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AHP Cost-Benefit Analysis is Justifiable

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Abstract: Several researchers think that the AHP Cost-Benefit Analysis method may give rise to misleading results in some situations. Hence they propose four alternative methods: (1) the aggregate $ method, (2) the cost-benefit ratio method, (3) the difference method and (4) the sum method. However, careful mathematical analysis, backed by numerical examples, show that the aggregate $ method and the cost-benefit ratio method imply nothing but the same results yielded by Saaty's original method (1994). More worrisome, we also show that the difference method and the sum method cause rank order reversal problems under some conditions.
Keyword — Analytic hierarchy process, Cost-benefit analysis

1. INTRODUCTION

In today's rapidly changing and global environment, decision-making plays a fundamental role in gaining and maintaining the competitive advantage in an enterprise or organization. Decision-making maybe considered an iterative process of choosing among alternatives in order to achieve goals and objectives. However, in many circumstances making the right decision is not easily done; it evolves many complex phases: analysis and synthesis, quantitative and qualitative, objective and subjective, linear and non-linear, etc. It is related to different decision-making models under conditions of uncertainty or multiple criteria (objectives). So in decision making it is crucial to employ a good logical method. Among those many related methods proposed by researchers and/or practitioners, Analytical Hierarchy Process (AHP) is one of the most important.

AHP, developed by Saaty (1996a), is a quantitative multi-criteria decision-making technique (Saaty (1990), Saaty (1994), Saaty and Kearns (1994), Saaty (1996b), Saaty and Vargas (1997)), which presents complex decision making problems in hierarchical terms. It merges biophysical, social and economic objectives which allow for more extensive management decision criteria. Moreover, it allows decision-makers to determine the relative importance of criteria, and to produce alternative solutions. Employing it in decision making involves four major steps. First, it decomposes the decision problem into a hierarchical model, i.e., Goals, Objectives and Alternatives; it then establishes priorities by using pair-wise comparison techniques, eigenvalues and eigenvectors, numerical judgments, graphical judgments and/or verbal judgments; once judgments have been entered for each part of the model, the information is synthesized to produce an overall preference which ranks the alternatives in relation to the overall goal; finally, a sensitivity analysis can be done to reveal how well the alternatives perform with respect to each of the objectives as well as how sensitive the alternatives are to changes in the importance of the objectives.

The AHP method has received considerable attention (Clayton and Wright (1993), Saaty and Cho (2001)), primarily because it places greater emphasis on the decision makers' preference structures and can rationally handle difficult

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decisions between complex and confusing alternatives. Many computer software tools such as Expert Choice, Vanguard Software, etc., have been developed to facilitate the researchers and practitioners in applying AHP to solve many different real and complex problems. A variety of successful applications in many different areas such as resources allocation, financial management, enterprise outsourcing selection, etc., has been reported (Blair et al. (2002), Bennett and Saaty (1993), Azis (1990)).

Although the AHP model is widely and successfully used in many fields, some researchers still challenge its appropriateness and completeness (Watson and Freeling (1982), Belton and Gear (1983), Dyer (1990), Apostolou and Hassell (1993), Finan and Hurley (1996), Schoner and Wedley (1989), Bernhard and Canada (1990), Wedley et al. (2001)). Fortunately, a series of more detailed analyses and expansions of the AHP model developed very soon to cope with these challenges (Saaty and Hu (1998), Saaty and Ozdemir (2003a), Saaty (2003), Millet and Saaty (2000), Saaty and Ozdemir (2003b), Saaty et al. (2005)). There are several papers that have worked to revise methods related to AHP that are worthy mentioning: Chang et al. (2008), Chao et al. (2004), Chu and Liu (2002), Deng et al. (2005), Yang et al. (2004), Jung et al. (2009), Ke et al. (2011), Lin et al. (2008a), Lin et al. (2008b), Chen et al. (2002), Tung et al. (2012). Wedley et al. (2001) discuss AHP Cost-Benefit Analysis and mention that in the AHP model, if cost-benefit priorities are derived from two separate hierarchies, then it is likely that the ratio of cost to benefit priorities will produce misleading results. To correct this, they think that adjustments must be made to put the numerator and denominator priorities into commensurate terms. Hence they propose four new methods to amend the original AHP Cost-Benefit Analysis method. For convenience in this paper, we designate their four new methods as follows: (1) *the aggregate $ method*, (2) *the cost-benefit ratio method*, (3) *the difference method* and (4) *the sum method*.

The purpose of this paper is to provide a deeper and more accurate understanding of the original AHP Cost-Benefit Analysis method so that the AHP model may be used more properly by researchers and practitioners. Through our careful mathematical analysis and thorough examination, we successfully show by mathematical formulation and demonstrate by numerical examples that *the aggregate $ method* and *the cost-benefit ratio method* imply nothing but the same results as Saaty's original method (Saaty (1994)). Even worse, we also show that *the difference method* and *the sum method* may cause rank order reversal problems.

2. REVIEW OF PREVIOUS WORK

In AHP Cost-Benefit Analysis, Wedley et al. (2001) illustrate how to put the commensurate ratios between costs and benefits using the example from Saaty (1994) regarding the best choice of word processing equipment among three brands: Lanier, Syntrex and Qyx. The cost and benefit scale weights for the alternatives are listed in the second and third rows in Table 1. The cost-benefit ratios are listed in the fourth row.

