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# Vector Product-Form Approach for $PH/PH/1/N$ Queueing Systems by Examples

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**Abstract:** In this paper, we study the $PH/PH/1/N$ open queueing system with finite capacity, $N$. We use the product-form method to solve the steady-state probabilities and give tables of numerical results in examples of $C_k/C_m/1/4$ and $C_k/C_m/1/6$. The merit of this method is that the computation time is independent of $N$. In our computational experiments, we have observed that when the capacity size of queueing system, $N > 100$, the computing efficiency of the product-form method is much better than that of a traditional method.
**Keyword —** phase-type distributions, matrix geometric solutions

## 1. INTRODUCTION

We consider a $C_k/C_m/1/N$ open queueing model which restrains a finite number of customers in the system. It is assumed that both interarrival and service time are Coxian distributions with $k$ and $m$ phases respectively. This system is a single server queueing system. Customers are served under the First-come First-served discipline (FCFS). The $C_k/C_m/1/N$ queueing system and the $PH/PH/1/N$ queueing system are the same.

The examples of $C_k/C_m/1/N$ can be found in many applications. In Neuts(1988), stationary probabilities was obtained in matrix-geometric form. The matrix-geometric method relies on determining the minimal nonnegative matrix solution $R$ of a matrix-quadratic equation; the invariant vector is expressed in terms of powers of $R$. Bertsimas(1990) studied a $C_k/C_m/s$ queue. He showed that the equilibrium probabilities for unboundary states are geometric in the number of waiting customers by using a generating function technique, we will take a different approach in this paper.

Le Boudec(1988) studied a $PH/PH/1$ queue. He showed that the stationary probability is a linear combination of product-forms which can be expressed in terms of roots of the associated characteristic polynomial. He showed that all eigenvectors used in the expression of the stationary probability of $PH/PH/1$ are Kronecker products and gave a simple formula for computing the stationary probability of the number of customers in the system. Luh(1999) used a similar approach to derive stationary probabilities in terms of linear combinations of product-forms in studying a system of two stations in tandem.

Wang(2002) considered a $PH/PH/1/N$ open queueing system containing finite number of customers $N$. She showed that the number of roots of the associated characteristic polynomial depends on the utilization factor, $\rho$, but independently of $N$. Liu(2004) established a procedure for solving stationary probabilities. It is easy to construct the product-form when the $m$ roots of the characteristic polynomial are distinct, each vectors used in the expression of stationary probabilities are described in terms of Kronecker products. One may refer to Liu(2004) for details.

Although, Le Boudec(1988), Wang(2002), have provided the product-form method to solve the stationary probabilities, they do not list the numerical results in detail and provide a solution procedure. In this paper, our goal is to calculate the stationary probabilities of $C_k/C_m/1/N$ numerically by using the product-form method and compare it with a traditional method. We modify the product-form method by perturbation when the utilization factor, $\rho$, equals to one. The Matlab software is the computing tool to solve the stationary probability. In this study, we focus on discussion the implementation of the product-form method. Precision and stability involutes numerical concept and schemes which would distract one's attention from the solution procedure itself. Thus, it is ignored in this study. The results of this study may be helpful to students and teachers who want to have a numerical solution of stationary probabilities of $C_k/C_m/1/N$ queueing system, since there is no existing close-form solution of such systems.

This paper is organized in the following. In section 2, we introduce the model of $C_k/C_m/1/N$ queues and the vector product-form solution. Section 3 establish an algorithm for solving stationary probabilities and give two examples to illustrate the stationary probabilities. In section 4, we describe the operational procedures of program and

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give the tables of numerical results of $C_k/C_m/1/4$ and $C_k/C_m/1/6$. The numerical results of the case of $\rho = 1$ is calculated by using the product-form method with small adjustment of constant which would give a satisfactory result from our experiments.

## 2. THE MODEL

### 2.1 Interarrival and Service Times

We assume that both interarrival and service times are of Coxian distribution with $k$ and $m$ stages respectively. It means that an arrival may go through at most up to $k$ phases, and the length of phase $j$ is exponentially distributed with a given rate $\lambda_j$ for $j = 1, \cdots, k$. After phase $j$, $j = 1, \cdots, k$, the interarrival time comes to an end with probability $p_j$, and it enters the next phase with probability $1 - p_j$. Obviously, $p_k = 1$. A similar notation for $\mu_j$ and $q_j$, $j = 1, \cdots, m$, in the service distribution is assumed.

Let $F_a(t)$ be the interarrival time distribution with phase $k$ and its mean by $\frac{1}{\lambda}$. Then, it is known that

$$F_a(t) = 1 - \boldsymbol{\tau}_1 \exp(\mathbf{T}_1 t)\mathbf{e}' = -\sum_{n=1}^{\infty} \boldsymbol{\tau}_1 \mathbf{T}_1^n \mathbf{e}' \frac{t^n}{n!},$$

where the $\boldsymbol{\tau}_1$ is the initial probability of $1 \times k$ vector

$$\boldsymbol{\tau}_1 = (1, 0, \cdots, 0),$$

$$\mathbf{T}_1 = \begin{bmatrix} -\lambda_1 & (1-p_1)\lambda_1 & & & 0 \\ & -\lambda_2 & (1-p_2)\lambda_2 & & \\ & & \ddots & \ddots & \\ & & & -\lambda_{k-1} & (1-p_{k-1})\lambda_{k-1} \\ 0 & & & & \lambda_k \end{bmatrix}$$

is a squared matrix of order $k$, and $\mathbf{e}'$ is a column vector of all entries equal to 1 in a proper dimension depending on its multiplier.

Similarly, the service time distribution $F_s(t)$ has mean $\frac{1}{\mu}$ and representation $(\boldsymbol{\tau}_2, \mathbf{T}_2)$ of dimension $m$, where

$$\boldsymbol{\tau}_2 = (1, 0, \cdots, 0)$$

is a $1 \times m$ vector and

$$\mathbf{T}_2 = \begin{bmatrix} -\mu_1 & (1-q_1)\mu_1 & & & 0 \\ & -\mu_2 & (1-q_2)\mu_2 & & \\ & & \ddots & \ddots & \\ & & & -\mu_{m-1} & (1-q_{m-1})\mu_{m-1} \\ 0 & & & & -\mu_m \end{bmatrix}$$

is the squared matrix of order $m$. The distribution is given by

$$F_a(t) = 1 - \boldsymbol{\tau}_2 \exp(\mathbf{T}_2 t)\mathbf{e}' = -\sum_{n=1}^{\infty} \boldsymbol{\tau}_2 \mathbf{T}_2^n \mathbf{e}' \frac{t^n}{n!}.$$

The Laplace transform of the interarrival time has the form

$$\begin{aligned} F_a{}^*(x) &= \int_0^{\infty} e^{-xt} dF_a(t) \\ &= \int_0^{\infty} e^{-xt} \left[-\boldsymbol{\tau}_1 \exp(\mathbf{T}_1 t)\mathbf{T}_1 \mathbf{e}'\right] dt \\ &= \int_0^{\infty} -\boldsymbol{\tau}_1 \exp\left[(\mathbf{T}_1 - x\mathbf{I}_1)t\right] \mathbf{T}_1 \mathbf{e}' dt \\ &= -\boldsymbol{\tau}_1 (\mathbf{T}_1 - x\mathbf{I}_1)^{-1} \exp\left[(\mathbf{T}_1 - x\mathbf{I}_1)\right]\Big|_0^{\infty} \mathbf{T}_1 \mathbf{e}' \\ &= \boldsymbol{\tau}_1 (x\mathbf{I}_1 - \mathbf{T}_1)^{-1} \boldsymbol{\gamma}_1 \end{aligned}$$

Where $\boldsymbol{\gamma}_1 = -\mathbf{T}_1\mathbf{e}'$, and $\mathbf{I}_1$ is an identity matrix with proper dimension in equation. Similarly, the Laplace transform of the service time has the form

$$F_a^*(x) = \boldsymbol{\tau}_2(x\mathbf{I}_2 - \mathbf{T}_2)^{-1}\boldsymbol{\gamma}_2,$$

where $\boldsymbol{\gamma}_2 = -\mathbf{T}_2\mathbf{e}'$. The utilization factor is defined as

$$\rho = \frac{\lambda}{\mu}$$

Since

$$\frac{1}{\lambda} = \int_0^\infty t dF_a(t) = -F_a^{*\prime}(0)$$

and

$$\frac{1}{\mu} = \int_0^\infty t dF_s(t) = -F_s^{*\prime}(0)$$

we have

$$\rho = \frac{F_s^{*\prime}(0)(0)}{F_a^{*\prime}(0)}.$$

### 2.2 Matrix of Transition Rates

The $C_k/C_m/1/N$ queueing system may be studied as a Quasi-Birth-Death process. A state of system is denoted by $(n, i, j)$, where $n$ is the number of customers in the system, $n \geq 0$, and $i$ (resp. $j$) is the phase of customer presents in the interarrival fictitious center (resp. the service center), $1 \leq i \leq k$, $1 \leq j \leq m$. We arrange the states $(n, i, j)$ in lexicographic order and partition of the state space according to the number of customers, $n$, i.e.

$$\mathcal{S}_n = \{(n, i, j) | 1 \leq i \leq k,\ 1 \leq j \leq m\}, n = 0, 1, 2, \cdots, N.$$

For fixed $n$ the state can be lexicographically in according with phase $i$ and $j$. The state space can be organized into three groups:

$$\mathcal{S}_0 = \{(0,1,0), (0,2,0), \cdots, (0,k,0)\},$$

$$\mathcal{S}_n = \begin{matrix} \{(n,1,1), & (n,1,2), & \cdots, & (n,1,m); \\ (n,2,1), & (n,2,2), & \cdots, & (n,2,m); \\ \vdots & \vdots & \cdots & \vdots \\ (n,k,1), & (n,k,2), & \cdots, & (n,k,m)\}, \end{matrix} \quad \text{where } 1 \leq n \leq N-1,$$

$$\mathcal{S}_N = \begin{matrix} \{(N,1,1), & (N,1,2), & \cdots, & (N,1,m); \\ (N,2,1), & (N,2,2), & \cdots, & (N,2,m); \\ \vdots & \vdots & \cdots & \vdots \\ (N,k,1), & (N,k,2), & \cdots, & (N,k,m)\}. \end{matrix}$$

