi_3(z))=1, p=0,\ \tilde{M}(\lambda b)=1$ in equation (49), we get

$$\omega(z)=[(1-\rho)(1-z)\tilde{B}_1(\lambda(1-z))\{q+p\tilde{V}(\lambda(1-z))\}][(\tilde{B}_1(\lambda(1-z))\{q+p\tilde{V}(\lambda(1-z))\}-z)]^{-1} \quad (79)$$

In this case, the result coincides with Medhi (2002).

Case (v): $M^X/G/1$ ***model with reliable server, uniform arrival, no optional service, no retrial, single vacation***

By setting $\alpha_1=\alpha_2=0$, $b=b_1=b_2=b_3=1,\ \tilde{D}_i(\varphi_3(z))=1$, $\tilde{M}(\lambda b)=1$ equation (49) gives

$$\omega(z)=[(1-\rho)(1-z)\tilde{B}_1(\lambda(1-z))\{\overline{r}_0+r_0\tilde{B}_2(\lambda(1-z))\}][(\tilde{B}_1(\lambda(1-z))\{\overline{r}_0+r_0\tilde{B}_2(\lambda(1-z))\}-z)]^{-1} \quad (80)$$

The same result was obtained in Choudhury and Madan (2006).

7. NUMERICAL ILLUSTRATION

For computation of numerical results in the congestion situation of industrial scenario encountered in manufacturing of electric appliances, an illustration which also validates the analytical results of the system is considered with different statistical distributions as follows:

(i) The flow of input as demand for specific product is in the batches and the batch size follows the geometrical distribution with first two moments $d^{(1)}=e(1-e)^{-1}$ and $d_{(2)}=d^{(1)}+d^{(2)}=e(1+e)(1-e)^{-2}$, respectively.

(ii) The both stages (essential/optional) of the service to fulfill the demand of the product with basic and luxury needs are considered to be $k-$ Erlangian, so that the first two moments are $\beta_i^{(1)}=[\mu_i]^{-1}, \beta_i^{(2)}=(k+1)[k\mu_i^{-2}]; i=1,2.$

(iii) On breakdown of the server, the delayed time to start the repair of failed server is assumed to follow the exponential distribution with parameters $\gamma_i (i=1,2)$. The first two moments of delayed time distribution are $\gamma_i^{(1)}=[\gamma_i]^{-1}, \gamma_i^{(2)}=2[\gamma_i^{-2}]$. For the computation purpose, we set the parameter $\gamma_i^{(1)}=g_{ij}^{(1)}/2$, $i=1,2$; $j=1,2,\ldots,m.$

(iv) During repair period, the repair time of failed server follows the gamma distribution with parameter g_{ij} and first two moments are considered as $g_{ij}^{(1)}=2[g_{ij}], g_{ij}^{(2)}=6[g_{ij}^{-2}]; i=1,2; j=1,2,\ldots,m$. We set the parameter g_{ij} as $g_{ij}^{(1)}=\beta_i^{(1)}/3$, $i=1,2; j=1,2,\ldots,m.$

(v) It is also proposed that, to perform some useful internal modification and other essential changes for efficient service, the vacation time of the server assumed to follow the exponential distribution with parameter v. The first and second moments of the vacation time distribution are taken as $E(V)=[v]^{-1}, E(V^2)=2[v^{-2}]$.

(vi) Due to retrial process of the service system, the customers may retry to get the service after some random time period. The distribution of retrial time is considered as exponential distribution with parameter θ, so $\tilde{M}(\lambda b)=\theta[\theta+\lambda b]^{-1}$.

The interpretation of the numerical results carried out for different performance measures are shown in the form of tables 1-5 and graphs 1-4. To compute the numerical results, the default parameters are considered as follows:

$$E(X)=2, \mu_1=\mu,=5,\ \mu_2=2\mu,\ m=3,\ k=2,\ \lambda=1.55,\ \alpha_1=\alpha=0.1,\ \alpha_2=2\alpha,\ r_0=0.6, v=20,$$

$$b=0.6, b_1=0.3, b_2=0.4, b_3=0.5.$$

7.1 Effects on Mean Queue Length

The queue length of the products demanded is affected by various parameters of the system namely service rate of the server, input flow of the demands, failure rate of the server and retrial rate of the customers, etc. The server provides the service only one unit at a time and other arrivals decides to either join the system after random period of time to get the service or not join the system. The effects of different parameters are presented in the tables 1-2. These tables exhibit the effects of arrival rate (λ) and service rates (μ) respectively, for the fixed values of other parameters on the queue length (L_q). It is observed that L_q increases with the growth of λ and α, but it in reverse trend with increase in μ for the fixed values of vacation probabilities or retrial time of the units. It is also seen that L_q increases (decreases) with increasing trend of $p(\theta)$.

