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A Manufacturing Inventory Model for Exponentially Increasing Demand with Preservation Technology and Shortage

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Abstract: An economic production quantity (EPQ) model plays a momentous role in production and manufacturing units. In this paper, we introduce four stages of production inventory model for deteriorating items in which three (Beginning, Developing and Maturity) different stages of production and one decline stage are considered. The demand is assumed as exponentially increasing and production is demand dependent. In an inventory cycle, the production facility can produce items in a determined number of replenishment cycles under finite time horizon. The shortages are permitted and completely backlogged. The purpose of this research is to investigate the optimal production lot size and total cost during the product life cycle (PLC) which consists of beginning, developing, maturity and decline stages. Necessary and sufficient conditions for a unique and optimal solution are derived. Finally numerical example and sensitivity analysis on parameters are made to validate the results of the proposed inventory system.
Keyword — Inventory, replenishment cycle, shortages, deteriorative items, preservation technology, different production stage.

1. INTRODUCTION

The Economic Production Quantity (EPQ) model is a simple mathematical model to deal with inventory management issues in a production inventory system. Inventory system is one of the main streams of the operation research which is essential in business enterprises and industries. It is considered to be one of the most popular inventory control model used in the industry (Osteryoung et al. 1986). In general, almost all products are found to be deteriorating overtime. Sometimes the rate of deterioration is too low, for items such as steel, hardware, glassware and toys, to cause consideration of deterioration in the determination of economic lot sizes. Usually, some items have significant rate of deterioration, such as blood, fish, strawberry, alcohol, gasoline, radioactive chemicals, medicine and food grains those deteriorate rapidly over time, which cannot be ignored in the decision making process of production lot size. The problem of deterioration of inventory has received considerable attention in recent years. In deteriorating inventory, most of the researchers have assumed the rate of deterioration as constant. The deterioration rate increases with age, that is, longer the items remain unused, the higher the rate at which they fail. Deterioration is defined as decay, damage, change or spoilage that prevents items from being used for its original purpose. Some examples of items that deteriorate are fashion goods, foods, mobile phones, chemicals, automobiles, drugs, etc.

A major issue in any business transaction is to control and maintain the inventories of deteriorating items. Goods are deteriorating because the values of the goods go down with time. The few common examples for deteriorating items are electronic products, fashion clothing, pharmaceuticals, paper-based materials, foods, vegetables, fruits and chemicals. Therefore, in practice, the loss due to deterioration cannot be ignored. Numerous studies have been carried out to address the problems of EPQ model for deteriorating items. A formulated mathematical model with a constant deterioration rate was originally presented by Ghare and Schrader (1963). After that, Teng and Chang (2005) presented an EPQ model for deteriorating items considering the demand rate depends on the selling price of the products and the stock level where the phenomena of inventory deterioration are categorized into three kinds such as direct spoilage, physical depletion and deterioration Ghoreishi et al. (2015) developed an EOQ model for non-instantaneous deteriorating items with a selling price and inflation induced demand rate under the effect of inflation and customer returns. The deterioration of an item cannot be prohibited but it may be reduced using preservation technology during the deterioration period. Dhandapani and Uthayakumar (2016) presented an EOQ model for fresh fruits with preservation technology investment for variable deterioration rate. Singh et al. (2016) studied that the retailers invest in preservation technology to reduce the rate of deterioration of products. Chowdhury et al (2017) presented an optimal inventory

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replenishment policy for deteriorating items where the rate of deterioration of items is directly proportional to time and the demand rate is a continuous quadratic function of time. Recently, Mashud et al. (2018) discussed a non-instantaneous inventory model for two constant deterioration rates under partially backlogged shortages. Rabbani and Aliabadi (2019) have developed an inventory model in a supply chain by considering shortages and delayed payments in partial form where demand rate was represented as a multivariate function of credit period, marketing expenditure and selling price.