$\mathcal{S}_0$ and $\mathcal{S}_N$ are defined for boundary states. Likewise, $\mathcal{S}_n$, $1 \leq n \leq N-1$ is defined for unboundary state. Denoted by $\mathbf{P}$ the stationary probability row-vector partitioned corresponding to $\mathcal{S}_n$ as:

$$\mathbf{P} = (\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_N),$$

where $\mathbf{P}_n$ is a stationary probability row-vector when $n$ customer in system. Define $\mathbf{Q}$ the transition rate matrix of the chain according to the arrangement of $\mathcal{S}_n$. Then $\mathbf{Q}$ is of the block-tridiagonal form and written as

$$\mathbf{Q} = \begin{matrix} \\ \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \vdots \\ \mathcal{S}_{N-2} \\ \mathcal{S}_{N-1} \\ \mathcal{S}_N \end{matrix} \begin{matrix} \begin{matrix} \mathcal{S}_0 & \mathcal{S}_1 & \mathcal{S}_2 & \cdots \mathcal{S}_{N-2} \mathcal{S}_{N-1} \mathcal{S}_N \end{matrix} \\ \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_0 & & & & & \\ \mathbf{C}_0 & \mathbf{B} & \mathbf{A} & & & & \\ & \mathbf{C} & \mathbf{B} & \mathbf{A} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \mathbf{C} & \mathbf{B} & \mathbf{A} & \\ & & & & \mathbf{C} & \mathbf{B} & \mathbf{A} \\ & & & & & \mathbf{C} & \mathbf{B}_1 \end{bmatrix} \end{matrix}$$

$\mathbf{A}_0$ is a $k \times km$ transition rate matrix from states of $\mathcal{S}_0$ to states of $\mathcal{S}_1$. $\mathbf{B}_0$ is a $k \times k$ transition rate matrix among states of $\mathcal{S}_0$. $\mathbf{C}_0$ is a $km \times k$ transition rate matrix from states of $\mathcal{S}_1$ to states of $\mathcal{S}_0$. $\mathbf{A}$ is a $km \times km$ transition rate matrix from states of $\mathcal{S}_n$ to states of $\mathcal{S}_{n+1}$, $1 \le n \le N-1$. $\mathbf{B}$ is a $km \times km$ transition rate matrix among states of $\mathcal{S}_n$, $1 \le n \le N-1$. $\mathbf{C}$ is a $km \times km$ transition rate matrix from states of $\mathcal{S}_n$ to states of $\mathcal{S}_{n-1}$, $2 \le n \le N-1$. $\mathbf{B}_1$ is a $km \times km$ transition rate matrix among states of $\mathcal{S}_N$.

The submatrices could be written as Kronecker product and Kronecker sum, defined in Bellman(1960), which were denoted by $\otimes$ and $\oplus$, respectively. Kronecker product and Kronecker sum were used to simplify the representation of the system of balance equations for queues by many researchers, for example Neuts(1981) and Neuts(1988). Here, the submatrices $\mathbf{A}_0$, $\mathbf{B}_0$, $\mathbf{C}_0$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{B}_1$ are given below:

$$\begin{array}{lll} \mathbf{A}_0 = \gamma_1\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_2 & \mathbf{B}_0 = \mathbf{T}_1 & \mathbf{C}_0 = \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2 \\ \mathbf{A} = \gamma_1\boldsymbol{\tau}_1 \otimes \mathbf{I}_2 & \mathbf{B} = \mathbf{T}_1 \oplus \mathbf{T}_2 & \mathbf{C} = \mathbf{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\tau}_2 \\ & \mathbf{B}_1 = (\mathbf{T}_1 + \mathbf{R}_1) \oplus \mathbf{T}_2 & \end{array} \tag{2.1}$$

where $\mathbf{R}_1$ is diag$(\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_k)$.

### 2.3 Balance Equations

For the balance equations $\mathbf{PQ} = \mathbf{0}$ and the normalization condition $\mathbf{Pe}' = 1$, we obtain the following equations:

$$\begin{cases} \mathbf{P}_0\mathbf{B}_0 + \mathbf{P}_1\mathbf{C}_0 = \mathbf{0} & (2.2) \\ \mathbf{P}_0\mathbf{A}_0 + \mathbf{P}_1\mathbf{B} + \mathbf{P}_2\mathbf{C} = \mathbf{0} & (2.3) \\ \mathbf{P}_{n-1}\mathbf{A} + \mathbf{P}_n\mathbf{B} + \mathbf{P}_{n+1}\mathbf{C} = \mathbf{0} \qquad 2 \le n \le N-1 & (2.4) \\ \mathbf{P}_{N-1}\mathbf{A} + \mathbf{P}_N\mathbf{B}_1 = \mathbf{0} & (2.5) \\ \mathbf{Pe}' = 1 & (2.6) \end{cases}$$

It is easy to rewrite the balance equations by substitute (2.1) into equations (2.2) $\sim$ (2.5):

$$\mathbf{P}_0\mathbf{T}_1 + \mathbf{P}_1(\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2) = 0, \tag{2.7}$$

$$\mathbf{P}_0(\gamma_1\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_2) + \mathbf{P}_1(\mathbf{T}_1 \oplus \mathbf{T}_2) + \mathbf{P}_2(\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\tau}_2) = 0, \tag{2.8}$$

$$\mathbf{P}_{n-1}(\gamma_1\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + \mathbf{P}_n(\mathbf{T}_1 \oplus \mathbf{T}_2) + \mathbf{P}_{n+1}(\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\tau}_2) = 0, \qquad 2 \le n \le N-1, \tag{2.9}$$

$$\mathbf{P}_{N-1}(\gamma_1\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + \mathbf{P}_N((\mathbf{T}_1 + \mathbf{R}_1) \oplus \mathbf{T}_2) = 0. \tag{2.10}$$

### 2.4 Vector Product-Form Solutions

#### 2.4.1 Case of Simple Roots

In Wang(2002), Wang expressed the unboundary stationary probabilities $(\mathbf{P}_n,\ n = 1, \cdots, N-1)$ of $C_k/C_m/1/N$ system can be written as a linear combination of product-forms. We review an important results from Wang(2002).

**Proposition 1** *(p.12 in Wang(2002)) The equation: $F_a^*(x)F_s^*(-x) = 1$ has $t$ solutions which we need. If $\rho < 1$, $t$ equals $m$ and the equation has $m$ solutions with positive real parts. If $\rho > 1$, $t$ equals $k$ and the equation has $k$ solutions with negative real parts.*

The proof is referred to Wang(2002).

According to Proposition 1, we assume the equation:

$$F_a^*(x)F_s^*(-x) = 1 \tag{2.11}$$

has $t$ solutions. For simplicity, we first assume all roots of (2.11) are simple. When the utilization factor, $\rho$, equals to one, we need to adjust equation (2.11) to avoid $x = 0$. For example, we can rewrite (2.11) as

$$F_a^*(x)F_s^*(-x) = 1.000001$$

Therefore, if $\rho < 1$, we need the roots of (2.11) with positive real parts. If $\rho > 1$, we need the roots of (2.11) with negative real part. If $\rho = 1$, we need all roots of (2.11) with small adjustment to avoid $x = 0$.

Let $x_\alpha$ be a solution of (2.11) which we need, $\alpha = 1, \cdots, t$ and set $w_\alpha = F_a^*(x_\alpha)$, for $w_\alpha \neq 0$. Given $x_\alpha$, we define $\mathbf{u}_\alpha$ and $\mathbf{v}_\alpha$ as follows,

$$\mathbf{u}_\alpha = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)^{-1}, \tag{2.12}$$

$$\mathbf{v}_\alpha = a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2(\mathbf{T}_2 - x_\alpha\mathbf{I}_2)^{-1}, \tag{2.13}$$

where $a_{\mathbf{u}_\alpha}, a_{\mathbf{v}_\alpha}$ are constants such that $\mathbf{u}_\alpha\mathbf{e}' = \mathbf{v}_\alpha\mathbf{e}' = 1$. Simply, set

$$a_{\mathbf{u}_\alpha} = \frac{x_\alpha}{w_\alpha - 1}, a_{\mathbf{v}_\alpha} = \frac{x_\alpha w_\alpha}{w_\alpha - 1}, \quad \text{for} \quad w_\alpha \neq 1.$$

Since there are $t$ solution, we define $\alpha = 1, 2, \cdots, t$,

$$\mathbf{w}_{\alpha,n} = w_\alpha^{n-1}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha), \quad 1 \leq n \leq N-1 \tag{2.14}$$

Therefore, by Wang(2002), we define the unboundary state probabilities are of the form,

$$\mathbf{P}_n = \sum_{\alpha=1}^{t} b_\alpha \mathbf{w}_{\alpha,n}, \quad b_\alpha \in \mathbb{C}, \quad 1 \leq n \leq N-1 \tag{2.15}$$

where $b_\alpha$ is the coefficients respect to $\mathbf{w}_{\alpha,n}$.