Table 1. Benefit weights, cost weights and cost-benefit ratios in two hierarchies

	Lanier	Syntrex	Qyx
Benefit priority (B_p)	0.42	0.37	0.21
Cost priority (C_p)	0.54	0.28	0.18
Cost-Benefit ratio (B_p/C_p)	$0.42/0.54{=}0.78$	$0.37/0.28{=}1.32$	$0.21/0.18{=}1.17$

Using cost-benefit ratios, Saaty selects Syntrex as the best word processing equipment. However, Wedley et al. (2001) consider that the normalized weights for the costs and benefits in the second and third rows of Table 1 could be produced by using different sets of financial data. They list three possible sets; for Case 1 $B\$>C\$$, for Case 2 mixed, and for Case 3 $B\$<C\$$, such that the normalized weights are predetermined as in Table 1. We quote their results in Table 2. For later discussion and mnemonics, we designate this aggregate benefit $ and the aggregate cost $ method as *the aggregate $ method*.

From their construction, in the first case in Table 2 where $B\$>C\$$, all $B\$/C\$$ are then larger than 1 that was constructed by multiplying ($5400, $2800, $1800) by 0.5 to imply that ($2700, $1400, $900). However, the relative weight remains the same as (0.54, 0.28, 0.18). In the second case, the $B\$/C\$$ and B_p/C_pratios are the same as Table 1. In the third case where the financial costs exceed the financial benefit, all $B\$/C\$$ are smaller than 1 that was constructed by multiplying ($5400, $2800, $1800) by 1.5 to imply that ($8100, $4200, $2700). However, the relative weight remains the same as (0.54, 0.28, 0.18). They claim that, in this case, the results produce three different sets of cost-benefit ratios with the same normalized weights.

Table 2. Cost-Benefit ratios with different sets of dollar amounts

Case	Measure	Lanier	Syntrex	Qyx
1. $B\$>C\$$	Aggregate benefit $	$4200	$3700	$2100
	Aggregate cost $	$2700	$1400	$900
	Cost$/Benefit $ ratio	1.56	2.64	2.33
2. Mixed	Aggregate benefit $	$4200	$3700	$2100
	Aggregate cost $	$5400	$2800	$1800
	Cost $/Benefit $ ratio	0.78	1.32	1.17
3. $B\$<C\$$	Aggregate benefit $	$4200	$3700	$2100
	Aggregate cost $	$8100	$4200	$2700
	Cost$/Benefit $ ratio	0.52	0.88	0.78

Therefore, Wedley et al. (2001) suggest the following amendment to tie the two hierarchies of cost-benefit within one hierarchy. Their amendment constructs a new hierarchy that contains the two cost-benefit hierarchies such that a new top level (B/C Goal) is constructed with relative weights (benefit, cost) = (0.667, 0.333). For later discussion, we abstractly define the relative weights of the cost-benefit as p and $1 - p$.

Within one hierarchy, when p=0.667 and $1-p$=0.333, the benefit and cost priorities are computed and the results are listed in the second, for example 0.42(0.667)=0.28 and third rows, for example 0.54(0.333)=0.18, of Table 3. They calculated the cost-benefit ratio, for example 0.28/0.18=1.56, and then expressed the results in the fourth row of Table 3.

Table 3. Benefit weights, cost weights and cost-benefit ratio within one hierarchy

	Lanier	Syntrex	Qyx
Benefit priority ($2B_p/3$)	0.42(0.667)=0.28	0.37(0.667)=0.247	0.21(0.667)=0.14
Cost priority ($C_p/3$)	0.54(0.333)=0.18	0.28(0.333)=0.093	0.18(0.333)=0.06
Cost-benefit ratio ($2B_p/C_p$)	0.28/0.18=1.56	0.247/0.093=2.64	0.14/0.06=2.33
Benefit-cost = Net benefit	0.28-0.18=0.1	0.247-0.094=0.154	0.14-0.06=0.08
1/cost priority	1/0.54=1.852	1/0.28=3.571	1/0.18=5.556
Normalized of sixth row	0.169	0.325	0.506
1/3 of seventh row	0.056	0.108	0.169
Sum of first and eighth row	0.336	0.355	0.309

From the cost-benefit ratio, Wedley et al. (2001) claim that Syntrex is still the best option. For simplicity, we will designate the cost-benefit ratio method within one hierarchy as *the cost-benefit ratio method*.

They define the net benefit as the difference in costs and benefits, calculate the net benefit, and then express the results in the fifth row of Table 3, for example 0.28-0.18=0.1. For simplicity, we will designate the net benefit method within one hierarchy as *the difference method*.

Wedley et al. (2001) consider the lower cost to be highly desirable so they take the inverse of the cost weights and normalize the results to offer another approach to put costs and benefits in a single hierarchy. For easy comparison, we quote their computation in the sixth, for example 1/0.54=1.852 and seventh rows, for example

$$[1.852/(1.852 + 3.571 + 5.556)] = 0.169 \tag{1}$$

in Table 3. They combine the benefits and the normalized inverse of the costs together using a linear combination with weights 2/3 and 1/3. The results are listed in the eighth, for example (0.169/3)=0.056 and ninth, for example 0.28+0.056=0.336, rows in Table 3. From the ninth row in Table 3, Wedley et al. (2001) conclude that Syntrex has the highest composite priority. For simplicity, we designate the sum of the benefits and the normalized inverse of the cost within one hierarchy as *the sum method*.