Product quality is not always perfect and actually depends on the state of the production process. The production process is subject to deterioration due to the occurrence of some assignable cause which may shift the process from an in-control state to an out-of control state and produce some defective items. But, in real situation the defective items are popular in many kinds of products. Hence, the defective rate cannot be ignored in the production process. The effect of an imperfect process on production run time and EPQ was initially studied by Rosenblatt and Lee (1986). In their study, the elapsed time until the process shift was assumed to be exponentially distributed. The optimal production run was found to be shorter than that of classical EPQ model. Sarker et al. (2008) have considered different variants of imperfect production processes. The common assumption of all the above mentioned models was that there were a fixed percentage of defective items produced during the out-of-control period. Haji et al. (2008) studied an imperfect production process with rework in which several products are produced on a single machine. Cardenas-Barron (2009) presented the mathematical expressions corrected in Sarker et al. (2008) are corrected and the appropriate solution to the numerical example and also established the closed forms for the optimal total inventory cost and the mathematical expressions for determining the total additional cost for working with a non-optimal solution for both policies that were not given by Sarker et al. (2008). Hsuch (2010) investigated inventory control policies in a manufacturing system during the PLC, the closed-form formulas of optimal production lot size, reorder point and safety stock in each phase of PLC are derived. Pour and Ghobadi (2018) organized the model of Mishra (2017) for a two echelon supply chain model for deteriorating items wherein the optimal selling price, production lot size, total cycle time, number of deliveries and delivery lot size are obtained simultaneously. Taleizadeh et al. (2018) developed four new sustainable economic production quantity models that consider different shortage situations. When shortages occur, the lost sale, full back-ordering and partial backordering models can be selected by the company managers depending on the manufacturer's motivation to get better service levels. Sekar and Uthayakumar (2018) developed an imperfect production inventory model for defective items where the defective items are reworked after several production processes and there is no break down during the rework process.

The purpose paper is to investigate the optimum production lot size and total cost during the PLC which consists of beginning, developing, maturity and decline stages with increasing demand rate, deterioration and shortages. The remainder of the paper is organized as follows. Section 2 presents the assumptions and notations. Section 3 presents the problem definition. The mathematical for the proposed inventory model is given Section 4. Numerical example is shown in Section 5. Finally, the paper summarizes and concludes in Section 6.

2. ASSUMPTIONS AND NOTATIONS

The assumptions and notations of this proposed inventory model (PLC) are as follows:

2.1 Assumptions

1. The demand rate is assumed to be increase exponentially and is given by the function $d\left(t\right) = ae^{bt}$ where $a > 0$ and $0 < b < 1$.
2. Shortages are allowed and completely backlogged.
3. Deterioration of products starts as soon as it comes into the inventory.
4. The production rate is always greater than or equal to demand rate and is given by the function $p\left(t\right) = \gamma ae^{bt}$ where $\gamma > 0$
5. Deterioration rate is known, constant.
6. During time t_1, inventory is built up due to demand and deteriorative items at the rate of x-times of $(p - d - \theta)$. A product enter growth and maturity stage at time t_2 and t_3, demand and production increases at the rate of y-times of $(p - d - \theta)$ and z-times of $(p - d - \theta)$ where x, y and z are constants thereafter inventory level declines continuously at a rate of $(d + \theta)$ and becomes zero at time T_1 $(t_1 + t_2 + t_3 + t_4 = T_1)$ and then shortages arises up to time T. Then the process is repeated.

2.2. Notations

p	Production rate in units per unit time
d	Demand rate in units per unit time
S	Shortage units per replenishment cycle
θ	Rate of deteriorative items
ϵ	Preservation technology cost for reducing deterioration rate in order to preserve the products, $\epsilon > 0$.
λ	Resultant deterioration rate, $\lambda = \theta - m(\epsilon)$.
i_c	Unit inspection cost
k_s	Setup cost
h_s	Holding cost per unit per unit time
c_p	Production/purchase cost per unit per unit time
D_c	Unit scrap/deterioration cost per unit per unit time
c_s	Cost of shortages per unit per unit time
TCF	Total cost function.
H	Finite time inventory horizon
m	Number of replenishment cycle where $H = mT$

3. PROBLEM DEFINITION

The mathematical model for optimal production lot size in this research is presented as follows. The cycle starts at time $t = 0$, in which the inventory level is zero. The production starts and increases at a rate p and simultaneously decreasing due to demand and deterioration during introduction stage for the period $[0, T_1]$. Thus the inventory accumulates at a rate of x-times of $(p - d - \theta)$. Then the product enters into growth stage during the period (t_1, t_2). Production and demand increases at the rate of 'y' times of $(p - d - \theta)$, where y is a constant, as more customers become aware of the product and its benefits and additional market segments are targeted up to time t_2 and the maturity stage during the period (t_2, t_3). The production and demand increases at the rate z times of $(p - d - \theta)$, where z is a constant and $z > y > x$. It is a most profitable stage and most common stage for all markets; because brand awareness is strong and advertising expenditure will be reduced. After that, the inventory level starts to decrease due to demand and deterioration during (t_3, t_4), the stage termed as decline stage, the market is shrinking, reducing overall amount of profit that can be shared amongst the remaining competitor. The product becomes technologically obsolete or customer tastes change. At time t_4 shortages starts and attained maximum level at t_5. Lastly during the period (t_5, T), inventory accumulates by the production and demand at the rate $(p - d)$ and reaches inventory zero level at time T. The inventory level attains levels Q_1, Q_2 and Q_3 at time t_1, t_2 and t_3 respectively.