**Proposition 2** *(Luh(1999)) Given* $x_\alpha,\ \mathbf{w}_{\alpha,n},\ 2 \leq n \leq N-2$ *satisfies equation* (2.9)

Proof:

(1) For any given $\alpha$, rewriting (2.12) and multiplying it by $(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)$, we have

$$\mathbf{u}_\alpha(\mathbf{T}_1 - x_\alpha\mathbf{I}_1) = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)^{-1}(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)$$
$$\Longrightarrow \mathbf{u}_\alpha\mathbf{T}_1 = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1 + x_\alpha\mathbf{u}_\alpha.$$

Similarly, rewriting (2.13), we have

$$\mathbf{v}_\alpha\mathbf{T}_2 = a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2 + x_\alpha\mathbf{v}_\alpha.$$

Therefore, it is derived

$$\mathbf{u}_\alpha\mathbf{T}_1 \otimes \mathbf{v}_\alpha + \mathbf{u}_\alpha \otimes \mathbf{v}_\alpha\mathbf{T}_2 = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1 \otimes \mathbf{v}_\alpha + \mathbf{u}_\alpha \otimes a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2.$$

(2) For any given $\alpha$, rewriting (2.12) and multiplying it by $\mathbf{T}_1\mathbf{e}'(= -\boldsymbol{\gamma}_1)$, we have

$$\begin{aligned}\mathbf{u}_\alpha\mathbf{T}_1\mathbf{e}' &= a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)^{-1}\mathbf{T}_1\mathbf{e}' \\ &= a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)^{-1}(-\boldsymbol{\gamma}_1) \\ &= a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(x_\alpha\mathbf{I}_1 - \mathbf{T}_1)^{-1}\boldsymbol{\gamma}_1 \\ &= a_{\mathbf{u}_\alpha}F_a^*(x_\alpha) \\ &= a_{\mathbf{u}_\alpha}w_\alpha.\end{aligned}$$

Similarly, rewriting (2.13), we have

$$\mathbf{v}_\alpha\mathbf{T}_2\mathbf{e}' = a_{\mathbf{v}_\alpha}F_s^*(-x) = \frac{a_{\mathbf{v}_\alpha}}{w_\alpha}.$$

Therefore, we can derive

$$\begin{aligned}&\tfrac{1}{w_\alpha}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1\mathbf{e}'\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + w_\alpha(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{I}_1 \otimes \mathbf{T}_2\mathbf{e}'\boldsymbol{\tau}_2) \\ = \quad &\tfrac{1}{w_\alpha}(a_{\mathbf{u}_\alpha}w_\alpha\boldsymbol{\tau}_1 \otimes \mathbf{v}_\alpha) + w_\alpha(\mathbf{u}_\alpha \otimes (\tfrac{a_{\mathbf{v}_\alpha}}{w_\alpha})\boldsymbol{\tau}_2).\end{aligned}$$

(3) Inserting (2.14) into (2.9) divided by $w_\alpha^{n-1}$, it becomes

$$\begin{aligned}&\tfrac{w_\alpha^{n-2}}{w_\alpha^{n-1}}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\boldsymbol{\gamma}_1\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + \tfrac{w_\alpha^{n-1}}{w_\alpha^{n-1}}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1 \oplus \mathbf{T}_2) + \tfrac{w_\alpha^{n}}{w_\alpha^{n-1}}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{I}_1 \otimes \boldsymbol{\gamma}_2\boldsymbol{\tau}_2) \\ = \quad &(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1 \oplus \mathbf{T}_2) - \left\{\tfrac{1}{w_\alpha}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1\mathbf{e}'\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + w_\alpha(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{I}_1 \otimes \mathbf{T}_2\mathbf{e}'\boldsymbol{\tau}_2)\right\} \\ = \quad &\mathbf{u}_\alpha\mathbf{T}_1 \otimes \mathbf{v}_\alpha + \mathbf{u}_\alpha \otimes \mathbf{v}_\alpha\mathbf{T}_2 - \left\{\tfrac{1}{w_\alpha}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{T}_1\mathbf{e}'\boldsymbol{\tau}_1 \otimes \mathbf{I}_2) + w_\alpha(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)(\mathbf{I}_1 \otimes \mathbf{T}_2\mathbf{e}'\boldsymbol{\tau}_2)\right\} \\ = \quad &a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1 \otimes \mathbf{v}_\alpha + \mathbf{u}_\alpha \otimes a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2 - \left\{\tfrac{1}{w_\alpha}(a_{\mathbf{u}_\alpha}w_\alpha\boldsymbol{\tau}_1 \otimes \mathbf{v}_\alpha) + w_\alpha(\mathbf{u}_\alpha \otimes (\tfrac{a_{\mathbf{v}_\alpha}}{w_\alpha})\boldsymbol{\tau}_2)\right\}\end{aligned}$$

Hence it balances the equation (2.9). □

We can rewrite the balance equations by substitute (2.15) into equations (2.2) $\sim$ (2.6). According to Proposition 2, any linear combination of $\mathbf{w}_{\alpha,n},\ 1 \leq n \leq N$ satisfies the balance equations (2.4). When we substitute (2.15) into balance equations, we can ignore the equations (2.4) excepted $n = N-1$.

$$\begin{cases}\mathbf{P}_0\mathbf{B}_0 + (\sum_{\alpha=1}^t b_\alpha\mathbf{w}_{\alpha,1})\mathbf{C}_0 = \mathbf{0} & (2.16) \\ \mathbf{P}_0\mathbf{A}_0 + (\sum_{\alpha=1}^t b_\alpha\mathbf{w}_{\alpha,1})\mathbf{B} + (\sum_{\alpha=1}^t b_\alpha\mathbf{w}_{\alpha,2})\mathbf{C} = \mathbf{0} & (2.17) \\ (\sum_{\alpha=1}^t b_\alpha\mathbf{w}_{\alpha,N-2})\mathbf{A} + (\sum_{\alpha=1}^t b_\alpha\mathbf{w}_{\alpha,N-1})\mathbf{B} + \mathbf{P}_N\mathbf{C} = \mathbf{0} & (2.18) \\ (\sum_{\alpha=1}^t b_\alpha\mathbf{w}_{\alpha,N-1})\mathbf{A} + \mathbf{P}_N\mathbf{B}_1 = \mathbf{0} & (2.19) \\ \mathbf{P}_0\mathbf{e}' + \sum_{\alpha=1}^t b_\alpha\mathbf{w}_{\alpha,1}\mathbf{e}' + \sum_{\alpha=1}^t b_\alpha\mathbf{w}_{\alpha,2}\mathbf{e}' + \cdots + \sum_{\alpha=1}^t b_\alpha\mathbf{w}_{\alpha,N-1}\mathbf{e}' + \mathbf{P}_N\mathbf{e}' = 1 & (2.20)\end{cases}$$

After rewriting above equations:

$$\begin{cases} \mathbf{P}_0\mathbf{B}_0 + b_1(\mathbf{w}_{1,1}\mathbf{C}_0) + b_2(\mathbf{w}_{2,1}\mathbf{C}_0) + \cdots + b_t(\mathbf{w}_{t,1}\mathbf{C}_0) = \mathbf{0} & (2.21) \\ \mathbf{P}_0\mathbf{A}_0 + b_1(\mathbf{w}_{1,1}\mathbf{B} + \mathbf{w}_{1,2}\mathbf{C}) + b_2(\mathbf{w}_{2,1}\mathbf{B} + \mathbf{w}_{2,2}\mathbf{C}) + \cdots + b_t(\mathbf{w}_{t,1}\mathbf{B} + \mathbf{w}_{t,2}\mathbf{C}) = \mathbf{0} & (2.22) \\ b_1(\mathbf{w}_{1,N-2}\mathbf{A} + \mathbf{w}_{1,N-1}\mathbf{B}) + \cdots + b_t(\mathbf{w}_{t,N-2}\mathbf{A} + \mathbf{w}_{t,N-1}\mathbf{B}) + \mathbf{P}_N\mathbf{C} = \mathbf{0} & (2.23) \\ b_1(\mathbf{w}_{1,N-1}\mathbf{A}) + b_2(\mathbf{w}_{2,N-1}\mathbf{A}) + \cdots + b_t(\mathbf{w}_{t,N-1}\mathbf{A}) + \mathbf{P}_N\mathbf{B}_1 = \mathbf{0} & (2.24) \\ \mathbf{P}_0\mathbf{e}' + b_1(\sum_{j=1}^{N-1}\mathbf{w}_{1,j}\mathbf{e}') + b_2(\sum_{j=1}^{N-1}\mathbf{w}_{2,j}\mathbf{e}') + \cdots + b_t(\sum_{j=1}^{N-1}\mathbf{w}_{t,j}\mathbf{e}') + \mathbf{P}_N\mathbf{e}' = 1 & (2.25) \end{cases}$$

Since the system is stable, at least one of the coefficient $b_\alpha$ must be nonull. Hence, for an appropriate choice of $b_\alpha$, we can solve boundary probabilities.

2.4.2 A Simple Case of Multiple Roots

In this section, we discuss the situation when multiple roots occur in (2.11). If the equation (2.11) has multiple roots, the expression of the unboundary state probabilities will be very complicated.

Let $x_1, x_2, \cdots, x_s$ be the $s$ distinct roots of (2.11) with multiplicity $r_1, r_2, \cdots, r_s$. According to Proposition 1, we know (2.11) has $t$ solutions. We assume $r_1 = 2$ and $r_2 = r_3 = \cdots = r_s = 1$, then we can get $s + 1 = t$. Since $x_1$ is a multiple root of (2.11), we can not define the unboundary state probabilities as (2.15). In Liu(2004), Liu provides the formula of vector product solution of unboundary stationary probabilities for it.

First, we set $w_\alpha = F_a^*(x_\alpha)$, for $\alpha = 1, 2, \cdots, s$, and define $\mathbf{u}_\alpha$ and $\mathbf{v}_\alpha$, $\alpha = 2, \cdots, s$, as (2.12), (2.13). Second, we define $\mathbf{u}_1^{(1)}, \mathbf{v}_1^{(1)}, \mathbf{u}_1^{(0)}, \mathbf{v}_1^{(0)}, \mathbf{u}_1^{(2)}, \mathbf{v}_1^{(2)}, \varphi_{11}, \varphi_{10}, \varphi_{12}$ as follows,

$$\begin{aligned} \mathbf{u}_1^{(1)} &= a_{\mathbf{u}_1}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-1}, \\ \mathbf{v}_1^{(1)} &= a_{\mathbf{v}_1}\boldsymbol{\tau}_2(\mathbf{T}_2 - x_1\mathbf{I}_2)^{-1}, \\ \mathbf{u}_1^{(0)} &= \frac{1}{w_1}\mathbf{u}_1^{(1)}(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-1}, \\ \mathbf{v}_1^{(0)} &= \frac{1}{w_1}\mathbf{v}_1^{(1)}(\mathbf{T}_2 + x_1\mathbf{I}_2)^{-1}, \\ \mathbf{u}_1^{(2)} &= \frac{b_{\mathbf{u}_1}}{w_1}\mathbf{u}_1^{(1)}(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-1}, \\ \mathbf{v}_1^{(2)} &= \frac{b_{\mathbf{v}_1}}{w_1}\mathbf{v}_1^{(1)}(\mathbf{T}_2 + x_1\mathbf{I}_2)^{-1}, \\ \varphi_{11} &= \mathbf{u}_1^{(1)} \otimes \mathbf{v}_1^{(1)}, \\ \varphi_{10} &= \mathbf{u}_1^{(0)} \otimes \mathbf{v}_1^{(1)} - \mathbf{u}_1^{(1)} \otimes \mathbf{v}_1^{(0)}, \\ \varphi_{12} &= \mathbf{u}_1^{(2)} \otimes \mathbf{v}_1^{(1)} - \mathbf{u}_1^{(1)} \otimes \mathbf{v}_1^{(2)}, \end{aligned}$$

where

$$\begin{aligned} b_{\mathbf{u}_1} &= \frac{-a_{\mathbf{u}_1}w_1}{\mathbf{u}_1^{(1)}(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-1}\boldsymbol{\gamma}_1} \\ b_{\mathbf{v}_1} &= \frac{-a_{\mathbf{v}_1}}{w_1\mathbf{v}_1^{(1)}(\mathbf{T}_2 + x_1\mathbf{I}_2)^{-1}\boldsymbol{\gamma}_2} \end{aligned}$$