We will show that their *aggregate $ method* and *the cost-benefit ratio method*with weights p and $1 - p$ can be handled abstractly in the next section, making Syntrex always the best choice. We will explain that *the aggregate $ method*and *the cost-benefit ratio method*will always imply the same results as that of Saaty. Hence, we will suggest that the decision-maker avoid using *the aggregate $ method*and *the cost-benefit ratio method*. Secondly, we will show why *the difference method*and *the sum method*are the best methods for preserving Syntrex. However, *the difference method*and *the sum method*might cause the rank order reversal problem. Numerical examples are provided to illustrate our findings. The criteria used to explain why the ranking order of Lanier and Qyx shift occurs are discussed. The explanations proposed by Wedley et al. (2001) for the shift will become questionable. We will show that the exact reason is dependent on the p value choice.

3. OUR ANALYSIS AND NUMERICAL EXAMPLES

In this section we will provide a careful analysis and clear numerical examples to demonstrate that the results of the four new amendment methods proposed by Wedley et al. (2001) are questionable.

3.1 The Aggregate $ Method and the Cost-benefit Ratio Method

First, we will abstractly discuss their *aggregate $ method*and *the cost-benefit ratio method.* From Table 1, we abstractly denote the benefit priority for Lanier, Syntrex and Qyx as b_1, b_2 and 3 and the cost priority for Lanier, Syntrex and Qyx as c_1, c_2 and c_3 respectively.

*The aggregate $ method*and *the cost-benefit ratio method*are multiplied by an amount x to b_i and an amount y to c_i. For example, in Case 1 in Table 2, $x = 10,000$ and $y = 5,000$; in Case 2 in Table 2, $x = 10,000$ and $y = 10,000$; in Case 3 in Table 2, $x = 10,000$ and $y = 15,000$; in Table 3, $x = p = 0.667$ and $y = 1 - p = 0.333$, where we compute that

$$5000(0.54) = 2700, 10000(0.54) = 5400, and 15000(0.54) = 8100. \tag{2}$$

Now, we consider the cost-benefit ratio for three word processing equipment alternatives: Lanier, Syntrex and Qyx, $xb_1/yc_1, xb_2/yc_2$ and xb_3/yc_3 respectively. Using different values for x and y, we will have different values for the cost-benefit ratio. However, the order of $xb_1/yc_1, xb_2/yc_2$ and xb_3/yc_3 is the same as $b_1/c_1, b_2/c_2$ and b_3/c_3, which is independent of the values of x and y. Therefore, *the aggregate $ method*and *the cost-benefit ratio method*will derive the same ordering as that proposed by Saaty (1994). Based on the above analysis, *the aggregate $ method*and *the cost-benefit ratio method*obviously imply the same and already known results proposed by Saaty (1994), so we suggest that the decision-maker should directly use Saaty's original method instead of *the aggregate $ method*and *the cost-benefit ratio method.*

3.2 The Difference Method

Wedley et al. (2001) claim that after adjustment, when the benefits and costs are in commensurate units, a permissible calculation in harmony with the regular cost-benefit analysis involves calculating the net benefit such that the cost priorities are treated as negative priorities. The cost can then be subtracted from the benefit, defined as *the difference method.* We quote their results in the fifth row of Table 2. They claim that Syntrex is still the best choice. We assume that within one hierarchy, the relative weights of the cost and benefit are p and $1 - p$. The weight p effect should also be investigated. Different weight ranges should be incorporated into the consideration.

From *the difference method*, the relative weight of Lanier is

$$pb_1 - (1 - p)c_1; \tag{3}$$

the relative weight of Syntrex is

$$pb_2 - (1 - p)c_1; \tag{4}$$

the relative weight of Qyx is

$$pb_3 - (1 - p)c_3. \tag{5}$$

To derive Lanier is the best choice, we require that

$$pb_1 - (1 - p)c_1 > pb_2 - (1 - p)c_2, \tag{6}$$

and

$$pb_1 - (1 - p)c_1 > pb_3 - (1 - p)c_3. \tag{7}$$

From Equation (6), we know the following inequalities are equivalent: (a) $pb_1 - (1 - p)c_1 > pb_2 - (1 - p)c_2$; (b) $(b_1 - b_2)p/(1 - p) > c_1 - c_2$; (c) $p/(1 - p) > (c_1 - c_2)/(b_1 - b_2)$, in this step, we use $b_1 > b_2$; and (d) $p > (c_1 - c_2)/(b_1 - b_2 + c_1 - c_2)$. By the same argument, from Equation (7), we derive that the following inequalities are equivalent: (e) $pb_1 - (1 - p)c_1 > pb_3 - (1 - p)c_3$; (f) $(b_1 - b_3)p/(1 - p) > c_1 - c_3$; (g) $p/(1 - p) > (c_1 - c_3)/(b_1 - b_3)$, in this step, we use $b_1 > b_3$; and (h) $p > (c_1 - c_3)/(b_1 - b_3 + c_1 - c_3)$. Hence, we obtain that there are two restrictions to guarantee that Lanier is the best choice,

$$p > (c_1 - c_2)/(b_1 - b_2 + c_1 - c_2) \quad \text{and} \quad p > (c_1 - c_3)/(b_1 - b_3 + c_1 - c_3). \tag{8}$$