7.2 Effects on Reliability Indices

The analytical results of availability measure (A_v) and failure frequency (F_f) are computed in order to study the effects of parameters and validate the results obtained for the queueing system with unreliable server in table 3. The effect of probability (r_0) to opt optional service on availability measure (A_v) and failure frequency (F_f) is presented in the table. It also displays that A_v decreases and F_f increases with the growth of α and θ.

7.3 Optimal Cost

To obtain optimal values of discrete parameters and corresponding cost parameters, the heuristic search method based on discrete allocation scheme is applied to find the optimal values of discrete system parameters and corresponding cost parameters. The MATLAB software is used to find the optimal values numerically. The total cost of the system depends on the cost incurred on different activities of the system. In order to explore the effects of different parameters on optimal cost, the default values of different costs elements are taken as $C_h = \$10/\text{day}$, $C_s = \$500/\text{day}$, $C_{b_i} = \$50/\text{unit}$, $C_{d_i} = \$20/\text{day}, C_{r_i} = \$30/\text{unit}$, $C_V = \$40/\text{day}$ and total cost are displayed in tables 4-5.

The effect of μ on total cost (*TC*) with the variation in p and θ are shown in table 4. From table, we observe that the total cost initially decreases with the growth in μ which shows the convexity of the cost function. It is also noticed that the total cost decreases with the increase in θ for different values of p. Further for fixed values of μ, total cost decreases (increases) with the increase in retrial rate θ (p) but after certain values of μ, this trend reveals a reverse pattern. Further from the table, it is also observed that for fixed values of p, total cost decreases with the increase values of θ. Table 5 shows the change in the cost for the varying the values of failure rates α of the server for different values of p and θ.

The effects of θ and r_0 on L_q for different service phases (k) are seen in figures 1-2. From these figures, it is clear that L_q decreases (increases) with the growth of the values of θ (r_0) for fixed values of some parameters. The effects of different parameters on total cost are presented in figures 3-4. From these figures, the optimal values for (λ^*, θ^*) and (μ^*, p^*) are obtained as (1.2, 3.0) and (5.0, 0.9), respectively with their optimal total costs \$260.93 and \$268.07 for fixed values of other parameters.

Table 1: L_q for different values of parameters (λ, p, θ)

λ	$p = 0.3$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$
1.40	15.23	14.33	13.55	19.14	17.78	16.64
1.45	19.04	17.67	16.51	25.53	23.19	21.30
1.50	25.05	22.74	20.89	37.58	32.76	29.15

1.55	35.97	31.43	28.02	69.01	54.39	45.18
1.60	62.07	49.76	41.78	359.39	150.27	96.31

Table 2: L_q for different values of parameters (μ, p, θ)

μ	$p = 0.3$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$
4.75	60.68	49.05	41.41	279.09	135.48	90.61
5.00	35.97	31.43	28.02	69.01	54.39	45.18
5.25	26.06	23.53	21.52	40.57	34.92	30.79
5.50	20.71	19.04	17.67	29.28	26.16	23.71
5.75	17.36	16.14	15.12	23.22	21.17	19.51

Table 3: Effects of parameters α, θ and r_0 on A_v and F_f

	$r_0 = 0.5$				$r_0 = 0.9$			
	$\theta = 3.5$		$\theta = 3.7$		$\theta = 3.5$		$\theta = 3.7$	
α	A_v	F_f	A_v	F_f	A_v	F_f	A_v	F_f
0.1	0.6810	0.0433	0.6940	0.0436	0.6786	0.0534	0.6917	0.0537
0.2	0.6735	0.0862	0.6864	0.0867	0.6701	0.1062	0.6831	0.1068
0.3	0.6660	0.1286	0.6789	0.1294	0.6617	0.1583	0.6747	0.1592
0.4	0.6586	0.1706	0.6714	0.1716	0.6534	0.2097	0.6663	0.2110
0.5	0.6513	0.2120	0.6640	0.2133	0.6452	0.2605	0.658	0.2621

Table 4: Effect of parameters μ, p and θ on *TC*

μ	$p = 0.5$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.7$
4.25	299.16	290.93	284.69	316.69	304.81	295.71
4.50	277.42	274.05	271.73	284.89	279.55	275.61
4.75	270.02	**269.26**	**269.11**	272.18	270.15	268.93
5.00	**269.63**	270.49	271.72	**268.51**	**268.44**	**268.88**
5.25	273.02	274.98	277.17	269.62	270.86	272.42
5.50	278.55	281.32	284.22	273.45	275.62	277.99
5.75	285.31	288.69	292.14	278.86	281.73	284.71
6.00	292.75	296.63	300.52	285.20	288.62	292.09