Third, since there are $s$ solutions, we define $\mathbf{w}_{\alpha,n}$, $\alpha = 2, \cdots, s$, as (2.14). By Liu(2004), we can define the unboundary state probailities in the following.
If $\boldsymbol{\tau}_1(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_1 = 0$, then

$$\mathbf{P}_n = b_{11}w_1^{n-1}\varphi_{11} + b_{12}w_1^{n-1}\varphi_{10} + \sum_{\alpha=2}^{s} b_\alpha\mathbf{w}_{\alpha,n}, \quad 1 \le n \le N-1. \tag{2.26}$$

If $\boldsymbol{\tau}_1(\mathbf{T}_1 - x_1\mathbf{I}_1)^{-2}\boldsymbol{\gamma}_2 \neq 0$, then

$$\mathbf{P}_n = \psi_n(w_1) + \sum_{\alpha=2}^{s} b_\alpha\mathbf{w}_{\alpha,n}, \quad 1 \le n \le N-1, \tag{2.27}$$

where

$$\begin{aligned}\phi_1(w_1) &= b_{11}\varphi_{11} + b_{12}\varphi_{12}\\ \phi_2(w_1) &= b_{11}w_1\varphi_{11} + b_{12}(w_1\varphi_{12} + \varphi_{11})\\ &\vdots\\ \phi_n(w_1) &= b_{11}w_1^{n-1}\varphi_{11} + b_{12}(w_1^{n-1}\varphi_{12} + (n-1)w_1^{n-2}\varphi_{11}).\end{aligned}$$

From (2.16), (2.17), we show the complexity of the case of multiple roots. In our numerical experience, we never find the multiple roots of (2.11). Therefore, In this paper, we focus on the case of simple roots and the algorithm in section 3 only fits for the case of simple roots.

### 2.5 Boundary State Probabilities

In this section, we want to compare the product-form method with traditional method. Note that $\mathbf{P}_0$ and $\mathbf{P}_N$ both are row vectors. Let

$$\mathbf{P}_0 = (P_{0,1,0}, P_{0,2,0}, \cdots, P_{0,k,0}). \tag{2.28}$$

and

$$\mathbf{P}_N = (P_{N,1,1}, P_{N,1,2}, \cdots, P_{N,1,m}, P_{N,2,1}, \cdots, P_{N,i,j}, \cdots, P_{N,k,m}). \tag{2.29}$$

Therefore the total number of unknowns of product-form method are $k + t + km$. Observe the equations (2.21) $\sim$ (2.25). The total number of equations is $k + 3km + 1$. It is noted that the number of unknowns and equations are independent of system size $N$. However, in the traditional method, the total number of unknowns is $k + kmN$. Observe the equations (2.2) $\sim$ (2.6), the total number of equations are $k + kmN + 1$. Obviously, the problem can be greatly reduced to a problem of solving a linear nonhomogenous system independent of $N$. Hence, the computing efficiency of the product-form method is much better than that of a traditional method when $N >> 3$.

According to proposition 1, $t$ depends on the condition of $\rho$. Whatever $\rho > 1$ or $\rho < 1$, the number $t$ is less or equal $km$. Therefore, the number of equations is greater than unknowns. Instead of checking the independent vector in (2.21) $\sim$ (2.25) and solving by Gaussian Elimination, the solution of this problem may be obtained by using some numerical methods.

### 2.6 Performance Measures

We denote the expected number of people waiting in the queue by $L_q$. Note that if 0 or 1 customer is present in the system, then nobody is waiting in line, but if $j$ people are present($j \geq 1$), there will be $j-1$ people waiting in line. Thus, we have

$$L_q = \sum_{j=1}^{j=N}(j-1)\pi_j.$$

where

$$\pi_j = \mathbf{P}_j\mathbf{e}', \quad 0 \leq j \leq N.$$

Also of interest is $L_s$, the expected number of customers in system. We have

$$L_s = \sum_{j=1}^{j=N}(j)\pi_j.$$

Often we are interested in the amount of time that a customer spends in a queueing system. We define $W_s$ as the expected time a customer spends in the queueing system, including time in line plus time in service, and $W_q$ as the expected time a customer spends waiting in line. Both $W_s$ and $W_q$ are computed under the assumption that the steady state has been reached. By using a powerful result known as *Little's formula*, $W_s$ and $W_q$ may be easily computed from $L_s$ and $L_q$.

**Proposition 3** *(Little's formula) For $C_k/C_m/1/N$ queueing system, the following relations hold:* $L_q = \lambda(1-\pi_N)W_q$, $L_s = \lambda(1-\pi_N)W_s$

Therefore, we can easily computed $W_s$ and $W_q$.

## 3. A SUMMARY OF THE ALGORITHM

In this section, we will solve the stationary probabilities by product-form method. Note the number of the unknowns is independent of $N$. In Section 3.1, we will introduce the algorithm which suits the case of simple roots. In Section 3.2, we will present two illustrative examples.

### 3.1 The Algorithm

We describe the algorithm for solving stationary probabilities of a $C_k/C_m/1/N$ system in the following steps.

**Step 1** Solve equation (2.11), $F_a^*(x)F_s^*(-x) = 1$. Let $x_\alpha$ be a solution of (2.11), $\alpha = 1, \cdots, t$.

**Step 2** Compute $w_\alpha, \mathbf{u}_\alpha, \mathbf{v}_\alpha$.

1. Compute $w_\alpha$ defined in $w_\alpha = F_a^*(x_\alpha)$.
2. Compute $\mathbf{u}_\alpha$ defined in (2.12), $\mathbf{u}_\alpha = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)^{-1}$.
3. Compute $\mathbf{v}_\alpha$ defined in (2.13), $\mathbf{v}_\alpha = a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2(\mathbf{T}_2 + x_\alpha\mathbf{I}_2)^{-1}$.

**Step 3** Compute $\mathbf{w}_{\alpha,n}$ defined in (2.14), $\mathbf{w}_{\alpha,n} = w_\alpha^n(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)$, $1 \leq n \leq N-1$.

**Step 4** Let $\mathbf{P}_n$ be a linear combination of $\mathbf{w}_{\alpha,n}$ that is $\mathbf{P}_n = \sum_{\alpha=1}^t b_\alpha \mathbf{w}_{\alpha,n}$, $b_\alpha \in \mathbb{C}$.

**Step 5** Set a linear nonhomogeneous system consisting of equations (2.21) $\sim$ (2.25).

**Step 6** Use the Cholesky factorization to solve the linear nonhomogeneous system and obtain coefficients $b_\alpha, \alpha = 1, \cdots, t$ and boundary stationary probabilities $\mathbf{P}_0$ and $\mathbf{P}_N$.

**Step 7** Substituting coefficients $b_\alpha, \alpha = 1, \cdots, t$, to (2.15) and obtain unboundary stationary probabilities $\mathbf{P}_n$, $1 \leq n \leq N-1$.

**Step 8** Compute the system-size probability $\boldsymbol{\pi}_n, n = 1, \cdots, N$.

It is important to note that no matter how large the system size $N$ is, we only need to solve coefficients $b_\alpha, \alpha = 1, \cdots, t$. Hence the computational complexity is greatly reduced.

### 3.2 Examples of $C_2/C_2/1/7$ System

<u>3.2.1 The example of Case 1 $\rho < 1$</u>

The system has the following features:

$$N = 7, \quad \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = (1, 0),$$
$$\lambda_1 = \lambda_2 = 4, \quad p_1 = 0.5, \quad p_2 = 1,$$
$$\mu_1 = \mu_2 = 5, \quad q_1 = 0.5, \quad q_2 = 1.$$

**Step 1** Solve equation (2.11), $F_a^*(x)F_s^*(-x) = 1$. Let $x_\alpha$ be a solution with positive real parts of (2.11), $\alpha = 1, 2$. We have

$$F_a^*(x) = \frac{2}{(x+4)} + \frac{8}{(x+4)^2},$$
$$F_s^*(x) = \frac{5}{2(x+5)} + \frac{25}{2(x+5)^2},$$

and the solutions of $F_a^*(x)F_s^*(-x) = 1$ are

$$x_1 = 6.5131, \quad x_2 = 0.8576.$$

**Step 2** Compute $w_\alpha, \mathbf{u}_\alpha, \mathbf{v}_\alpha$.

1. Compute $w_\alpha$ defined in $w_\alpha = F_a^*(x_\alpha)$.

$$w_1 = F_a^*(x_1) = 0.2626,$$
$$w_2 = F_a^*(x_2) = 0.7508.$$

2. Compute $\mathbf{u}_\alpha$ defined in (2.12), $\mathbf{u}_\alpha = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha\mathbf{I}_1)^{-1}$.

$$\mathbf{u}_1 = (0.8402, 0.1598),$$

$$\mathbf{u}_2 = (0.7084, 0.2916).$$

3. Compute $\mathbf{v}_\alpha$ defined in (2.13), $\mathbf{v}_\alpha = a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2(\mathbf{T}_2 + x_\alpha\mathbf{I}_2)^{-1}$.

$$\mathbf{v}_1 = (-1.5331, 2.5331),$$

$$\mathbf{v}_2 = (0.6236, 0.3764).$$

**Step 3** Compute $\mathbf{w}_{\alpha,n}$ defined in (2.14), $\mathbf{w}_{\alpha,n} = w_\alpha^n(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)$, $1 \le n \le 6$.

$$\begin{aligned}
\mathbf{w}_{1,1} &= (-1.2881, 2.1282, -0.2450, 0.4049),\\
\mathbf{w}_{1,2} &= (-0.3383, 0.5589, -0.0644, 0.1063),\\
\mathbf{w}_{1,3} &= (-0.0888, 0.1468, -0.0169, 0.0279),\\
\mathbf{w}_{1,4} &= (-0.0233, 0.0385, -0.0044, 0.0073),\\
\mathbf{w}_{1,5} &= (-0.0061, 0.0101, -0.0012, 0.0019),\\
\mathbf{w}_{1,6} &= (-0.0016, 0.0027, -0.0003, 0.0005),\\
\mathbf{w}_{2,1} &= (0.4417, 0.2666, 0.1819, 0.1098),\\
\mathbf{w}_{2,2} &= (0.3316, 0.2002, 0.1365, 0.0824),\\
\mathbf{w}_{2,3} &= (0.2490, 0.1503, 0.1025, 0.0619),\\
\mathbf{w}_{2,4} &= (0.1869, 0.1128, 0.0770, 0.0464),\\
\mathbf{w}_{2,5} &= (0.1403, 0.0847, 0.0578, 0.0349),\\
\mathbf{w}_{2,6} &= (0.1054, 0.0636, 0.0434, 0.0262),
\end{aligned}$$

**Step 4** Let $\mathbf{P}_n$ be a linear combination of $\mathbf{w}_{\alpha,n}$ that is $\mathbf{P}_n = \sum_{\alpha=1}^{2} b_\alpha \mathbf{w}_{\alpha,n}$, $b_\alpha \in \mathbb{C}$.