To derive Syntrex is the best choice, we require that

$$pb_2 - (1 - p)c_2 > pb_1 - (1 - p)c_1, \tag{9}$$

and

$$pb_2 - (1 - p)c_2 > pb_3 - (1 - p)c_3. \tag{10}$$

We know the following inequalities are equivalent: (i) $pb_2-(1-p)c_2 > pb_1-(1-p)c_1$; (j) $(b_2-b_1)p/(1-p) > c_2-c_1$; (k) $p/(1-p) < (c_2-c_1)/(b_2-b_1)$, in this step, we use $b_1 > b_2$; and (l) $p < (c_1-c_2)/(b_1-b_2+c_1-c_2)$. By the same argument, we derive that the following inequalities are equivalent: (m) $pb_2-(1-p)c_2 > pb_3-(1-p)c_3$; (n) $(b_2-b_3)p/(1-p) > c_2-c_3$; (o) $p/(1-p) > (c_2-c_3)/(b_2-b_3)$, in this step, we use $b_2 > b_3$; and (p) $p > (c_2-c_3)/(b_2-b_3+c_2-c_3)$. Hence, we obtain that there are two restrictions to guarantee that Syntrex is the best choice,

$$(c_1-c_2)/(b_1-b_2+c_1-c_2) > p \quad \text{and} \quad p > (c_2-c_3)/(b_2-b_3+c_2-c_3). \tag{11}$$

To derive Qyx is the best choice, we require that

$$pb_3-(1-p)c_3 > pb_1-(1-p)c_1, \tag{12}$$

and

$$pb_3-(1-p)c_3 > pb_2-(1-p)c_2. \tag{13}$$

We know the following inequalities are equivalent: (q) $pb_3-(1-p)c_3 > pb_1-(1-p)c_1$; (r) $(b_3-b_1)p/(1-p) > c_3-c_1$; (s) $p/(1-p) < (c_3-c_1)/(b_3-b_1)$, in this step, we use $b_1 > b_3$; and (t) $p < (c_3-c_1)/(b_3-b_1+c_3-c_1)$. By the same argument, we derive that the following inequalities are equivalent: (u) $pb_3-(1-p)c_3 > pb_2-(1-p)c_2$; (v) $(b_3-b_2)p/(1-p) > c_3-c_2$; (w) $p/(1-p) < (c_3-c_2)/(b_3-b_2)$, in this step, we use $b_2 > b_3$; and (x) $p < (c_3-c_2)/(b_3-b_2+c_3-c_2)$. Hence, we obtain that there are two restrictions to guarantee that Qyx is the best choice,

$$(c_1-c_3)/(b_1-b_3+c_1-c_3) > p \quad \text{and} \quad (c_2-c_3)/(b_2-b_3+c_2-c_3) > p. \tag{14}$$

Now, we compare our derivations of Equations (8), (11) and (14), to put them together to observe that

$$p > (c_1-c_2)/(b_1-b_2+c_1-c_2) \quad \text{and} \quad p > (c_1-c_3)/(b_1-b_3+c_1-c_3),$$

$$(c_1-c_2)/(b_1-b_2+c_1-c_2) > p \quad \text{and} \quad p > (c_2-c_3)/(b_2-b_3+c_2-c_3),$$

and

$$(c_1-c_3)/(b_2-b_3+c_1-c_3) > p \quad \text{and} \quad (c_2-c_3)/(b_2-b_3+c_2-c_3) > p,$$

to derive a workable plan to partition the range of p, then we can claim that the following two restrictions,

$$(c_1-c_2)/(b_1-b_2+c_1-c_2) > (c_1-c_3)/(b_1-b_3+c_1-c_3) \tag{15}$$

and

$$(c_1-c_3)/(b_1-b_3+c_1-c_3) > (c_2-c_3)/(b_2-b_3+c_2-c_3) \tag{16}$$

simplify the expressions of Equations (8), (11) and (14) as

$$p > (c_1-c_2)/(b_1-b_2+c_1-c_2), \tag{17}$$

$$(c_1-c_2)/(b_1-b_2+c_1-c_2) > p \quad \text{and} \quad p > (c_2-c_3)/(b_2-b_3+c_2-c_3),$$

and

$$(c_2-c_3)/(b_2-b_3+c_2-c_3) > p, \tag{18}$$

which is a partition of $1 > p > 0$.

Based on the results of Equations (17), (11) and (18), we can partition the range of $1 > p > 0$ into three disjointed parts to obtain which word processing equipment will be the best choice.

Based on observations of Equations (15) and (16), we derive that the following are equivalent: (a) $(c_1-c_2)/(b_1-b_2+c_1-c_2) > (c_1-c_3)/(b_1-b_3+c_1-c_3)$; (b) $b_2 > [b_1(c_2-c_3)/(c_1-c_3)] + [b_3(c_1-c_2)/(c_1-c_3)]$; and (c) $(c_1-c_3)/(b_1-b_3+c_1-c_3) > (c_2-c_3)/(b_2-b_3+c_2-c_3)$.