4. MATHEMATICAL FORMULATION OF THE INVENTORY MODEL

The graphical presentation of the deteriorating inventory level of serviceable items at time t_i $(i = 1, 2, \cdots, 5)$ in a replenishment cycle is shown in the Fig 1. The governing differential equations of the i^{th} replenishment cycle for the four different stages of the model are as follows:

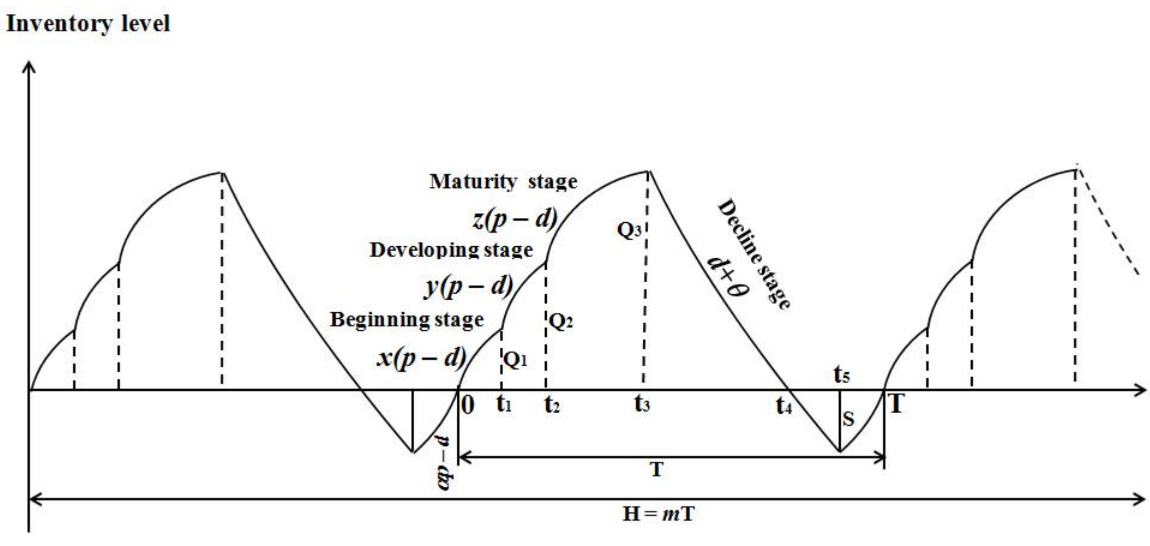

Fig 1: Inventory level of finished goods for i^{th} replenishment cycle

The governing differential equations of the i^{th} replenishment cycle for the four different stages of the model are as follows:
The differential equation for the beginning stage of the production during $(0, t_1)$ is given by:

$$\frac{dI_{1i}(t)}{dt} + x\lambda I_{1i}(t) = x(\gamma - 1)ae^{bt}, \qquad 0 < t < t_1 \tag{1}$$

The differential equation for the developing stage of the production during (t_1, t_2) is written by:

$$\frac{dI_{2i}(t)}{dt} + y\lambda I_{2i}(t) = y(\gamma - 1)ae^{bt}, \qquad t_1 < t < t_2 \tag{2}$$

The differential equation for the maturity stage of the production during (t_2, t_3) is presented by:

$$\frac{dI_{3i}(t)}{dt} + z\lambda I_{3i}(t) = z(\gamma - 1)ae^{bt}, \qquad t_2 < t < t_3 \tag{3}$$

The differential equation for the decline stage of the non-production during (t_3, t_4) is drawn as:

$$\frac{dI_{4i}(t)}{dt} + \lambda I_{4i}(t) = -ae^{bt}, \qquad t_3 < t < t_4 \tag{4}$$

The differential equation for the shortage accumulating stage during (t_4, t_5) is given by:

$$\frac{dI_{5i}(t)}{dt} = -ae^{bt}, \qquad t_4 < t < t_5 \tag{5}$$

The differential equation for the shortage backlogging stage during (t_5, t_6) is stated by:

$$\frac{dI_{6i}(t)}{dt} + \lambda I_{6i}(t) = (\gamma - 1)ae^{bt}, \qquad t_5 < t < T \tag{6}$$

The initial and boundary conditions are as follows:

$$I_{1i}(0) = I_{4i}(T_1) = 0, I_{1i}(t_1) = Q_1, I_{2i}(t_2) = Q_2, I_{3i}(t_3) = Q_3, I_{5i}(t_5) = S, I_{6i}(T) = 0.$$

Now, we solve the above six differential equations using the initial and boundary conditions. Since all the above equations are linear equation, we can easily able to solve the above six differential equations.
The solution of equation (1), that is, the inventory level of beginning stage at any time 't' is obtained as:

$$I_{1i}(t) = \frac{xa(\gamma - 1)}{b + \lambda}\left\{e^{bt} - e^{\lambda t}\right\}, \quad 0 \le t \le t_1 \tag{7}$$