**Step 5** Set a linear nonhomogeneous system consisting of equations (2.21) $\sim$ (2.25).

$$\mathbf{z}\cdot\begin{bmatrix}
-4 & 0 & 7.4210 & 2.4374 & 0 & 0 & 0 & 0\\
2 & -4 & 1.4118 & 1.0035 & 0 & 0 & 0 & 0\\
2 & 4 & 13.5414 & -2.1459 & 0 & 0 & 0 & 0\\
0 & 0 & -22.3741 & -1.2951 & 0 & 0 & 0 & 0\\
0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0\\
0 & 0 & 0.0000 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -0.0024 & -0.4364 & 2.5 & 5 & 0 & 0\\
0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0\\
0 & 0 & -0.0005 & -0.1797 & 0 & 0 & 2.5 & 5\\
0 & 0 & 0.0000 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -0.0044 & 0.3842 & -7.0000 & 0 & 0 & 0\\
0 & 0 & 0.0073 & 0.2319 & 2.5 & -7 & 0 & 0\\
0 & 0 & 0 & 0 & 2 & 0 & -5 & 0\\
0 & 0 & 0 & 0 & 0 & 2 & 2.5 & -5\\
1 & 1 & 1.3557 & 3.2937 & 1 & 1 & 1 & 1
\end{bmatrix}^t = \mathbf{b}.$$

where $\mathbf{z} = (P_{0,1,0}, \cdots, P_{0,k,0}, b_1, \cdots, b_t, P_{N,1,1}, \cdots, P_{N,i,j}, \cdots, P_{N,k,m})$, and $\mathbf{b} = (0, 0, \cdots, 0, 1)$.

**Step 6** Use the Cholesky factorization to solve the linear nonhomogeneous system and obtain coefficients $b_\alpha$, $\alpha = 1, 2$, and boundary stationary probabilities $\mathbf{P}_0$ and $\mathbf{P}_7$.

$$\mathbf{P}_0 = (0.1152, 0.1105),$$

$$\mathbf{P}_7 = (0.0129, 0.0126, 0.0045, 0.0066),$$

$$b_1 = -0.0133, \quad b_2 = 0.2294.$$

**Step 7** Substituting coefficients $b_\alpha,\ \alpha = 1,\ 2$, to (2.15) and obtain unboundary stationary probabilities $\mathbf{P}_n$, $1 \leq n \leq 6$.

$$\mathbf{P}_1 = (0.1185, 0.0329, 0.0450, 0.0198),$$
$$\mathbf{P}_2 = (0.0806, 0.0385, 0.0322, 0.0175),$$
$$\mathbf{P}_3 = (0.0583, 0.0325, 0.0237, 0.0138).$$
$$\mathbf{P}_4 = (0.0432, 0.0254, 0.0177, 0.0106).$$
$$\mathbf{P}_5 = (0.0323, 0.0193, 0.0133, 0.0080).$$
$$\mathbf{P}_6 = (0.0242, 0.0146, 0.0100, 0.0060).$$

**Step 8** Compute the system-size probability $\boldsymbol{\pi}_n,\ n = 0, \cdots, 7$.

$$\boldsymbol{\pi}_0 = 0.2256,$$
$$\boldsymbol{\pi}_1 = 0.2162,$$
$$\boldsymbol{\pi}_2 = 0.1688,$$
$$\boldsymbol{\pi}_3 = 0.1284,$$
$$\boldsymbol{\pi}_4 = 0.0968,$$
$$\boldsymbol{\pi}_5 = 0.0728,$$
$$\boldsymbol{\pi}_6 = 0.0547,$$
$$\boldsymbol{\pi}_7 = 0.0366,$$

We know that the idle probability of the system is $\boldsymbol{\pi}_0 = 0.2256$, and the blocking probability of the $C_2/C_2/1/7$ queueing system is $\boldsymbol{\pi}_7 = 0.0366$.

3.2.2 The example of Case 2 $\rho > 1$

The system has the following features:

$$N = 7, \quad \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2 = (1, 0),$$
$$\lambda_1 = \lambda_2 = 5, \quad p_1 = 0.5, \quad p_2 = 1,$$
$$\mu_1 = \mu_2 = 4, \quad q_1 = 0.5, \quad q_2 = 1.$$

**Step 1** Solve equation (2.11), $F_a^*(x)F_s^*(-x) = 1$. Let $x_\alpha$ be a solution with negative real parts of (2.11), $\alpha = 1,\ 2$. We have

$$F_a^*(x) = \frac{5}{2(x+5)} + \frac{25}{2(x+5)^2},$$
$$F_s^*(x) = \frac{2}{(x+4)} + \frac{8}{(x+4)^2},$$

and the solutions of $F_a^*(x)F_s^*(-x) = 1$ are

$$x_1 = -6.5131, \quad x_2 = -0.8576.$$

**Step 2** Compute $w_\alpha, \mathbf{u}_\alpha, \mathbf{v}_\alpha$.

1. Compute $w_\alpha$ defined in $w_\alpha = F_a^*(x_\alpha)$.

$$w_1 = F_a^*(x_1) = 3.8078,$$
$$w_2 = F_a^*(x_2) = 1.3320.$$

2. Compute $\mathbf{u}_\alpha$ defined in (2.12), $\mathbf{u}_\alpha = a_{\mathbf{u}_\alpha}\boldsymbol{\tau}_1(\mathbf{T}_1 - x_\alpha \mathbf{I}_1)^{-1}$.

$$\mathbf{u}_1 = (-1.5331, 2.5331),$$
$$\mathbf{u}_2 = (0.6236, 0.3764).$$

3. Compute $\mathbf{v}_\alpha$ defined in (2.13), $\mathbf{v}_\alpha = a_{\mathbf{v}_\alpha}\boldsymbol{\tau}_2(\mathbf{T}_2 + x_\alpha \mathbf{I}_2)^{-1}$.

$$\mathbf{v}_1 = (0.8402, 0.1598),$$

$$\mathbf{v}_2 = (0.7084, 0.2916).$$

**Step 3** Compute $\mathbf{w}_{\alpha,n}$ defined in (2.14), $\mathbf{w}_{\alpha,n} = w_\alpha^{n-1}(\mathbf{u}_\alpha \otimes \mathbf{v}_\alpha)$, $1 \le n \le 6$.

$$\mathbf{w}_{1,1} = (-1.2881, -0.2450, 2.1282, 0.4049),$$

$$\mathbf{w}_{1,2} = (-4.9046, -0.9330, 8.1038, 1.5417),$$

$$\mathbf{w}_{1,3} = (-18.6755, -3.5528, 30.8571, 5.8702),$$

$$\mathbf{w}_{1,4} = (-71.112, -13.528, 117.5, 22.352),$$

$$\mathbf{w}_{1,5} = (-270.78, -51.512, 447.4, 85.113),$$

$$\mathbf{w}_{1,6} = (-1031.1, -196.15, 1703.6, 324.09),$$

$$\mathbf{w}_{2,1} = (0.4417, 0.1819, 0.2666, 0.1098),$$

$$\mathbf{w}_{2,2} = (0.5884, 0.2423, 0.3551, 0.1462),$$

$$\mathbf{w}_{2,3} = (0.7838, 0.3227, 0.4730, 0.1948),$$

$$\mathbf{w}_{2,4} = (1.044, 0.42982, 0.63005, 0.25941),$$

$$\mathbf{w}_{2,5} = (1.3905, 0.57252, 0.83923, 0.34553),$$

$$\mathbf{w}_{2,6} = (1.8522, 0.76259, 1.1178, 0.46024),$$

**Step 4** Let $\mathbf{P}_n$ be a linear combination of $\mathbf{w}_{\alpha,n}$ that is $\mathbf{P}_n = \sum_{\alpha=1}^{2} b_\alpha \mathbf{w}_{\alpha,n}$, $b_\alpha \in \mathbb{C}$.

**Step 5** Set a linear nonhomogeneous system consisting of equations (2.21) $\sim$ (2.25).

$$\mathbf{z} \cdot \begin{bmatrix} -5 & 0 & -3.5563 & 1.611 & 0 & 0 & 0 & 0 \\ 2.5 & -5 & 5.8759 & 0.97228 & 0 & 0 & 0 & 0 \\ 2.5 & 5 & -1.9489 & -1.8299 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.37076 & -0.75341 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10840 & -8.9973 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0.0000 & 0.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & -17910 & -5.4301 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5940.3 & 10.22 & -6.5 & 0 & 0 & 0 \\ 0 & 0 & 1130.1 & 4.2077 & 2 & -6.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.5 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.5 & 2 & -4 \\ 1 & 1 & 1085.2 & 13.81 & 1 & 1 & 1 & 1 \end{bmatrix}^t = \mathbf{b},$$

where $\mathbf{z} = (P_{0,1,0}, \cdots, P_{0,k,0}, b_1, \cdots, b_t, P_{N,1,1}, \cdots, P_{N,i,j}, \cdots, P_{N,k,m})$, and $\mathbf{b} = (0, 0, \cdots, 0, 1)$.