Therefore, we can abstractly describe the results in the following theorem. The task of eliciting relationships between the best choice in *the difference method*and the weight range are not trivial. They are characterized by several technical criteria. Here, we list one possible result that is suitable for the example in Saaty (1994).

Take the example of Wedley et al. (2001) with $b_1 = 0.42, b_2 = 0.37, b_3 = 0.21, c_1 = 0.54, c_2 = 0.28$, and $c_3 = 0.18$, then $[b_1(c_2-c_3)/(c_1-c_3)] + [b_3(c_1-c_2)/(c_1-c_3)] = 0.2683$ that is less than $b_2 = 0.37$ to indicate that our extra condition in Theorem 1 was supported by data from Wedley et al. (2001).

Theorem 1 *For the difference method, under the conditions* $b_2 > [b_1(c_2-c_3)/(c_1-c_3)] + [b_3(c_1-c_2)/(c_1-c_3)]$, $b_1 > b_2 > b_3$ *and* $c_1 > c_2 > c_3$, *we show that*

(1) *when* $1 > p > (c_1-c_2)/(b_1-b_2+c_1-c_2)$, *Lanier is the best choice;*
(2) *when* $(c_1-c_2)/(b_1-b_2+c_1-c_2) > p > (c_2-c_3)/(b_2-b_3+c_2-c_3)$, *Syntrex is the best choice;*
(3) *when* $(c_2-c_3)/(b_2-b_3+c_2-c_3) > p > 0$, *Qyx is the best choice.*

To help ordinary practitioners to visualize our result, we sketch our findings of Theorem 1 in Figure 1.

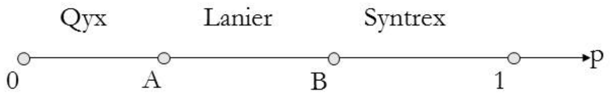

Figure 1. The best Choice for word processing equipment by the difference method, where A $= (c_2 - c_3)/(b_2 - b_3 + c_2 - c_3)$, and B $= (c_1 - c_2)/(b_1 - b_2 + c_1 - c_2)$.

From Theorem 1, we can obviously find that since different best choices can be derived according to different p values, which means that *the difference method*might cause the rank order reversal problem in many cases.

Example 1 From Table 1, we have $b_1 = 0.42, b_2 = 0.37, b_3 = 0.21, c_1 = 0.54, c_2 = 0.28$, and $c_3 = 0.18$ then $(c_1 - c_2)/(b_1 - b_2 + c_1 - c_2) = 26/31 = 0.839$. If we take $p = 17/20 = 0.85$, by *the difference method*, the weight of Lanier $= 552/2000 > Syntrex = 545/2000$. This coincides with our Theorem 1 (1). Using different p values, the rank order reversal problem may occur. Hence, *the difference method*is heavily dependent upon the p value. Unless Wedley et al. (2001) can give us some further explanation why their choice for the p value satisfies some criteria (for example, $(c_1 - c_2)/(b_1 - b_2 + c_1 - c_2) > p > (c_2 - c_3)/(b_2 - b_3 + c_2 - c_3)$, otherwise their difference method might cause the rank order reversal problem.

Wedley et al. (2001) mentioned the shift in the second best choice between (i) *the aggregate $ method*and (ii) the net benefit method. Wedley et al. (2001) claimed that Syntrex remains the best choice, although Lanier and Qyx switched places in rank order. The explanation of Wedley et al. (2001) for the switch is that Lanier has high costs but also high benefits. In Lanier's ratio (B_p/C_p) the high costs are lower compared to Qyx, but the difference between the benefits and costs is higher. In the following, we will provide our explanation for why that shift, or its lack, depends on the value of p.

Under the same assumption of Theorem 1, with $b_2 > [b_1(c_2 - c_3)/(c_1 - c_3)] + [b_3(c_1 - c_2)/(c_1 - c_3)]$, $b_1 > b_2 > b_3$ and $c_1 > c_2 > c_3$, we consider the following cases:

(C1) $1 > p > (c_1 - c_2)/(b_1 - b_2 + c_1 - c_2)$,
(C2) $(c_1 - c_2)/(b_1 - b_2 + c_1 - c_2) > p > (c_2 - c_3)/(b_2 - b_3 + c_2 - c_3)$, and
(C3) $(c_2 - c_3)/(b_2 - b_3 + c_2 - c_3) > p > 0$.

For case (C1), we already established that Lanier is the best choice, and then Syntrex is the second best choice if and only if

$$pb_2 - (1 - p)c_2 > pb_3 - (1 - p)c_3. \tag{19}$$

We recall the equivalence relation between inequalities (m) and (p) to derive that the inequality of (19) is

$$p > (c_2 - c_3)/(b_2 - b_3 + c_2 - c_3). \tag{20}$$

Under the condition of case(C1), then $p > (c_1 - c_2)/(b_1 - b_2 + c_1 - c_2)$. With the restriction of $b_2 > [b_1(c_2 - c_3)/(c_1 - c_3)] + [b_3(c_1 - c_2)/(c_1 - c_3)]$, we obtain that inequalities of (15) and (16) are valid to imply the inequality of (20) holds. Hence, for case (C_1), Syntrex is the second best choice.