Table 5: Effect of parameters α, p and θ on *TC*

α	$p = 0.5$			$p = 0.7$		
	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.5$	$\theta = 3.6$	$\theta = 3.5$	$\theta = 3.6$
0.1	269.63	270.49	271.72	268.51	268.44	268.88
0.2	268.30	268.71	269.56	**268.07**	267.48	267.47
0.3	267.40	267.33	267.77	268.13	266.98	266.49
0.4	**266.97**	266.38	266.38	268.74	**266.96**	265.94
0.5	267.03	265.88	265.40	269.92	267.46	**265.87**
0.6	267.62	**265.86**	264.86	271.74	268.53	266.31

0.7	268.78	266.36	**264.79**	274.25	270.20	267.29
0.8	270.57	267.41	265.21	277.53	272.55	268.86
0.9	273.05	269.06	266.18	281.68	275.64	271.09

Fig. 1: Effect of θ on L_q

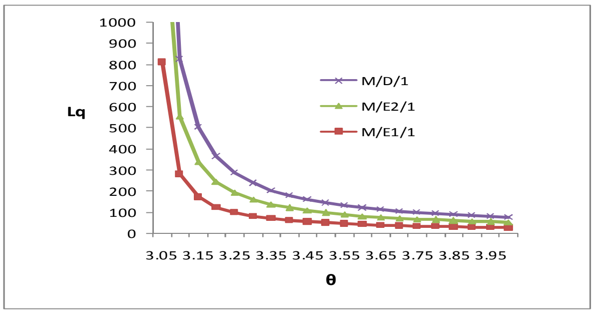


Fig. 2: Effect of r_0 on L_q

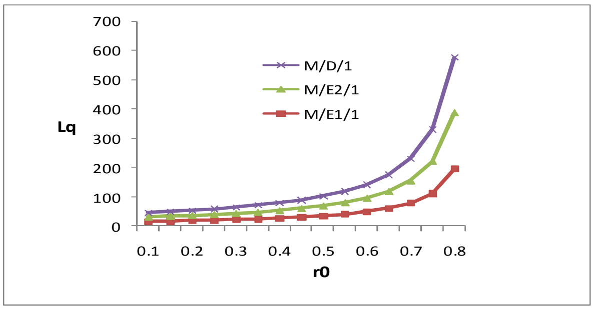


Fig. 3: Effects of λ and θ on *TC*

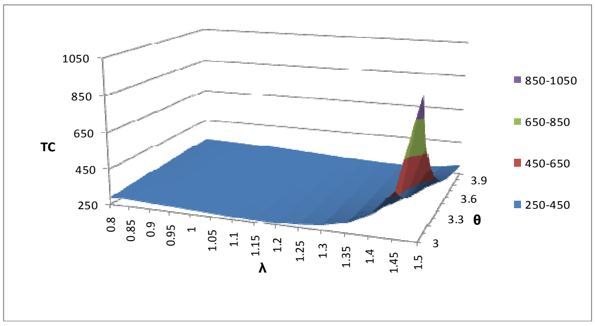
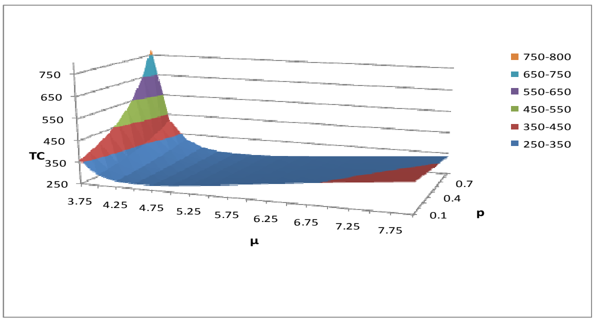


Fig. 4: Effects of μ and p on *TC*



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8. CONCLUSION

The stochastic modelling of unreliable retrial queueing system investigated in present study has the provision for the waiting of the units in retrial orbit in order to try again after a random period of time if server is busy. The incorporation of the additional optional service with essential service makes the system more versatile and can experienced in many congestion situations of real life activities. The congestion situations encountered at flexible manufacturing systems/production system, health care clinics, and many other places and may also attract the units to get the service at one place as per their requirements. The present study can be further extended and discouragement behaviours are also studied for the multi-server queue, queue with delayed repair under $N-$policy, etc.

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