**Step 6** Use the Cholesky factorization to solve the linear nonhomogeneous system and obtain coefficients $b_\alpha$, $\alpha = 1, 2$, and boundary stationary probabilities $\mathbf{P}_0$ and $\mathbf{P}_7$.

$$\mathbf{P}_0 = (0.0139, 0.0143),$$

$$\mathbf{P}_7 = (0.0845, 0.0598, 0.0529, 0.0639),$$

$$b_1 = 0.000005, \quad b_2 = 0.050952.$$

**Step 7** Substituting coefficients $b_\alpha,\ \alpha = 1,\ 2$, to (2.15) and obtain unboundary stationary probabilities $\mathbf{P}_n$, $1 \le n \le 6$.

$$\mathbf{P}_1 = (0.0225, 0.0093, 0.0136, 0.0056),$$
$$\mathbf{P}_2 = (0.0300, 0.0123, 0.0181, 0.0075),$$
$$\mathbf{P}_3 = (0.0398, 0.0164, 0.0242, 0.0100).$$
$$\mathbf{P}_4 = (0.0529, 0.0218, 0.0327, 0.0133).$$
$$\mathbf{P}_5 = (0.0696, 0.0289, 0.0449, 0.0180).$$
$$\mathbf{P}_6 = (0.0895, 0.0379, 0.0650, 0.0250).$$

**Step 8** Compute the system-size probability $\boldsymbol{\pi}_n,\ n = 0, \ldots, 7$.

$$\boldsymbol{\pi}_0 = 0.0282,$$
$$\boldsymbol{\pi}_1 = 0.0510,$$
$$\boldsymbol{\pi}_2 = 0.0679,$$
$$\boldsymbol{\pi}_3 = 0.0905,$$
$$\boldsymbol{\pi}_4 = 0.1207,$$
$$\boldsymbol{\pi}_5 = 0.1614,$$
$$\boldsymbol{\pi}_6 = 0.2174,$$
$$\boldsymbol{\pi}_7 = 0.2610,$$

We know that the idle probability of the system is $\boldsymbol{\pi}_0 = 0.0282$, and the blocking probability of the $C_2/C_2/1/7$ queueing system is $\boldsymbol{\pi}_7 = 0.2610$.

## 4. THE NUMERICAL RESULT

### 4.1 Case 1 $C_k/C_m/1/4$ System

The following tables are constructed by the product-form method, except the case of $\rho = 1$. The case of $\rho = 1$ is constructed by the product-form method with small adjustment in (2.11). $L_q(pd)$ means the expected queue length which is calculated from the equations (2.21) $\sim$ (2.25). $L_q(td)$ means the expected queue length which is calculated from the equations (2.2) $\sim$ (2.6). The difference comes from approximation used by vector product-form approach. The mean service time is $0.1$, i.e. $\mu = 10$.

In the Table 1, we list the stationary probabilities of $M/E_2/1/4$ queueing system, and $\mu_1 = \mu_2 = 20$.

Table 1: $M/E_2/1/4$

| $\rho$ | $\lambda_1$ | $\boldsymbol{\pi}_0$ | $\boldsymbol{\pi}_1$ | $\boldsymbol{\pi}_2$ | $\boldsymbol{\pi}_3$ | $\boldsymbol{\pi}_4$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 2 | 0.8001 | 0.168 | 0.0273 | 0.0040 | 0.0005 | 0.0370 | 0.0370 |
| 0.5 | 5 | 0.5097 | 0.2867 | 0.1294 | 0.0549 | 0.0193 | 0.2971 | 0.02972 |
| 0.8 | 8 | 0.2748 | 0.2638 | 0.2093 | 0.1587 | 0.0935 | 0.8071 | 0.8071 |
| 1 | 10 | 0.1698 | 0.2122 | 0.2228 | 0.2255 | 0.1698 | 1.1832 | 1.1832 |
| 2 | 20 | 0.0097 | 0.0372 | 0.1143 | 0.3142 | 0.5141 | 2.2851 | 2.2687 |
| 3 | 30 | 0.0017 | 0.0107 | 0.0546 | 0.2639 | 0.6686 | 2.5883 | 2.5850 |

In the Table 2, we list the stationary probabilities of $M/E_3/1/4$ queuing system, and $\mu_1 = \mu_2 = \mu_3 = 30$.

Table 2: $M/E_3/1/4$

| $\rho$ | $\lambda_1$ | $\boldsymbol{\pi}_0$ | $\boldsymbol{\pi}_1$ | $\boldsymbol{\pi}_2$ | $\boldsymbol{\pi}_3$ | $\boldsymbol{\pi}_4$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 2 | 0.8001 | 0.1709 | 0.0274 | 0.0035 | 0.0004 | 0.0354 | 0.0330 |
| 0.5 | 5 | 0.5076 | 0.2985 | 0.1317 | 0.0510 | 0.0152 | 0.2792 | 0.2746 |
| 0.8 | 8 | 0.2661 | 0.2747 | 0.2192 | 0.1605 | 0.0827 | 0.7883 | 0.7842 |
| 1 | 10 | 0.1579 | 0.2163 | 0.2339 | 0.2365 | 0.1579 | 1.1774 | 1.1774 |
| 2 | 20 | 0.0096 | 0.0354 | 0.1115 | 0.3383 | 0.5055 | 2.3047 | 2.3019 |
| 3 | 30 | 0.0010 | 0.0071 | 0.0449 | 0.2799 | 0.6671 | 2.6062 | 2.6057 |

In the Table 3, we list the stationary probabilities of $E_2/E_2/1/4$ queuing system, and $\mu_1 = \mu_2 = 20$.

Table 3: $E_2/E_2/1/4$

| $\rho$ | $\lambda_1 = \lambda_2$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 4 | 0.8 | 0.1885 | 0.0110 | 0.0005 | 0 | 0.0121 | 0.0121 |
| 0.5 | 10 | 0.5017 | 0.3565 | 0.1073 | 0.0283 | 0.0062 | 0.1824 | 0.1843 |
| 0.8 | 16 | 0.2347 | 0.3123 | 0.2219 | 0.1450 | 0.0702 | 0.7225 | 0.7662 |
| 1 | 20 | 0.1165 | 0.2132 | 0.2309 | 0.2406 | 0.1988 | 1.3085 | 1.3085 |
| 2 | 40 | 0.0024 | 0.0146 | 0.0601 | 0.2607 | 0.6608 | 2.5639 | 2.5678 |
| 3 | 60 | 0.0002 | 0.0019 | 0.0177 | 0.1785 | 0.8018 | 2.7800 | 2.7800 |

In the Table 4, we list the stationary probabilities of $E_2/E_3/1/4$ queuing system, $\mu_1 = \mu_2 = \mu_3 = 30$.

Table 4: $E_2/E_3/1/4$

| $\rho$ | $\lambda_1 = \lambda_2$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 4 | 0.8000 | 0.1908 | 0.0115 | 0.0003 | 0.0000 | 0.0122 | 0.0095 |
| 0.5 | 10 | 0.5009 | 0.3750 | 0.1055 | 0.0216 | 0.0035 | 0.1595 | 0.1532 |
| 0.8 | 16 | 0.2270 | 0.3354 | 0.2363 | 0.1408 | 0.0577 | 0.6911 | 0.717 |
| 1 | 20 | 0.1021 | 0.2157 | 0.2446 | 0.2541 | 0.1865 | 1.3087 | 1.3087 |
| 2 | 40 | 0.0005 | 0.0067 | 0.0436 | 0.2699 | 0.6792 | 2.6210 | 2.6157 |
| 3 | 60 | 0.0000 | 0.0006 | 0.0100 | 0.1779 | 0.8115 | 2.8003 | 2.8001 |

In the Table 5, we list the stationary probability of $E_2/C_2/1/4$ queuing system. After phase 1, the service time comes to an end with probability $0.2$, and it enters the next phase with probability $0.8$, i.e. $q_1 = 0.2$. And $\mu_1 = \mu_2 = 18$.

Table 5: $E_2/C_2/1/4$

| $\rho$ | $\lambda_1 = \lambda_2$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 4 | 0.8 | 0.1870 | 0.0123 | 0.0007 | 0 | 0.0137 | 0.0131 |
| 0.5 | 10 | 0.5023 | 0.3455 | 0.1118 | 0.0324 | 0.0079 | 0.2004 | 0.2029 |
| 0.8 | 16 | 0.2393 | 0.2984 | 0.2168 | 0.1472 | 0.0775 | 0.7436 | 0.7945 |
| 1 | 20 | 0.1252 | 0.2098 | 0.2247 | 0.2337 | 0.2067 | 1.3122 | 1.3122 |
| 2 | 40 | 0.0041 | 0.0203 | 0.0699 | 0.2519 | 0.6512 | 2.5273 | 2.5340 |
| 3 | 60 | 0.0004 | 0.0036 | 0.0247 | 0.1766 | 0.7947 | 2.7619 | 2.7619 |

In the Table 6, we list the stationary probabilities of $E_2/C_3/1/4$ queuing system. After phase 1, the service time come to an end with probability $0.2$, and enter the next phase with probability $0.8$. After phase 2, the service time comes to an end with probability $0.5$, and it enters the next phase with probability $0.5$, i.e. $p_1 = q_1 = 0.2$ and $q_2 = 0.5$. And $\mu_1 = \mu_2 = \mu_3 = 22$.

Table 6: $E_2/C_3/1/4$

| $\rho$ | $\lambda 1 = \lambda_2$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 4 | 0.8000 | 0.1876 | 0.0140 | 0.0007 | 0.0000 | 0.0155 | 0.0131 |
| 0.5 | 10 | 0.5020 | 0.3485 | 0.1156 | 0.0316 | 0.0072 | 0.2003 | 0.1968 |
| 0.8 | 16 | 0.2382 | 0.3011 | 0.2229 | 0.1487 | 0.0758 | 0.7477 | 0.7897 |
| 1 | 20 | 0.1228 | 0.2088 | 0.2283 | 0.2371 | 0.2053 | 1.3155 | 1.3155 |
| 2 | 40 | 0.0037 | 0.0196 | 0.0689 | 0.2523 | 0.6527 | 2.5316 | 2.5383 |
| 3 | 60 | 0.0004 | 0.0036 | 0.0246 | 0.1762 | 0.7951 | 2.7624 | 2.7625 |

In the Table 7, we list the stationary probabilities of $C_2/C_2/1/4$ queuing system. After phase 1, the interarrival time and service time come to an end with probabilities $0.2$, and enter the next phase with probabilities $0.8$, i.e. $p_1 = q_1 = 0.2$. And $\mu_1 = \mu_2 = 18$.

