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Multicommodity Network Flows: A Survey, Part I: Applications and Formulations

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Abstract: The multicommodity network flow (MCNF) problem arises often in logistics and telecommunication applications such as network design or routing problems. In particular, MCNF appears when more than one commodity (cargo, packet, or personnel) needs to be shipped between specific node pairs without violating the capacity constraints associated with the arcs. This problem has spurred much research in the field of linear, combinatorial, and integer programming. As the first part of our MCNF survey over the related literature of recent three decades, this paper summarizes the applications and different mathematical formulations for modeling the linear or integral multicommodity network flow problems.
Keyword — Multicommodity, Network Optimization, Linear Programming, Integer Programming

1. INTRODUCTION

The multicommodity network flow (MCNF) problem is defined over a network where more than one commodity needs to be shipped from specific origin nodes to destination nodes while not violating the capacity constraints associated with the arcs. Many network routing and network design problems can be modeled as MCNF problems. For example, message routing in telecommunication, scheduling and routing in logistics and transportation, production scheduling and planning, global routing in VLSI design, and distribution system design problems are all MCNF problems. De Loera and Onn (2005) have shown that arbitrary linear and integer programs can be encoded as two-commodity flow problems, suggesting that even more applications might be related to MCNF.

$|N| = 5, |A| = 10, |K| = 3$

K^1: $(s^1, t^1) = (1, 4)$, $B^1 = 6$
K^2: $(s^2, t^2) = (5, 3)$, $B^2 = 4$
K^3: $(s^3, t^3) = (3, 1)$, $B^3 = 8$

origins, destinations, and demands
of three commodities

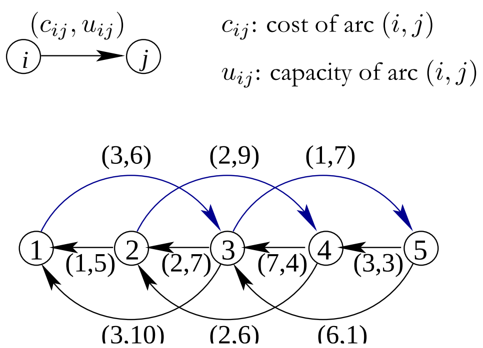

Figure 1: An MCNF example problem

In addition to its wide applicability, MCNF also has an important theoretical history. Since the 1960s, the study of MCNF has motivated many important advances in optimization research. For example, *column generation* developed by Ford and Fulkerson (1958) was originally designed to solve max MCNF problems, and is still a common technique for solving large-scale LP problems. Seymour (1981) proposed the *max-flow min-cut matroid* and several important matroid theorems as a result of his study of multicommodity flow. The nonintegrality property of MCNF spurred much research in integral LP related to matroids, algorithmic complexity, and polyhedral combinatorics (see Grötschel, Lovász, & Schrijver, 1993 for details). The block-angular constraint structure (see Figure 2) of the MCNF constraints serves

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$$\widetilde{N} = \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Node-arc incidence matrix $\widetilde{N}$

$$\begin{bmatrix} \widetilde{N} & & \\ & \widetilde{N} & \\ & & \widetilde{N} \\ I & I & I \end{bmatrix} X \begin{array}{l} = b_1 = [\,6\ 0\ 0\ \text{-}6\ 0\,]^T \\ = b_2 = [\,0\ 0\ \text{-}4\ 0\ 4\,]^T \\ = b_3 = [\,\text{-}8\ 0\ 8\ 0\ 0\,]^T \\ \leq u = [\,6\ 5\ 9\ 10\ 7\ 7\ 6\ 4\ 1\ 3\,]^T \end{array}$$

$$X = [\,x^1\ x^2\ x^3\,]^T,\ x^k \in R_+^{|A|}\ \forall k \in K$$

MCNF constraints

Figure 2: Block angular structure of the example problem in Figure 1

as a best-practice example for decomposition techniques such as *Dantzig-Wolfe decomposition* (Dantzig & Wolfe, 1961) and *Benders decomposition* (Benders, 1962), and basis-partitioning methods such as *generalized upper bounding* (Hartman & Lasdon, 1972; Lasdon, 1970) and *key variables* (Rosen, 1964). Recently, a very good survey on theoretical issues in MCNF and related problems can be found in (Schrijver, 2003).