The solution of equation (2), that is, the inventory level of developing stage at any time 't' is derived as:

$$I_{2i}(t) = Q_1 e^{y\lambda(t_1 - t)} + \frac{ya(\gamma - 1)}{b + y\lambda}\left\{e^{(b+y\lambda)t} - e^{(b+y\lambda)t_1}\right\}e^{-y\lambda t}, \quad t_1 \le t \le t_2 \tag{8}$$

The solution of equation (3), that is, the inventory level of maturity stage at any time 't' is presented as:

$$I_{3i}(t) = Q_2 e^{z\lambda(t_2 - t)} + \frac{za(\gamma - 1)}{b + y\lambda}\left\{e^{(b+z\lambda)t} - e^{(b+\lambda z)t_2}\right\}e^{-z\lambda t}, \quad t_2 \le t \le t_3 \tag{9}$$

The solution of equation (4), that is, the inventory level of decline stage at any time 't' is derived as:

$$I_{4i}(t) = Q_3 e^{\lambda(t_3 - t)} + \frac{a}{b + \lambda}\left\{e^{(b+\lambda)t} - e^{(b+\lambda)t_3}\right\}e^{-\lambda t}, \quad t_3 \le t \le t_4 \tag{10}$$

The solution of equation (5), that is, the inventory level of shortage accumulating stage at any time 't' is obtained as:

$$I_{5i}(t) = \frac{a}{b}\left\{e^{bt_4} - e^{bt}\right\}, \quad t_4 \le t \le t_5 \tag{11}$$

The solution of equation (6), that is, the inventory level of shortage backlogging stage at any time 't' is presented as:

$$I_{6i}(t) = S + \frac{a(\lambda - 1)}{b}\left\{e^{bt} - e^{bt_5}\right\}, \quad t_5 \le t \le T \tag{12}$$

Maximum inventory: Q_1
During the time t_1, the maximum inventory is computed using equation (7) and the boundary condition $I_{1i}(t_1) = Q_1$.

$$Q_1 = \frac{xa(\gamma - 1)}{b + \lambda}\left\{e^{bt_1} - e^{\lambda t_1}\right\} \tag{13}$$

Maximum inventory: Q_2
During the time t_2, the maximum inventory is computed using equation (8) and the boundary condition $I_{2i}(t_2) = Q_2$.

$$Q_2 = Q_1 e^{y\lambda(t_1 - t_2)} + \frac{ya(\gamma - 1)}{b + y\lambda}\left\{e^{(b+y\lambda)t_2} - e^{(b+y\lambda)t_1}\right\}e^{-y\lambda t_2} \tag{14}$$

Maximum inventory: Q_3
During the time t_3, the maximum inventory is computed using equation (9) and the boundary condition $I_{3i}(t_3) = Q_3$.

$$Q_3 = Q_2 e^{z\lambda(t_2 - t_3)} + \frac{za(\gamma - 1)}{b + y\lambda}\left\{e^{(b+z\lambda)t_3} - e^{(b+\lambda z)t_2}\right\}e^{-z\lambda t_3} \tag{15}$$

Shortage level: S
The shortage level (S) is obtained by using the boundary condition $I_{6i}(T) = 0$ in the equation (12)

$$S = \frac{a(\lambda - 1)}{b}\left\{e^{bt_5} - e^{bT}\right\} \tag{16}$$

Total cost function (TCF):

The total cost comprises of the sum of the production cost, ordering cost, inspection cost, holding cost, deteriorating cost and shortage cost. They are grouped together after evaluating the above cost individually,

(i) *Production cost* :
Production cost includes, labor, raw materials, advertisement, consumable manufacturing supplies, general overhead etc. The items are produced during the periods $(0, \mathrm{t}_1)$, $(\mathrm{t}_1, \mathrm{t}_2)$, $(\mathrm{t}_2, \mathrm{t}_3)$, $(\mathrm{t}_5, \mathrm{T})$. Hence, the production cost during the above periods is calculated by:

$$PC = c_p\left\{\int_0^{t_1} p(t)\,dt + \int_{t_1}^{t_2} p(t)\,dt + \int_{t_2}^{t_3} p(t)\,dt + \int_{t_5}^{T} p(t)\,dt\right\}$$

$$PC = \frac{a\gamma}{b}c_p\left\{\left(e^{bt} - 1\right) + \left(e^{bt_2} - e^{bt_1}\right) + \left(e^{bt_3} - e^{bt_2}\right) + \left(e^{bT} - e^{bt_5}\right)\right\} \tag{17}$$