For case (C2), we already established that Syntrex is the best choice, and then Lanier is the second best choice if and only if

$$pb_1 - (1 - p)c_1 > pb_3 - (1 - p)c_3. \tag{21}$$

We recall the equivalence relation between inequalities (e) and (h) to derive that the inequality of (21) is

$$p > (c_1 - c_3)/(b_1 - b_3 + c_1 - c_3). \tag{22}$$

Similarly, Qyx is the second best choice if and only if

$$pb_3 - (1 - p)c_3 > pb_1 - (1 - p)c_1. \tag{23}$$

We recall the equivalence relation between inequalities (q) and (t) to derive that the inequality of (23) is

$$p < (c_1 - c_3)/(b_1 - b_3 + c_1 - c_3). \tag{24}$$

Under the conditions of case (C2), then $(c_1 - c_2)/(b_1 - b_2 + c_1 - c_2) > p > (c_2 - c_3)/(b_2 - b_3 + c_2 - c_3)$.

With the restriction of $b_2 > [b_1(c_2-c_3)/(c_1-c_3)] + [b_3(c_1-c_2)/(c_1-c_3)]$, we recall Equations (15) and (16) to derive that

$$(c_1-c_2)/(b_1-b_2+c_1-c_2) > (c_1-c_3)/(b_1-b_3+c_1-c_3),$$

and

$$(c_1-c_3)/(b_1-b_3+c_1-c_3) > (c_2-c_3)/(b_2-b_3+c_2-c_3).$$

Hence, if $(c_1-c_2)/(b_1-b_2+c_1-c_2) > p > (c_1-c_3)/(b_1-b_3+c_1-c_3)$, Lanier is the second best choice. If $(c_1-c_3)/(b_1-b_3+c_1-c_3) > p > (c_2-c_3)/(b_2-b_3+c_2-c_3)$, Qyx is the second best choice.

For case (C3), we already know that Qyx is the best choice, and then Syntrex is the second best choice if and only if

$$pb_2-(1-p)c_2 > pb_1-(1-p)c_1. \tag{25}$$

We recall the equivalence relation between inequalities (i) and (l) to derive that the inequality of (25) is

$$p < (c_1-c_2)/(b_1-b_2+c_1-c_2). \tag{26}$$

Under the condition of case (C3), then $(c_2-c_3)/(b_2-b_3+c_2-c_3) > p$. With the restriction of $b_2 > [b_1(c_2-c_3)/(c_1-c_3)] + [b_3(c_1-c_2)/(c_1-c_3)]$, we obtain that inequalities of (15) and (16) are valid to imply the inequality of (26) holds. Hence, for case (C3), Syntrex is the second best choice.

Based on our above discussion, we derive more robust results for *the difference method*, such that we abstractly derive the following results for the second best choice of *the difference method.*

Theorem 2 *For the difference method, under the conditions* $b_2 > [b_1(c_2-c_3)/(c_1-c_3)] + [b_3(c_1-c_2)/(c_1-c_3)]$, $b_1 > b_2 > b_3$ *and* $c_1 > c_2 > c_3$, *we show that*

(1) *when* $1 > p > (c_1-c_2)/(b_1-b_2+c_1-c_2)$, *Syntrex is the second best choice;*
(2) *when* $(c_1-c_2)/(b_1-b_2+c_1-c_2) > p > (c_1-c_3)/(b_1-b_3+c_1-c_3)$, *Lanier is the second best choice;*
(3) *when* $(c_1-c_3)/(b_1-b_3+c_1-c_3) > p > (c_2-c_3)/(b_2-b_3+c_2-c_3)$, *Qyx is the second best choice;*
(4) *when* $(c_2-c_3)/(b_2-b_3+c_2-c_3) > p > 0$, *Syntrex is the second best choice.*

To help ordinary practitioners to visualize our result, we sketch our findings of Theorem 2 in Figure 2.

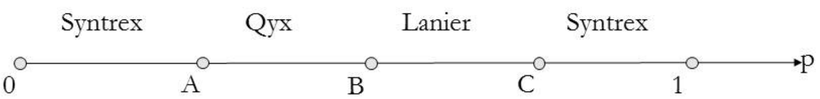

Figure 2. The second best choice by the difference method, where A $= (c_2-c_3)/(b_2-b_3+c_2-c_3)$, B $= (c_1-c_3)/(b_1-b_3+c_1-c_3)$, and C $= (c_1-c_2)/(b_1-b_2+c_1-c_2)$.

Example 2 *With the same data as Example 1, Wedley et al. (2001) took p=0.667 and found that Lanier was the second best choice, which is a shift since in Table 1, the second best choice is Qyx. As* $(c_1-c_2)/(b_1-b_2+c_1-c_2) = 26/31 = 0.839$, $(c_1-c_3)/(b_1-b_3+c_1-c_3) = 12/19 = 0.632$, *and* $(c_2-c_3)/(b_2-b_3+c_2-c_3) = 10/26 = 0.385$, *their result coincides with Theorem 2 (2). If we change* p *from 0.667 to 0.4, then Qyx is the second best choice, as indicated in Theorem 2 (3), such that the shift in the second best choice will not occur. Therefore, their explanation for the shift of the second best choice is questionable. Actually, the reason for a shift in the second best choice is the different value of* p.