The MCNF problem extends the *single commodity network flow* (SCNF) problem in a sense that if we disregard the *bundle constraints*, the arc capacity constraints that tie together flows of different commodities passing through the same arc, an MCNF problem can be viewed as several independent SCNF problems. Although they might seem like a minor change on the surface, the existence of the bundle constraints makes MCNF problems much more difficult to solve than SCNF problems. First, many SCNF algorithms exploit the single commodity flow property in which flows in opposition direction on an arc can be canceled out; on the other hand, flows of different commodities do not cancel in MCNF problems. Second, in SCNF problems, the max-flow min-cut theorem of Ford and Fulkerson (1962) guarantees that the maximum flow equals to the minimum cut, but such a property no longer holds for MCNF problems. Third, the *total unimodularity* of the constraint matrix in the SCNF LP formulation guarantees that optimal *integer-valued* flows (important in many applications) can be found by linear programming if the node supplies/demands and arc capacities are integers, but this integrality property can not be extended to MCNF LP formulations, except for several special planar graphs (Lomonosov, 1985; Nagamochi & Ibaraki, 1989; Okamura, 1983; Okamura & Seymour, 1981), or two-commodity flows with even integral demands and capacities (T. Hu, 1963; Rothschild & Whinston, 1966; Sakarovitch, 1973).

In general, there are three major MCNF problems in the literature: the *max MCNF* problem, the *max-concurrent flow* problem, and the *min-cost MCNF* problem. The max MCNF problem is to maximize the sum of flows for all commodities between their respective origins and destinations. The max-concurrent flow problem maximizes the fraction (throughput) of the satisfied demands d^k for all commodities k. In other words, the max concurrent flow problem maximizes the fraction λ for which the flows do not exceed arc capacities while the fractional demands λd^k between all pairs of nodes are satisfied. The min-cost MCNF problem is to find the flow assignment satisfying the demands of all commodities with minimum cost without violating the capacity constraints on all arcs. This paper will focus on min-cost MCNF problem. Hereafter, when we refer to the MCNF problem, we mean the min-cost MCNF problem.

Although there has been much research relating to the MCNF problem during the past four decades, MCNF is still considered to be a difficult mathematical programming problem (Bienstock, 2002), especially when there is a large number of commodities. Solving the general integral MCNF problem is NP-complete (Karp, 1975). In fact, even solving the two-commodity integral flow problem is NP-complete (Even, Itai, & Shamir, 1976). On the other hand, the linear MCNF problem can theoretically be solved in strongly polynomial time (Tardos, 1986). Computationally speaking, however, the MCNF problems, especially the integral MCNF problems may take much computational time to solve. For example, a telecommunication bandwidth packing problem of 29 nodes, 61 arcs, and 68 commodities tested in (Barnhart, Hane, & Vance, 2000) is 12.34% away from its optimal objective value after one hour of computation, although 68 SCNF problems of similar size (if we disregard the bundle constraints) could be solved to optimality in less than a second. A large-scale network design case of 33966 constraints and 588618 variables tested in (Bienstock, 2002) has been reported to be 20% away from its optimum even after several days of computation using the CPLEX dual simplex method as the LP relaxation solver.

The previous survey papers in multicommodity network flow were written almost three decades ago. During the past thirty years, many new methods and applications in the field have been proposed and researched. We have surveyed over hundreds of references and summarized most of the applications, formulations and solution methods in the literature for solving the MCNF problems. This paper covers the first part of our survey aiming at the MCNF applications and formulations. The second part of our survey that focus on MCNF solution methods can be found in Wang (2018). This paper contains four Sections. Section 2 introduces applications of MCNF problems, including the network routing and design problems. In Section 3, we present different formulations of MCNF problems and discuss their pros and cons. Section 4 concludes this paper.

2. APPLICATIONS

MCNF models arise in many real-world applications. Most of them are network routing and network design problems.

2.1 Network Routing Problems

2.1.1 Message Routing in Telecommunication:

Consider each requested OD pair to be a commodity. The problem is to find a min-cost flow routing for all demands of requested OD pairs while satisfying the arc capacities. This appears often in telecommunication. For example, message routing for many OD pairs (Barnhart, Johnson, Hane, & Sigismondi, 1995; McBride & Mamer, 2001), packet routing on virtual circuit data network (Lin & Yee, 1992) known as unsplittable flows (Kleinberg, 1996) (all of the packets in a session are transmitted over exactly one path between the OD to minimize the average number of packets in the network), or routing on a ring network (Shepherd & Zhang, 2001) (any cycle is of length n), are MCNF problems.