(ii) *Production setup cost/ordering cost* :
At the starting of the production process, the supplier has to get the equipment ready. The production setup cost occurs when $t = 0$ in the interval $(0, t_1)$. Therefore the production setup cost is given by:

$$SC = K_s \tag{18}$$

(iii) *Shortages cost* :
In this inventory model shortages hold during (t_4, t_5) and to overcome the backorders due to shortages items are produced during (t_5, T). Therefore, the total time periods for shortages are (t_4, t_5) and (t_5, T).

$$CS = c_s\left\{\int_{t_4}^{t_5} I_{5i}(t)\,dt + \int_{t_5}^{T} I_{5i}(t)\,dt\right\}$$

$$CS = c_s\left[\int_{t_4}^{t_5} \frac{a}{b}\left(e^{bt_4} - e^{bt}\right)dt + \int_{t_5}^{T}\left\{S + \frac{a(\gamma - 1)}{b}\left(e^{bt} - e^{bt_5}\right)\right\}dt\right]$$

$$CS = c_s\begin{bmatrix} \frac{a}{b}\left\{(t_5 - T)e^{bt_4} - \frac{\left(e^{bt_5} - e^{bt_4}\right)}{b}\right\} + S(T - t_5) \\ + \frac{a(\gamma-1)}{b}\left\{\frac{\left(e^{bT} - e^{bt_5}\right)}{b} - (T - t_5)e^{bt_5}\right\} \end{bmatrix} \tag{19}$$

(iv) $\boldsymbol{Deterioration\ cost}$**:**
The items are stored in an inventory during $(0, t_1)$, (t_1, t_2), (t_2, t_3) and (t_3, t_4) to meet customers demand. The items are deteriorating during the above storage periods. Hence, the cost of deterioration for the above periods is calculated as follows:

$$DC = D_c\lambda\left\{\int_0^{t_1} I_{1i}(t)\,dt + \int_{t_1}^{t_2} I_{2i}(t)\,dt + \int_{t_2}^{t_3} I_{3i}(t)\,dt + \int_{t_3}^{t_4} I_{4i}(t)\,dt\right\}$$

$$DC = D_c\lambda\begin{bmatrix} \frac{a(\gamma-1)}{b\lambda(b+\lambda)}\left\{\lambda(e^{bt_1}-1) - b(e^{\lambda t_1}-1)\right\} - \frac{Q_1}{y\lambda}(e^{y\lambda(t_1-t_2)}-1) \\ +\frac{ay(\gamma-1)}{b+y\lambda}\left\{\frac{(e^{bt_2}-e^{bt_1})}{b} + \frac{e^{(b+y\lambda)t_1}}{y\lambda}(e^{-y\lambda t_2}-e^{-y\lambda t_1})\right\} \\ -\frac{Q_2}{z\lambda}(e^{z\lambda(t_2-t_3)}-1) + \frac{Q_3}{\lambda}(1-e^{\lambda(t_3-t_4)}) \\ +\frac{az(\gamma-1)}{b+z\lambda}\left\{\frac{(e^{bt_3}-e^{bt_2})}{b} + \frac{e^{(b+z\lambda)t_2}}{z\lambda}(e^{-z\lambda t_3}-e^{-z\lambda t_2})\right\} \\ +\frac{a}{b+\lambda}\left\{\frac{(e^{bt_4}-e^{bt_3})}{b} - \frac{e^{-\lambda t_3}}{b+\lambda}(e^{(b+\lambda)t_4}-e^{(b+\lambda)t_3})\right\} \end{bmatrix} \tag{20}$$

(v) $\boldsymbol{Holding\ cost}$**:**
Since it is necessary to keep the items in stock during the periods $(0, t_1)$, $(t_1,\ t_2)$, $(t_2,\ t_3)$ and (t_3, t_4), the cost of holding is calculated in these periods. During (t_4, t_5), the items undergoes shortage so there is no item to stock and during (t_5, T), the items are produced which are used to overcome the backorder so no items are required to be held or stored during this period.

$$HC = h_s\left\{\int_0^{t_1} I_{1i}(t)\,dt + \int_{t_1}^{t_2} I_{2i}(t)\,dt + \int_{t_2}^{t_3} I_{3i}(t)\,dt + \int_{t_3}^{t_4} I_{4i}(t)\,dt\right\}$$