Table 7: $C_2/C_2/1/4$

| $\rho$ | $\lambda_1 = \lambda_2$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 3.6 | 0.8000 | 0.1805 | 0.0178 | 0.0016 | 0.0001 | 0.0213 | 0.0213 |
| 0.5 | 9 | 0.5038 | 0.3262 | 0.1187 | 0.0398 | 0.0114 | 0.2326 | 0.2345 |
| 0.8 | 14.4 | 0.2489 | 0.2878 | 0.2146 | 0.1513 | 0.0846 | 0.7711 | 0.8037 |
| 1 | 18 | 0.1369 | 0.2110 | 0.2240 | 0.2309 | 0.1972 | 1.2774 | 1.2774 |
| 2 | 36 | 0.0058 | 0.0254 | 0.0808 | 0.2647 | 0.6188 | 2.4666 | 2.4778 |
| 3 | 54 | 0.0006 | 0.0049 | 0.0302 | 0.1919 | 0.7723 | 2.7308 | 2.7310 |

In the Table 8, we list the stationary probabilities of $C_2/C_3/1/4$ queuing system. After phase 1, the interarrival time and service time come to an end with probabilities 0.2, and enter the next phase with probabilities 0.8. After phase 2,the service time comes to an end with probability 0.5, and it enters the next phase with probability 0.5, i.e. $p_1 = q_1 = 0.2$ and $q_2 = 0.5$. And $\mu_1 = \mu_2 = \mu_3 = 22$.

Table 8: $C_2/C_3/1/4$

| $\rho$ | $\lambda_1 = \lambda_2$ | $\boldsymbol{\pi}_0$ | $\boldsymbol{\pi}_1$ | $\boldsymbol{\pi}_2$ | $\boldsymbol{\pi}_3$ | $\boldsymbol{\pi}_4$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 3.6 | 0.8000 | 0.1811 | 0.0192 | 0.0016 | 0.0001 | 0.0227 | 0.0206 |
| 0.5 | 9 | 0.5035 | 0.3287 | 0.1218 | 0.0392 | 0.0106 | 0.2320 | 0.2291 |
| 0.8 | 14.4 | 0.2477 | 0.2900 | 0.2196 | 0.1529 | 0.0830 | 0.7744 | 0.7995 |
| 1 | 18 | 0.1346 | 0.2104 | 0.2273 | 0.2342 | 0.1955 | 1.2795 | 1.2795 |
| 2 | 36 | 0.0051 | 0.0247 | 0.0797 | 0.2655 | 0.6201 | 2.4711 | 2.4824 |
| 3 | 54 | 0.0006 | 0.0049 | 0.0300 | 0.1916 | 0.7729 | 2.7318 | 2.7319 |

In the Table 9, we list the stationary probabilities of $C_4/C_3/1/4$ queuing system. After phase $i$, $i = 1, 2, 3$, the interarrival time come to an end with probability 0.2, and enter the next phase with probability 0.8. After phase $j$, $j = 1, 2$, the service time comes to an end with probability 0.5, and it enters the next phase with probability 0.5, i.e. $p_i = 0.2$, $i = 1, 2, 3$ and $q_j = 0.5$, $j = 1, 2$. And $\mu_1 = \mu_2 = \mu_3 = 17.5$.

Table 9: $C_4/C_3/1/4$

| $\rho$ | $\lambda_i, i = 1, \cdots, 4$ | $\boldsymbol{\pi}_0$ | $\boldsymbol{\pi}_1$ | $\boldsymbol{\pi}_2$ | $\boldsymbol{\pi}_3$ | $\boldsymbol{\pi}_4$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|
| 0.2 | 5.905 | 0.8000 | 0.1769 | 0.0225 | 0.0025 | 0.0002 | 0.0281 | 0.0258 |
| 0.5 | 14.76 | 0.5046 | 0.3175 | 0.1242 | 0.0443 | 0.0140 | 0.2548 | 0.2509 |
| 0.8 | 23.616 | 0.2542 | 0.2810 | 0.2133 | 0.1533 | 0.0943 | 0.8028 | 0.8316 |
| 1 | 29.52 | 0.1420 | 0.2059 | 0.2146 | 0.2207 | 0.2169 | 1.3066 | 1.3066 |
| 2 | 59.04 | 0.0068 | 0.0278 | 0.0765 | 0.2060 | 0.6574 | 2.4606 | 2.5095 |
| 3 | 88.56 | 0.0010 | 0.0061 | 0.0288 | 0.1315 | 0.8083 | 2.7168 | 2.7642 |

### 4.2 Case 2 $C_k/C_m/1/6$ System

The following tables are constructed by the product-form method, but the case of $\rho = 1$. The case of $\rho = 1$ is constructed by the product-form method with small adjustment in (2.11). $L_q(pd)$ means the expected queue length which is calculated from the equations (2.21) $\sim$ (2.25). $L_q(td)$ means the expected queue length which is calculated from the equations (2.2) $\sim$ (2.6). The difference comes from approximation used by vector product-form approach. The mean of service time distribution is 0.1, i.e. $\mu = 10$.

In the Table 10, we list the stationary probabilities of $M/E_2/1/6$ queuing system, and $\mu_1 = \mu_2 = 20$.

Table 10: $M/E_2/1/6$

| $\rho$ | $\lambda_1$ | $\boldsymbol{\pi}_0$ | $\boldsymbol{\pi}_1$ | $\boldsymbol{\pi}_2$ | $\boldsymbol{\pi}_3$ | $\boldsymbol{\pi}_4$ | $\boldsymbol{\pi}_5$ | $\boldsymbol{\pi}_6$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 2 | 0.8000 | 0.1680 | 0.0273 | 0.0040 | 0.0006 | 0.0001 | 0.0000 | 0.0375 | 0.0375 |
| 0.5 | 5 | 0.5016 | 0.2822 | 0.1274 | 0.0540 | 0.0224 | 0.0092 | 0.0032 | 0.3557 | 0.3557 |
| 0.8 | 8 | 0.2356 | 0.2262 | 0.1795 | 0.1361 | 0.1019 | 0.0761 | 0.0446 | 1.2846 | 1.2846 |
| 1 | 10 | 0.1169 | 0.1461 | 0.1534 | 0.1552 | 0.1557 | 0.1558 | 0.1169 | 2.1386 | 2.1386 |
| 2 | 20 | 0.0015 | 0.0054 | 0.0164 | 0.0447 | 0.1183 | 0.3107 | 0.5030 | 4.2185 | 4.2065 |
| 3 | 30 | 0.0001 | 0.0005 | 0.0024 | 0.0115 | 0.0551 | 0.2636 | 0.6668 | 4.5793 | 4.5788 |

In the Table 11, we list the stationary probabilities of $M/E_3/1/6$ queuing system, and $\mu_1 = \mu_2 = \mu_3 = 30$.

Table 11: $M/E_3/1/6$

| $\rho$ | $\lambda_1$ | $\boldsymbol{\pi}_0$ | $\boldsymbol{\pi}_1$ | $\boldsymbol{\pi}_2$ | $\boldsymbol{\pi}_3$ | $\boldsymbol{\pi}_4$ | $\boldsymbol{\pi}_5$ | $\boldsymbol{\pi}_6$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 2 | 0.8000 | 0.1709 | 0.0274 | 0.0035 | 0.0004 | 0.0000 | 0.0000 | 0.0358 | 0.0333 |
| 0.5 | 5 | 0.5010 | 0.2946 | 0.1300 | 0.0503 | 0.0189 | 0.0071 | 0.0021 | 0.3260 | 0.3210 |
| 0.8 | 8 | 0.2296 | 0.2370 | 0.1891 | 0.1385 | 0.0999 | 0.0719 | 0.0370 | 1.2385 | 1.2344 |
| 1 | 10 | 0.1071 | 0.1468 | 0.1587 | 0.1605 | 0.1607 | 0.1607 | 0.1071 | 2.1379 | 2.1379 |
| 2 | 20 | 0.0010 | 0.0038 | 0.0121 | 0.0366 | 0.1108 | 0.3351 | 0.5006 | 4.2611 | 4.2606 |
| 3 | 30 | 0.0000 | 0.0002 | 0.0012 | 0.0072 | 0.0450 | 0.2798 | 0.6667 | 4.6030 | 4.6029 |

In the Table 12, we list the stationary probabilities of $E_2/E_2/1/6$ queuing system, and $\mu_1 = \mu_2 = 20$, $\lambda_1 = \lambda_2$.

Table 12: $E_2/E_2/1/6$

| $\rho$ | $\lambda_i$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 4 | 0.8000 | 0.1885 | 0.0110 | 0.0005 | 0.0000 | 0.0000 | 0.0000 | 0.0121 | 0.0121 |
| 0.5 | 10 | 0.5001 | 0.3554 | 0.1070 | 0.0282 | 0.0072 | 0.0018 | 0.0004 | 0.1940 | 0.1942 |
| 0.8 | 16 | 0.2137 | 0.2844 | 0.2021 | 0.1321 | 0.0849 | 0.0544 | 0.0262 | 1.0695 | 1.0939 |
| 1 | 20 | 0.0795 | 0.1453 | 0.1566 | 0.1587 | 0.1598 | 0.1645 | 0.1356 | 2.2897 | 2.2897 |
| 2 | 40 | 0.0002 | 0.0009 | 0.0036 | 0.0146 | 0.0600 | 0.2605 | 0.6602 | 4.5559 | 4.5559 |
| 3 | 60 | 0.0000 | 0.0000 | 0.0002 | 0.0019 | 0.0177 | 0.1785 | 0.8017 | 4.7795 | 4.7795 |

In the Table 13, we list the stationary probabilities of $E_2/E_3/1/6$ queuing system, $\mu_1 = \mu_2 = \mu_3 = 30$, $\lambda_1 = \lambda_2$.