2.1.2 Scheduling and Routing in Logistics and Transportation:

In logistics or transportation problems, commodities may be objects such as products, cargo, or even personnel. The commodity scheduling and routing problem is often modeled as an MCNF problem on a time-space network where a commodity may be a tanker (Bellmore, Bennington, & Lubore, 1971), aircraft (Hane et al., 1995), crew (Cappanera & Gallo, 2001), rail freight (Assad, 1980; Crainic, Ferland, & Rousseau, 1984; Kwon, Martland, & Sussman, 1998), or Less-than Truck-Load (LTL) shipment (Barnhart & Sheffi, 1993; Farvolden, Powell, & Lustig, 1993). Dynamic MCNF problems (Aronson, 1989; Fleischer & Skutella, 2002; Köhler, Möhring, & Skutella, 2002) deals with flows over time in which each arc is associated with a transit time while the capacity over an arc limits the rate of flow into the arc at each point in time.

Golden (1975) gives an MCNF model for port planning that seeks optimal simultaneous routing where commodities are export/import cargo, nodes are foreign ports (as origins), domestic hinterlands (as destinations) and US ports (as transshipment nodes), and arcs are possible routes for cargo traffic. Similar problems appear in grain shipment networks (Ali et al., 1984; Barnett, Binkley, & McCarl, 1984).

A disaster relief management problem is formulated as a multicommodity multimodal network flow problem with time windows by Haghani and Oh (1996) where commodities (food, medical supplies, machinery, and personnel) from different areas are to be shipped via many modes of transportation (car, ship, helicopter,...,etc.) in the most efficient manner to minimize the loss of life and maximize the efficiency of the rescue operations.

2.1.3 Production Scheduling and Planning:

Jewell (1957) solves a warehousing and distribution problem for seasonal products by formulating it as an MCNF model where each time period is a transshipment node, one dummy source and one dummy sink node exist for each product, and arcs connect from source nodes to transshipment nodes, earlier transshipment nodes to later transshipment nodes, and transshipment nodes to sink nodes. Commodities are products to be shipped from sources to sinks with minimum cost.

D'Amours et al. (1996) solve a planning and scheduling problem in a Symbiotic Manufacturing Network (SMN) for a multiproduct order. A broker receives an order of products in which different parts of the products may be manufactured by different manufacturing firms, stored by some storage firms, and shipped by a few transportation firms between manufacturing firms, storage firms and customers. The problem is to design a planning and scheduling bidding scheme for the broker to make decisions on when the bids should be assigned and who they should be assigned to, such that the total cost is minimized. They propose an MCNF model where each firm (manufacturing or storage) at each period represents a node, an arc is either a possible transportation link or possible manufacturing (storage) decision, and a commodity represents an order for different product.

Aggarwal et al. (1995) use an MCNF model to solve an equipment replacement problem which determines the time and amount of the old equipment to be replaced to minimize the cost.

2.1.4 Other Routing Problems:

1. VLSI design (Albrecht, 2000, 2001; Carden & Cheng, 1991; J. Hu & Sapatnekar, 2001; Raghavan, 1992; Sarrafzadeh & Wong, 1996; Vygen, 2004): Global routing in VLSI design can be modeled as an origin-destination MCNF problem where nodes represent collections of terminals to be wired, arcs correspond to the channels through which the wires run, and commodities are OD pairs to be routed.
2. Caching and prefetching problems in disk systems (Albers, Garg, & Leonardi, 2000; Albers & Witt, 2001): Since computer processors are much faster than memory access, caching and prefetching are common techniques used in modern computer disk systems to improve the performance of their memory systems. In prefetching, memory blocks are loaded from the disk (slow memory) into the cache (fast memory) before actual references to the blocks, so the waiting time is reduced in accessing memory from the disk. Caching tries to keep actively referenced memory blocks in fast memory. At any time, at most one fetch operation is executed. Albers and Witt (2001) model this problem as a min-cost MCNF problem that decides when to initiate a prefetch, and what blocks to fetch and evict.
3. Traffic equilibrium (Ferris, Meeraus, & Rutherford, 1999; LeBlanc, Morlok, & Pierskalla, 1975; Roughgarden, 2002): The traffic equilibrium law proposed by Wardrop (1952) states that at equilibrium, for each origin-destination pair, the travel times on all routes used are equal, and are less than the travel times on all nonused routes. Given the total demand for each OD pair, the equilibrium traffic assignment problem is to predict the traffic flow pattern on each arc for each OD pair that follows Wardrop's equilibrium law. It can be formulated as a nonlinear MCNF problem where the bundle constraints are eliminated but the nonlinear objective function is designed to capture the flow congestion effects.
4. Graph Partitioning (Sensen, 2001): The graph partitioning problem is to partition a set of nodes of a graph into disjoint subsets of a given maximal size such that the number of arcs with endpoints in different subsets is minimized. It is an NP-hard problem. Based on a lower bounding method for the graph bisection problem, which corresponds to a MCNF problem, Sensen (2001) proposes three linear MCNF models to obtain lower bounds for the graph partitioning problem. He then uses branch-and-bound to compute the exact solution.
   Similarly, many NP-hard problems such as min-cut linear arrangement, crossing number, minimum feedback arc set, minimum 2D area layout, and optimal matrix arrangement for nested disection can be approximately solved by MCNF algorithms (Klein, Agrawal, Ravi, & Rao, 1990; Klein, Rao, Agrawal, & Ravi, 1995; F. Leighton & Rao, 1988; T. Leighton & Rao, 1999).