3.3 The Sum Method

Now let us consider *the sum method.* We take the inverse of the costs as $1/c_1, 1/c_2 and 1/c_3$. The costs are then normalized as $c_2c_3/(c_1c_2+c_2c_3+c_3c_1)$, $c_1c_3/(c_1c_2+c_2c_3+c_3c_1)$ and $c_1c_2/(c_1c_2+c_2c_3+c_3c_1)$.

With a given pair of values of relative weight p and $1-p$, the sum of the benefit and the normalized inverse of the cost for the three alternatives: Lanier, Syntrex and Qyx are considered.

From *the sum method*, the relative weight of Lanier is

$$pb_1+(1-p)[c_2c_3/(c_2c_3+c_1c_3+c_1c_2)]; \tag{27}$$

the relative weight of Syntrex is

$$pb_2+(1-p)[c_1c_3/(c_2c_3+c_1c_3+c_1c_2)]; \tag{28}$$

the relative weight of Qyx is

$$pb_3+(1-p)[c_1c_2/(c_2c_3+c_1c_3+c_1c_2)]. \tag{29}$$

To derive that Lanier is the best choice, requires that

$$pb_1 + (1-p)[c_2c_3/(c_2c_3 + c_1c_3 + c_1c_2)] > pb_2 + (1-p)[c_1c_3/(c_2c_3 + c_1c_3 + c_1c_2)], \tag{30}$$

and

$$pb_1 + (1-p)[c_2c_3/(c_2c_3 + c_1c_3 + c_1c_2)] > pb_3 + (1-p)[c_1c_2/(c_2c_3 + c_1c_3 + c_1c_2)]. \tag{31}$$

We can rewrite Equations (30) and (31) as

$$p > c_3(c_1 - c_2)/[c_3(c_1 - c_2) + (b_1 - b_2)(c_1c_2 + c_2c_3 + c_3c_1)], \tag{32}$$

and

$$p > c_2(c_1 - c_3)/[c_2(c_1 - c_3) + (b_1 - b_3)(c_1c_2 + c_2c_3 + c_3c_1)], \tag{33}$$

where we have used $b_1 > b_2$ and$b_1 > b_2$.

To derive Syntrex is the best choice, we require that

$$pb_2 + (1-p)[c_1c_3/(c_2c_3 + c_1c_3 + c_1c_2)] > pb_1 + (1-p)[c_2c_3/(c_2c_3 + c_1c_3 + c_1c_2)], \tag{34}$$

and

$$pb_2 + (1-p)[c_1c_3/(c_2c_3 + c_1c_3 + c_1c_2)] > pb_3 + (1-p)[c_1c_2/(c_2c_3 + c_1c_3 + c_1c_2)]. \tag{35}$$

We can rewrite Equations (34) and (35) as

$$c_3(c_1 - c_2)/[c_3(c_1 - c_2) + (b_1 - b_2)(c_1c_2 + c_2c_3 + c_3c_1)] > p, \tag{36}$$

and

$$p > c_1(c_2 - c_3)/[c_1(c_2 - c_3) + (b_2 - b_3)(c_1c_2 + c_2c_3 + c_3c_1)], \tag{37}$$

where we have used $b_1 > b_2$ and$b_2 > b_3$.

To derive Qyx is the best choice, we require that

$$pb_3 + (1-p)[c_1c_2/(c_2c_3 + c_1c_3 + c_1c_2)] > pb_1 + (1-p)[c_2c_3/(c_2c_3 + c_1c_3 + c_1c_2)], \tag{38}$$

and

$$pb_3 + (1-p)[c_1c_2/(c_2c_3 + c_1c_3 + c_1c_2)] > pb_2 + (1-p)[c_1c_3/(c_2c_3 + c_1c_3 + c_1c_2)]. \tag{39}$$

We can rewrite Equations (38) and (39) as

$$c_2(c_1 - c_3)/[c_2(c_1 - c_3) + (b_1 - b_3)(c_1c_2 + c_2c_3 + c_3c_1)] > p, \tag{40}$$

and

$$c_1(c_2 - c_3)/[c_1(c_2 - c_3) + (b_2 - b_3)(c_1c_2 + c_2c_3 + c_3c_1)] > p, \tag{41}$$

where we have used $b_1 > b_3$ and $b_2 > b_3$.

At this point, there are six inequalities for *the sum method*to decide the best choice listed as Equations (32-33, 36-37, 40-41). We try to find additional conditions to synthesize above mentioned six inequalities such that we can partition for $1 > p > 0$ into three disjointed parts to obtain which word processing equipment will be the best choice by *the sum method*.