Table 13: $E_2/E_3/1/6$

| $\rho$ | $\lambda_i$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 4 | 0.8 | 0.1908 | 0.0115 | 0.0003 | 0.0000 | 0.0000 | 0.0000 | 0.0122 | 0.0095 |
| 0.5 | 10 | 0.5000 | 0.3744 | 0.1054 | 0.0216 | 0.0043 | 0.0008 | 0.0001 | 0.1654 | 0.1582 |
| 0.8 | 16 | 0.2089 | 0.3088 | 0.2175 | 0.1296 | 0.0762 | 0.0448 | 0.0184 | 0.9763 | 0.9869 |
| 1 | 20 | 0.0685 | 0.1446 | 0.1633 | 0.1646 | 0.1654 | 0.1705 | 0.1252 | 2.2939 | 2.2939 |
| 2 | 40 | 0.0000 | 0.0002 | 0.0014 | 0.0078 | 0.0443 | 0.2695 | 0.6768 | 4.6118 | 4.6114 |
| 3 | 60 | 0.0000 | 0.0000 | 0.0000 | 0.0006 | 0.0100 | 0.1779 | 0.8114 | 4.8001 | 4.8001 |

In the Table 14, we list the stationary probabilities of $E_2/C_2/1/6$ queuing system. After phase 1, the service time comes to an end with probability 0.2, and it enters the next phase with probability 0.8, i.e. $q_1 = 0.2$. And $\mu_1 = \mu_2 = 18$, $\lambda_1 = \lambda_2$.

Table 14: $E_2/C_2/1/6$

| $\rho$ | $\lambda_i$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 4 | 0.8 | 0.1870 | 0.0123 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0137 | 0.0137 |
| 0.5 | 10 | 0.5002 | 0.3441 | 0.1113 | 0.0323 | 0.0091 | 0.0025 | 0.0006 | 0.2161 | 0.2164 |
| 0.8 | 16 | 0.2168 | 0.2705 | 0.1965 | 0.1334 | 0.0892 | 0.0594 | 0.0311 | 1.1236 | 1.1531 |
| 1 | 20 | 0.0861 | 0.1442 | 0.1536 | 0.1556 | 0.1569 | 0.1613 | 0.1423 | 2.2921 | 2.2921 |
| 2 | 40 | 0.0004 | 0.0018 | 0.0060 | 0.0203 | 0.0698 | 0.2515 | 0.6503 | 4.5134 | 4.5135 |
| 3 | 60 | 0.0000 | 0.0001 | 0.0005 | 0.0036 | 0.0247 | 0.1765 | 0.7945 | 4.7606 | 4.7606 |

In the Table 15, we list the stationary probabilities of $E_2/C_3/1/6$ queuing system. After phase 1, the service time comes to an end with probability 0.2, and it enters the next phase with probability 0.8. After phase 2, the service time comes to an end with probability 0.5, and it enters the next phase with probability 0.5. And $\mu_1 = \mu_2 = \mu_3 = 22$, $\lambda_1 = \lambda_2$.

Table 15: $E_2/C_3/1/6$

| $\rho$ | $\lambda_i$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 4 | 0.8 | 0.1876 | 0.0140 | 0.0007 | 0.0000 | 0.0000 | 0.0000 | 0.0155 | 0.0131 |
| 0.5 | 10 | 0.5001 | 0.3472 | 0.1152 | 0.0315 | 0.0083 | 0.0022 | 0.0005 | 0.2141 | 0.2086 |
| 0.8 | 16 | 0.2156 | 0.2727 | 0.2019 | 0.1347 | 0.0885 | 0.0580 | 0.0295 | 1.1161 | 1.1384 |
| 1 | 20 | 0.0842 | 0.1430 | 0.1555 | 0.1573 | 0.1583 | 0.1627 | 0.1408 | 2.2973 | 2.2973 |
| 2 | 40 | 0.0003 | 0.0017 | 0.0057 | 0.0196 | 0.0688 | 0.2520 | 0.6519 | 4.5188 | 4.5189 |
| 3 | 60 | 0.0000 | 0.0001 | 0.0005 | 0.0036 | 0.0246 | 0.1762 | 0.7950 | 4.7613 | 4.7612 |

In the Table 16, we list the stationary probabilities of $C_2/C_2/1/6$ queuing system. After phase 1, the interarrival time and service time come to an end with probabilities 0.2, and enter the next phase with probabilities 0.8, i.e. $p_1 = q_1 = 0.2$. And $\mu_1 = \mu_2 = 18$, $\lambda_1 = \lambda_2$.

Table 16: $C_2/C_2/1/6$

| $\rho$ | $\lambda_i$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 3.6 | 0.8000 | 0.1805 | 0.0178 | 0.0016 | 0.0001 | 0.0000 | 0.0000 | 0.0213 | 0.0213 |
| 0.5 | 9 | 0.5004 | 0.3239 | 0.1179 | 0.0395 | 0.0129 | 0.0042 | 0.0012 | 0.2581 | 0.2585 |
| 0.8 | 14.4 | 0.2215 | 0.2561 | 0.1910 | 0.1347 | 0.0936 | 0.0648 | 0.0361 | 1.1808 | 1.2012 |
| 1 | 18 | 0.0942 | 0.1451 | 0.1534 | 0.1554 | 0.1565 | 0.1595 | 0.1358 | 2.2508 | 2.2508 |
| 2 | 36 | 0.0006 | 0.0026 | 0.0081 | 0.0254 | 0.0807 | 0.2644 | 0.6181 | 4.4494 | 4.4496 |
| 3 | 54 | 0.0000 | 0.0001 | 0.0008 | 0.0049 | 0.0302 | 0.1918 | 0.7721 | 4.7290 | 4.7290 |

In the Table 17, we list the stationary probabilities of $C_2/C_3/1/4$ queuing system. After phase 1, the interarrival time and service time come to an end with probabilities $0.2$, and enter the next phase with probabilities $0.8$. After phase 2, the service time comes to an end with probability $0.5$, and it enters the next phase with probability $0.5$, i.e. $p_1 = q_1 = 0.2$ and $q_2 = 0.5$. And $\mu_1 = \mu_2 = \mu_3 = 22$, $\lambda_1 = \lambda_2$.

Table 17: $C_2/C_3/1/6$

| $\rho$ | $\lambda_i$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 3.6 | 0.8000 | 0.1810 | 0.0192 | 0.0016 | 0.0001 | 0.0000 | 0.0000 | 0.0227 | 0.0206 |
| 0.5 | 9 | 0.5003 | 0.3267 | 0.1211 | 0.0390 | 0.0121 | 0.0037 | 0.0010 | 0.2555 | 0.2509 |
| 0.8 | 14.4 | 0.2203 | 0.2580 | 0.1954 | 0.1360 | 0.0933 | 0.0638 | 0.0345 | 1.1748 | 1.1891 |
| 1 | 18 | 0.0923 | 0.1442 | 0.1551 | 0.1570 | 0.1579 | 0.1609 | 0.1341 | 2.2547 | 2.2547 |
| 2 | 36 | 0.0005 | 0.0024 | 0.0077 | 0.0247 | 0.0796 | 0.2653 | 0.6196 | 4.4554 | 4.4555 |
| 3 | 54 | 0.0000 | 0.0001 | 0.0008 | 0.0049 | 0.0300 | 0.1915 | 0.7727 | 4.7301 | 4.7299 |

In the Table 18, we list the stationary probabilities of $C_4/C_3/1/4$ queuing system. After phase $i$, $i = 1, 2, 3$, the interarrival time come to an end with probability $0.2$, and enter the next phase with probability $0.8$. After phase $j$, $j = 1, 2$, the service time comes to an end with probability $0.5$, and it enters the next phase with probability $0.5$, i.e. $p_i = 0.2$, $i = 1, 2, 3$ and $q_j = 0.5$, $j = 1, 2$. And $\mu_1 = \mu_2 = \mu_3 = 17.5$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$.

Table 18: $C_4/C_3/1/6$

| $\rho$ | $\lambda_i$ | $\pi_0$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $L_q(pd)$ | $L_q(td)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.2 | 5.905 | 0.8000 | 0.1769 | 0.0225 | 0.0025 | 0.0003 | 0.0000 | 0.0000 | 0.0283 | 0.0259 |
| 0.5 | 14.76 | 0.5005 | 0.3150 | 0.1232 | 0.0439 | 0.0152 | 0.0052 | 0.0016 | 0.2859 | 0.2800 |
| 0.8 | 23.616 | 0.2244 | 0.2481 | 0.1884 | 0.1354 | 0.0961 | 0.0681 | 0.0418 | 1.2289 | 1.2469 |
| 1 | 29.52 | 0.0982 | 0.1434 | 0.1491 | 0.1511 | 0.1519 | 0.1545 | 0.1511 | 2.2805 | 2.2805 |
| 2 | 59.04 | 0.0009 | 0.0036 | 0.0100 | 0.0275 | 0.0753 | 0.2055 | 0.6544 | 4.3850 | 4.4728 |
| 3 | 88.56 | 0.0000 | 0.0003 | 0.0013 | 0.0060 | 0.0280 | 0.1297 | 0.8079 | 4.6552 | 4.7609 |

## Conclusions and Remarks

In the thesis, we have studied the $C_k/C_m/1/N$ open queueing system containing finite number of customers, $N$. We list the stationary probabilities of $C_k/C_m/1/N$ by using the product-form method and compare it with a traditional method. We use the Matlab software to solve the stationary probability. We suggest some problems for further investigation. For example, one can discuss the precision and stability of the product-form method. In this case, we can choose other algorithms to solve the linear nonhomogeneous system consisting of equations (2.21) $\sim$ (2.25) We also suggest that the methods presented here may be extended to a multi-server open system containing finite number of customers with phase type interarrival and service times distributions.

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# Appendix A

**Definition 1** *The Kronecker product of* $A \in M_{m,n}(F)$ *and* $B \in M_{p,q}(F)$ *is denoted by* $A \otimes B$ *and is defined to be the block matrix.*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in M_{mp,nq}(F)$$

We also mention another Kronecker operation, the Kronecker sum, $A \oplus B$ is defined by square matrices $A$ and $B$ and is given by

$$A \oplus B \triangleq A \otimes I_m + I_n \otimes B,$$

where $A \in M_n$ and $B \in M_m$. Thus, $A \otimes I_m$, $I_n \otimes B$, and their sum are in $M_{mn}$.