2.2 Network Design Problems

Given a graph G, a set of commodities K to be routed according to known demands, and a set of facilities L that can be installed on each arc, the *capacitated network design* problem is to route flows and install facilities at minimum cost. This is an MCNF problem which involves flow conservation and bundle constraints plus some side constraints related to the installation of the new facilities. The objective function may be nonlinear or general discontinuous step-increasing (Gabrela, Knippelb, & Minoux, 1999).

Problems such as the design of a network where the maximum load on any edge is minimized (Bienstock & Günlük, 1995) or the service quality and survivability constraints are met with minimum cost of additional switches/transport pipes in ATM networks (Bienstock & Saniee, 2001) appear often in the telecommunication literature. Bienstock et al. use metric inequalities, aggregated formulations (Bienstock, Chopra, Günlük, & Tsai, 1998) and facet-defining inequalities (Bienstock & Muratore, 2000) to solve capacitated survivable network design problems. Gouveia (1996) discusses MCNF formulations for a specialized *terminal layout problem* which seeks a minimum spanning tree with hop constraints (a limit on the number of hops (links) between the computer center and any terminal in the network).

Maurras et al. (2000; 1997) study network design problems with jump constraints (i.e., each path has no more than a fixed number of arcs). Girard and Sansò (1998) show that the network designed using an MCNF model significantly improves the robustness of the heuristic solutions at a small cost increase. Gendron et al. (1999) write a comprehensive survey paper in multicommodity capacitated network design.

Similar problems also appear in transportation networks, such as locating vehicle depots in a freight transportation system so that the client demands for empty vehicles are satisfied while the depot opening operating costs and other transportation costs are minimized. Crainic et al. (1989) have solved this problem by various methods such as branch-and-bound (Crainic, Delorme, & Dejax, 1993) and its parallelization (Bourbeau, Crainic, & Gendron, 2000; Gendron & Crainic, 1997), dual-ascent (Crainic & Delorme, 1993), and tabu-search (Crainic, Gendreau, Soriano, & Toulouse, 1993). Crainic also writes a survey paper (Crainic, 2000) about service network design in freight transportation. In fact, these problems are facility location problems. Geoffrion and Graves (1974) model a distribution system design problem as a fixed charge MCNF problem. More MCNF models for facility location are surveyed by Aikens (1985).

3. FORMULATIONS

Let N denote the set of all nodes in G, A the set of all arcs, and K the set of all commodities. For commodity k with origin s_k and destination t_k, c_{ij}^k represents its per unit flow cost on arc (i,j) and x_{ij}^k the flow on arc (i,j). Let b_i^k be the supply/demand at node i, and B^k be the total demand units of commodity k. Let u_{ij} be the arc capacity on arc (i,j). Without loss of generality, we assume each unit of each commodity consumes one unit of capacity from each arc on which it flows.

3.1 Node-Arc Form

The node-arc form of MCNF problem is a direct extension of the conventional SCNF formulation. It can be formulated as follows:

$$\min \sum_{k\in K} \sum_{(i,j)\in A} c_{ij}^k x_{ij}^k = Z^*(x) \quad \text{(Node-Arc)}$$

$$s.t. \sum_{(i,j)\in A} x_{ij}^k - \sum_{(j,i)\in A} x_{ji}^k = b_i^k \;\; \forall i \in N, \forall k \in K \quad (1)$$

$$\sum_{k\in K} x_{ij}^k \leq u_{ij} \;\; \forall (i,j) \in A \text{ (bundle constraints)} \quad (2)$$

$$x_{ij}^k \geq 0 \;\; \forall (i,j) \in A, \forall k \in K$$

where $b_i^k = B^k$ if $i = s_k$, $b_i^k = -B^k$ if $i = t_k$, and $b_i^k = 0$ if $i \in N \backslash \{s_k, t_k\}$. This formulation has $|K||A|$ variables and $|N||K| + |A|$ nontrivial constraints.