$$HC = h_s\begin{bmatrix} \frac{a(\gamma-1)}{b\lambda(b+\lambda)}\left\{\lambda(e^{bt_1}-1) - b(e^{\lambda t_1}-1)\right\} - \frac{Q_1}{y\lambda}(e^{y\lambda(t_1-t_2)}-1) \\ +\frac{ay(\gamma-1)}{b+y\lambda}\left\{\frac{(e^{bt_2}-e^{bt_1})}{b} + \frac{e^{(b+y\lambda)t_1}}{y\lambda}(e^{-y\lambda t_2}-e^{-y\lambda t_1})\right\} \\ -\frac{Q_2}{z\lambda}(e^{z\lambda(t_2-t_3)}-1) + \frac{Q_3}{\lambda}(1-e^{\lambda(t_3-t_4)}) \\ +\frac{az(\gamma-1)}{b+z\lambda}\left\{\frac{(e^{bt_3}-e^{bt_2})}{b} + \frac{e^{(b+z\lambda)t_2}}{z\lambda}(e^{-z\lambda t_3}-e^{-z\lambda t_2})\right\} \\ +\frac{a}{b+\lambda}\left\{\frac{(e^{bt_4}-e^{bt_3})}{b} - \frac{e^{-\lambda t_3}}{b+\lambda}(e^{(b+\lambda)t_4}-e^{(b+\lambda)t_3})\right\} \end{bmatrix} \tag{21}$$

(vi) $\boldsymbol{Inspection\ cost\ (IC)}$**:**
Since the inspection is carried out during the production periods $(0, t_1)$, $(t_1,\ t_2)$ and $(t_2,\ t_3)$, the inspection cost is calculated as follows:

$$IC = i_c\left\{\int_0^{t_1} p(t)\,dt + \int_{t_1}^{t_2} p(t)\,dt + \int_{t_2}^{t_3} p(t)\,dt\right\}$$

$$IC = \frac{a\gamma}{b} i_c\left\{\left(e^{bt}-1\right) + \left(e^{bt_2}-e^{bt_1}\right) + \left(e^{bt_3}-e^{bt_2}\right)\right\} \tag{22}$$

Total cost function $\boldsymbol{(TCF)}$**:**
The total cost function for the proposed inventory model is defined as follows:

$TCF =$ Production cost + Inspection cost + Holding cost + Deterioration cost + Production Setup cost +Shortage cost + Cost of defective items.

Our objective is to minimize the total cost function (TCF). The problem is given by

$$\text{Minimize} \quad TCF = PSC + [PC + CS + DC + HC + IC]\sum_{i=1}^{m} e^{-(i-1)RT}$$

$$TCF = SC + [PC + CS + DC + HC + IC]\left[\frac{1-\mathrm{e}^{-\mathrm{RH}}}{1-\mathrm{e}^{-\mathrm{RT}}}\right]$$

$$TFC = K_s + \begin{bmatrix} \frac{a\gamma}{b}c_p\left\{\left(e^{bt}-1\right) + \left(e^{bt_2}-e^{bt_1}\right) + \left(e^{bt_3}-e^{bt_2}\right) + \left(e^{bT}-e^{bt_5}\right)\right\} \\ +c_s\begin{bmatrix} \frac{a}{b}\left\{(t_5-T)\,e^{bt_4} - \frac{(e^{bt_5}-e^{bt_4})}{b}\right\} + S\,(T-t_5) \\ +\frac{a(\gamma-1)}{b}\left\{\frac{(e^{bT}-e^{bt_5})}{b} - (T-t_5)\,e^{bt_5}\right\} \end{bmatrix} \\ +D_c\lambda\xi + h_s\xi + \frac{a\gamma}{b}i_c\left\{\left(e^{bt}-1\right) + \left(e^{bt_2}-e^{bt_1}\right) + \left(e^{bt_3}-e^{bt_2}\right)\right\} \end{bmatrix}\left[\frac{1-e^{-RH}}{1-e^{-RT}}\right] \tag{23}$$

$$\text{where} \quad \xi = \begin{bmatrix} \frac{a(\gamma-1)}{b\lambda(b+\lambda)}\left\{\lambda(e^{bt_1}-1)-b(e^{\lambda t_1}-1)\right\}-\frac{Q_1}{y\lambda}(e^{y\lambda(t_1-t_2)}-1) \\ +\frac{ay(\gamma-1)}{b+y\lambda}\left\{\frac{(e^{bt_2}-e^{bt_1})}{b}+\frac{e^{(b+y\lambda)t_1}}{y\lambda}(e^{-y\lambda t_2}-e^{-y\lambda t_1})\right\} \\ -\frac{Q_2}{z\lambda}\left(e^{z\lambda(t_2-t_3)}-1\right)+\frac{Q_3}{\lambda}\left(1-e^{\lambda(t_3-t_4)}\right) \\ +\frac{az(\gamma-1)}{b+z\lambda}\left\{\frac{(e^{bt_3}-e^{bt_2})}{b}+\frac{e^{(b+z\lambda)t_2}}{z\lambda}(e^{-z\lambda t_3}-e^{-z\lambda t_2})\right\} \\ +\frac{a}{b+\lambda}\left\{\frac{(e^{bt_4}-e^{bt_3})}{b}-\frac{e^{-\lambda t_3}}{b+\lambda}\left(e^{(b+\lambda)t_4}-e^{(b+\lambda)t_3}\right)\right\} \end{bmatrix}$$

Let us assume that (as assumed in Viji and Karthikeyan 2016) $t_1 = lt_4,\ \ t_2 = mt_4,\ t_3 = nt_4$ and $t_5 = \mu\,(T - T_1)$.