We try to claim the following two additional conditions

$$\begin{aligned} &c_3(c_1 - c_2)/[c_3(c_1 - c_2) + (b_1 - b_2)(c_1c_2 + c_2c_3 + c_3c_1)] > \\ &\qquad c_2(c_1 - c_3)/[c_2(c_1 - c_3) + (b_1 - b_3)(c_1c_2 + c_2c_3 + c_3c_1)], \end{aligned} \tag{42}$$

and

$$\begin{aligned} &c_2(c_1 - c_3)/[c_2(c_1 - c_3) + (b_1 - b_3)(c_1c_2 + c_2c_3 + c_3c_1)] > \\ &\qquad c_1(c_2 - c_3)/[c_1(c_2 - c_3) + (b_2 - b_3)(c_1c_2 + c_2c_3 + c_3c_1)]. \end{aligned} \tag{43}$$

We know that inequalities of Equations (42) and (43) are equivalent to

$$(b_1 - b_3)/c_2(c_1 - c_3) > (b_1 - b_2)/c_3(c_1 - c_2), \tag{44}$$

and

$$(b_2 - b_3)/c_1(c_2 - c_3) > (b_1 - b_3)/c_2(c_1 - c_3). \tag{45}$$

Based on inequalities of Equations (44) and (45), we claim that inequalities of Equations (42) and (43) are equivalent to the following new condition,

$$b_2 > [b_1c_1(c_2 - c_3)/c_2(c_1 - c_3)] + [b_3c_3(c_1 - c_2)/c_2(c_1 - c_3)]. \tag{46}$$

In Theorems 1 and 2, the results for the choice are highly dependent upon the weight. This leads to a need to study the relationship between the weight range and the best choice of *the sum method*. We then abstractly handle the problem and derive the following results.

Theorem 3 *Using the sum method, under the conditions of* $b_2 > [b_1c_1(c_2-c_3)/c_2(c_1-c_3)]+[b_3c_3(c_1-c_2)/c_2(c_1-c_3)]$, $b_1 > b_2 > b_3$ *and* $c_1 > c_2 > c_3$*, we show that*

(1) *If* $1 > p > c_3(c_1-c_2)/[c_3(c_1-c_2)+(b_1-b_2)(c_1c_2+c_2c_3+c_3c_1)]$*, then Lanier is the best choice*
(2) *If* $c_3(c_1-c_2)/[c_3(c_1-c_2)+(b_1-b_2)(c_1c_2+c_2c_3+c_3c_1)] > p >$
$c_1(c_2-c_3)/[c_1(c_2-c_3)+(b_2-b_3)(c_1c_2+c_2c_3+c_3c_1)]$*, then Syntrex is the best choice;*
(3) *If* $c_1(c_2-c_3)/[c_1(c_2-c_3)+(b_2-b_3)(c_1c_2+c_2c_3+c_3c_1)] > p > 0$*, then Qyx is the best choice.*

To help ordinary practitioners to visualize our result, we sketch our findings of Theorem 3 in the next Figure 3.

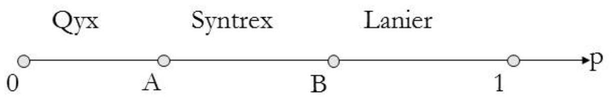

Figure 3. The best choice for word processing equipment by the sum method, where
A $= c_1(c_2-c_3)/[c_1(c_2-c_3)+(b_2-b_3)E]$, B $= c_3(c_1-c_2)/[c_3(c_1-c_2)+(b_1-b_2)E]$, and
$E = c_1c_2+c_2c_3+c_3c_1$.

Example 3 As in Example 2, we have $c_3(c_1-c_2)/[c_3(c_1-c_2)+(b_1-b_2)(c_1c_2+c_2c_3+c_3c_1)]$=0.738 and $c_3(c_1-c_2)/[c_3(c_1-c_2)+(b_1-b_2)(c_1c_2+c_2c_3+c_3c_1)]$=0.530. In Wedley et al. (2001), they take p=0.667 and claim that Syntrex is the best choice as indicated in Theorem 3 (2). However, using different p values, for example p=0.5, Qyx is the best choice as in Theorem 3 (3). Hence, *the sum method*might cause the rank order reversal problem.

When constructing one hierarchy to combine two cost-benefit hierarchies, Wedley et al. (2001) did not give any explanation for the reason why they chose p=2/3 such that (p, 1-p)=0.667,0.333). From our previous study, *the aggregate $ method*and *the cost-benefit ratio method*imply the already known results. Hence, we might advise the decision-maker not to use these two methods. On the other hand, for *the difference method*and *the sum method*, the choice of p will influence the result; so the selection of the p value requires further investigation. Up to now, how to select a proper p value has not been explained in detail for methods proposed in Wedley et al. (2001). Consequently, we suggest that the decision-maker not apply *the difference method*and *the sum method*to avoid the rank order reversal problem.

4. CONCLUSION

Based on our detailed analysis, theoretical derivations and numerical examples in the above section, we successfully demonstrate that the four new methods proposed by some researchers for adjusting the original AHP Cost-benefit Analysis method should be reconsidered. In other words, our careful mathematical analysis and thorough examination, demonstrated by numerical examples, show that *the aggregate $ method*and *the cost-benefit ratio method* imply nothing but the same results as Saaty's original method (1994). Even worse, we have also shown that *the difference method*and *the sum method*cause rank order reversal problems under some criteria. We hope that the analytic results presented in this paper can provide a deeper and more accurate understanding of the original AHP Cost-benefit Analysis method so that the AHP model may be used more properly by researchers and practitioners.

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