3.2 Arc-Path Form

First suggested by Ford and Fulkerson (1958), the arc-path form of the min-cost MCNF problem is based on the fact that any network flow solution can be decomposed into path flows and cycle flows. Under the assumption that no cycle has negative cost, any arc flow vector can be expressed optimally by simple path flows.

For commodity k, let P^k denote the set of all possible simple paths from s_k to t_k, f_p the units of flow on path $p \in P^k$, and PC_p^c the cost of path p using c_{ij}^k as the unit flow cost along arc (i,j). δ_a^p is a binary indicator which equals to 1 if path p passes through arc a, and 0 otherwise. It can be formulated as follows:

$$\min \sum_{k\in K} \sum_{p\in P^k} PC_p^c f_p = Z^*(f) \quad \text{(Arc-Path)}$$

$$s.t. \quad \sum_{p\in P^k} f_p = 1 \;\; \forall k \in K \quad (3)$$

$$\sum_{k\in K} \sum_{p\in P^k} (B^k \delta_a^p) f_p \leq u_a \;\; \forall a \in A \text{ (bundle constraints)} \quad (4)$$

$$f_p \geq 0 \;\; \forall p \in P^k, \forall k \in K$$

Inequalities (4) are the bundle constraints, and (3) are the convexity constraints which force the optimal solution for each commodity k to be a convex combination of some simple paths in P^k. The optimal solution for this formulation may be fractional. For the binary MCNF problem where each commodity can only be shipped on one path, f_p will be be binary variables.

This formulation has $\sum_{k\in K} |P^k|$ variables and $|K| + |A|$ nontrivial constraints.

3.3 Comparison of Formulations

Many telecommunication MCNF problems treat OD pairs as commodities; thus the number of commodities $|K|$ may be $O(|N|^2)$ in the worst case. In such case, the node-arc form may have $O(|N|^3)$ constraints which make its computations more difficult and memory management less efficient.

The arc-path formulation, on the other hand, has at most $O(|N|^2)$ constraints but exponentially many variables. The problem caused by a huge number of variables can be resolved by column-generation techniques. In particular, when using the revised simplex method to solve the arc-path form, at most $|K| + |A|$ of the $\sum_{k \in K} |P^k|$ variables are positive in any basic solution. In each iteration of the revised simplex method, a shortest path subproblem for each commodity can be solved to find a new path for a simplex pivot-in operation if it corresponds to a nonbasic column with negative reduced cost.

Dantzig-Wolfe (DW) decomposition is a common technique used to solve problems with block-angular structure. The algorithm decomposes the problem into two smaller sets of problems, a *restricted master problem* (RMP) and k subproblems, one for each commodity. It first solves the k subproblems and uses their basic columns to solve the RMP to optimality. Then it uses the dual variables of the RMP to solve the k subproblems, which produce nonbasic columns with negative reduced cost to be added to the RMP. The procedures are repeated until no more profitable columns are generated for the RMP. More details about DW will be discussed in our follow-up survey paper focus on MCNF solution methods.

4. CONCLUSIONS

Due to the increasingly wide applications and important impacts of the multicommodity network flows, this paper surveys the applications and mathematical formulations of the minimum cost multicommodity network flow problems. We summarize the important findings, especially in the applications and mathematical formulations of MCNF, appeared in the literature over the recent three decades. Most of the applications are variants of network design or network routing problems related with scheduling, planning, or logistics problems. As the technologies of unmanned vehicles become more advanced and popular, we expect to see many more MCNF applications. Indeed, most of the routing problems for fleets of unmanned vehicles are variants of MCNF problems. To solve the MCNF problems by mathematical programming, we introduce and compare two common formulations: the node-arc form is suitable for problems of smaller size, whereas the arc-path form with column generation techniques works better for large-scale MCNF problems. Note that different formulations lead to different solution methods. In our follow-up survey paper, we will further classify MCNF solution methods (primal-dual methods, basis-partitioning methods, resource-directive methods, price-directive methods, interior-point methods, convex-programming methods, and polynomial-time approximation methods) in the literature, briefly introduce how these methods work, and then summarize their computational performance.

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