The objective is to determine the optimum value of t_4 and T so that $TC\,(t_4,\ T)$ is minimum. The value of t_4 and T, for which the total cost $TC\,(t_4,\ T)$ is minimum, is the solution of equations $\frac{\partial TC(t_4,T)}{\partial t_4} = 0$ and $\frac{\partial TC(t_4,T)}{\partial T} = 0$ satisfy the condition

$$\left\{\left(\frac{\partial^2 TC\,(t_4,T)}{\partial t_4}\right)\left(\frac{\partial^2 TC\,(t_4,T)}{\partial T^2}\right) - \left(\frac{\partial^2 TC\,(t_4,T)}{\partial t_4 \partial T}\right)\right\} > 0$$

The optimum solution of equation (23) is obtained by using mathematical software MATLAB.

5. NUMERICAL EXAMPLE

Let the inventory system has the following parameter values: $a = 135, b = 0.3, x = 0.1, y = 0.2, z = 0.3, \gamma = 1.3, k_s = 300, h_s = 9$ per unit, $c_p = 80$, $D_c = 7$, $c_s = 10$, $\theta = 0.4$, $\epsilon = 2.35$, $m\,(\epsilon) = \theta\,(1 - \exp\,(-0.1\epsilon))$, $l = 0.2, m = 0.4, n = 0.6$ and $\mu = 0.33$.

The optimum value of $t_1,\ t_2,\ t_3,\ t_4,\ t_5$ and T is obtained and they are given below:

$$t_1^* = 0.1023, t_2^* = 0.1688, t_3^* = 0.1139, t_4^* = 0.2196, t_5^* = 0.3074 \text{ and } T^* = 0.4065.$$

The optimum quantities at the end of initial, middle and maturity stages are respectively given by:

$$Q_1^* \approx 29, \quad Q_2^* \approx 76 \quad \text{and} \quad Q_3^* \approx 127.$$

The economic production quantity is given by:

$$Q^* \approx 86.$$

The shortage units of items are given by:

$$S \approx 9.$$

The optimum inventory cost of the model is given by:

$$TC^* = 7551.53$$

The other inventory costs are given below:

Production cost $\approx 7,974$

Holding cost ≈ 189

Shortage cost ≈ 45.

Sensitivity analysis

The Sensitivity analysis is performed by keeping one parameter changing while all other parameters as constant and corresponding change of TC with that variable and the corresponding graphical figures are given. Here we have performed the sensitivity analysis by changing the five parameters $a,\ b,\ h_s, c_p$ and c_s one by one. The result shows that the parameters a, b and h_s are highly sensitive with the total average inventory cost and the parameters c_p and c_s are slightly sensitive with the total average inventory cost.

Table 1: Sensitivity with various parameters v/s total inventory cost

a	TC	b	TC	h_s	TC	c_p	TC	c_s	TC
135	7551.53	0.3	7551.53	3	6877.25	80	7551.53	4	7454.97
140	7952.36	0.4	7456.73	4	6991.26	82	7572.28	5	7478.57
145	8325.98	0.5	7302.89	5	7109.22	84	7595.41	6	7491.63
150	8739.05	0.6	7125.35	6	7255.12	86	7605.83	7	7508.72
155	8889.32	0.7	6985.65	7	7358.09	88	7628.32	8	7521.39
160	9395.54	0.8	6794.40	8	7465.72	90	7659.01	9	7537.57
165	9835.07	0.9	6568.33	9	7551.53	92	7687.17	10	7551.53

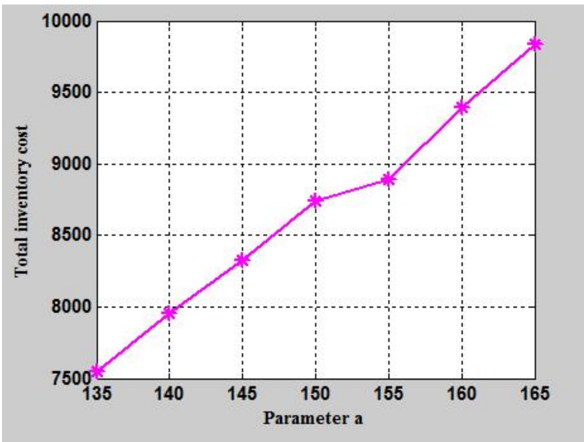

Fig 2: Sensitivity: parameter a v/s TC

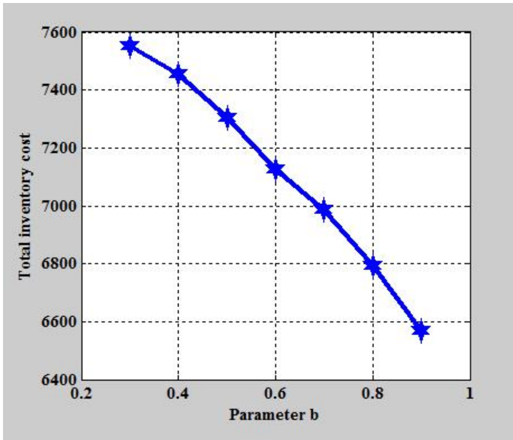

Fig 3: Sensitivity: parameter b v/s TC

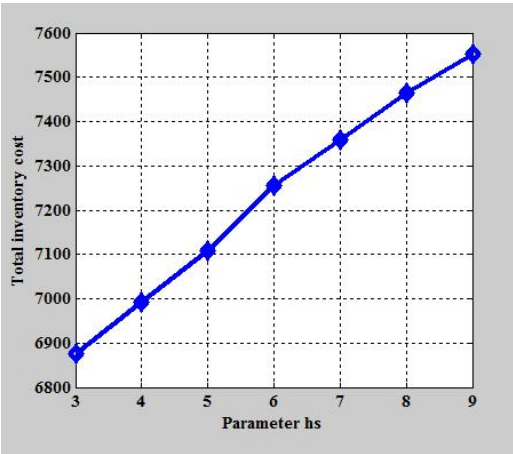

Fig 4: Sensitivity: parameter h_s v/s TC

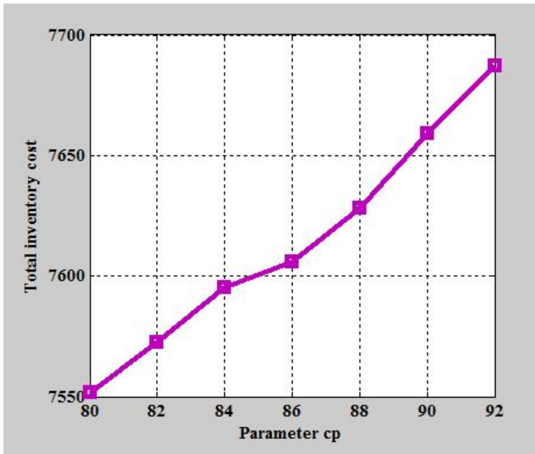

Fig 5: Sensitivity: parameter c_p v/s TC

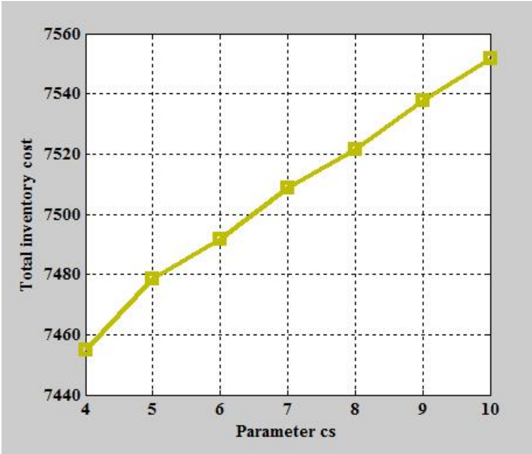

Fig 6: Sensitivity: parameter c_s v/s TC

Observations:

From the Table 1 and the Fig 2, 3, 4, 5 and 6, it is observed that:

I. Increase in the value of the parameter a, the total inventory cost increases.

II. As we increase the value of the parameter b, the total inventory cost decreases.

III. As we increase the holding cost (h_s) per unit, the total inventory cost also increase.

IV. When the cost of production per unit increases (c_p), the total inventory cost increases.

V. If we increase shortage cost (c_s) per unit, then the total inventory cost increases.

6. CONCLUSION

In this article, we introduced an inventory model for deteriorating items in which four different levels of production are considered. We assumed that the demand is an increasing function and the rate of deterioration is constant. The proposed model is suitable for newly launched product with constant pattern up to a point in time. Such situation is desirable since by starting at low rate of production, large quantum of stock of manufacturing items at the initial stage to be avoided which will be leading to reduction in the holding cost. Due to this, we will get consumer satisfaction and earn potential profit. Here we established mathematical model and its solution. To demonstrate the model, numerical example and its sensitivity analysis are given. The proposed inventory model can assist the manufacturer and retailer in determining the optimal order quantity, cycle time and total inventory cost accurately. For further research, this model can be extended in several ways involving different demand rates such as linear, quadratic, cubic, Weibull deterioration with three parameters, time discounting and rework of defective items.

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