International Journal of
Operations Research

International Journal of Operations Research Vol. 16, No. 4, 105-118 (2019)

Linear and Conditional Logit Models of Demand Shifting for the Time-varying Capacity-Demand-Imbalance Problem

Christian John Immanuel S. Boydon*, Lowell L. Lorenzo, Iris Ann G. Martinez

Department of Industrial Engineering and Operations Research, University of the Philippines
Diliman, Quezon City, 1101 Metro Manila, Philippines

Received September 2019; *Revised December* 2019; *Accepted December* 2019

Abstract: We study the capacity-demand-imbalance problem occurring in manufacturing and service systems in which capacity is constrained but demand is time-varying. We construct profit optimization models using three different demand shifting functions and two different approaches. The demand shifting functions differ in the behavior (i.e. linear or conditional logit) as well as in the factors that affect this shift (i.e. price, demand difference, time difference). The approaches differ in the situation mainly as seen from how customers react upon seeing a fully utilized system (i.e. either they balk/leave immediately or wait in line until a server is free). We then perform numerical experiments on derived data from a real-life service system to study the practicality of these models in terms of profit improvement and computational effort. We find out that the parameters of the models affect profit improvement in a certain trend and that price settings which result to profit improvement can be computed in a fast time. In practice, a manufacturing or service system should aim to estimate these parameters based on actual demand shifting behavior and then determine the optimal price setting using the model that best approximates the real-life system.
Keyword — revenue management, demand shifting, nonlinear programming, cost analysis

1. INTRODUCTION

Demand is naturally fluctuating in most manufacturing and service systems. Given a manufacturing system with fixed resources for production or a service system with fixed number of servers to fulfil demand of incoming customers, demand that is time-varying can have severe repercussions. If we divide time into finite discrete time periods, for a manufacturing system we would see that some time periods will have overutilized resources with work-in-process piling while some time periods will have underutilized resources. In the former, the system may incur shortage cost due to delays in production while in the latter the system still has opportunities for profit since overhead is being paid without utilizing capacity. Similarly, for a service system we would see that some time periods will have busy servers seeing a growing queue while some time periods will have servers that are idle. In the former, the system may incur a certain amount of waiting cost due to customers in queue while in the latter idle servers translate to salaries paid without earning revenue. For systems in which capacity is constrained and demand is time-varying, we call this the capacity-demand-imbalance problem.

Revenue management has a variety of ways to solve related problems regarding influencing demand to maximize revenues or profit. But while successful and extensively researched in the airline and hotel industry, little mathematical research has been done on the specific aspect of demand shifting (Ingold & Yeoman, 2000; Shen & Su, 2007). Studies of demand shifting in the electric grid industry, e.g. Joe-Wong, Sen, Ha, and Chiang (2012), have a lot of difference with the manufacturing and service industry that make their frameworks not directly applicable. Thus, it is interesting to see how such demand shifting models could possibly be applied to manufacturing systems (e.g. make-to-order production system) such that when price is manipulated, demand is influenced to move into less occupied time periods resulting to shorter lead time for buyers; or to service systems (e.g. bank, restaurant, tourist spot, amusement park, post office, parking system) such that when price is manipulated, demand is influenced to move into less busy time periods resulting to less waiting time for customers. The motivation for optimization lies in the trade-off between lower revenues due to price reduction, higher revenues due to more demand served, and lower cost due to reduced shortage cost penalty. The basic decision of the optimization model is by how much

*Corresponding authors e-mail: csboydon@up.edu.ph

http://doi.org/10.6886/IJOR.201912_16(4).0002

should price be reduced and when should price be reduced.
In this paper, we focus on the construction of simple linear and conditional logit functions of demand shifting for the time-varying capacity-demand-imbalance problem. We then incorporate these into two different profit optimization models and perform numerical experiments to get insights on their mathematical structures and practicality. Further, we discuss some possible extensions to make the models more realistic. We end the paper with some discussions regarding the use of these models as well as managerial insights.

2. LITERATURE REVIEW

Possibly the oldest model that involved pricing policies in managing capacity is Whitin (n.d.) which provided an extended inventory management model wherein optimal price is determined assuming a probabilistic demand related to price. It finds motivation from the classical Newsboy Problem which also becomes the basis of one of the mathematical models presented later. McAfee and Te Velde (2006), focusing on the airline industry, provides a survey of revenue management research and discusses the different operations research models concerning dynamic pricing applicable also in the hotel, electricity, and retail industries. While it is an extensive review of existing models and practical applications, all those models assume only a single period (i.e. booking horizon) in which price is manipulated dynamically to influence potential customers to buy until that inventory unit (e.g. airline seat, hotel room) is consumed. Finally, papers in real-time pricing often used in the electric grid industry, e.g. Faria and Vale (2011); Kobayashi, Maruta, Sakurama, and Azuma (2014), change prices per time period but does so dynamically such that demand shifting is not considered, using only a relationship of price and demand. To the authors' knowledge, there is only one paper, i.e. Joe-Wong et al. (2012), that models the shifting of demand along multiple time periods, but which uses only a linear function, does not maximize profit, and provides a different situation not applicable to manufacturing and service systems.
More recent review papers in revenue management (M. Chen & Chen, 2015; Shen & Su, 2007) discuss the different intertemporal substitution models in dynamic pricing but still assume a single period (i.e. booking horizon). Other review papers (Chao & Zhou, 2006; X. Chen & Simchi-Levi, 2004) discuss inventory and pricing strategies in the infinite-horizon or multiple-time period, thus tackling the capacity-demand-imbalance problem, but do not include demand shifting along time periods. A very recent review paper (Klein, Koch, Steinhardt, & Strauss, 2019) mentioned the real-time multi-period planning horizon and discussed possible extensions of revenue management in the manufacturing sector, but focuses on availability/capacity control (make-to-order) and inventory control (make-to-stock), thus not covering dynamic pricing and demand shifting.
In summary, none of the papers discuss profit optimization models with demand shifting for the capacity-demand-imbalance problem applicable to manufacturing and service systems. Traditional revenue management assumes only a single period (i.e. booking horizon) and thus demand shifting along selling time periods do not happen. Recent extensions of revenue management consider multiple selling periods that have demand as a function of price but do not construct models that incorporate the phenomenon of demand shifting while maximizing profit which is usually the goal of most manufacturing and service systems.

3. MATHEMATICAL MODEL

3.1 Demand shifting function

First, we discuss possible demand shifting functions that can be used. As in Joe-Wong et al. (2012), the structure of the demand shifting function is simple factors are placed in the equation depending on whether they are positively or negatively related to demand shift with a parameter to adjust the magnitude of relationship. Literature has provided some factors aside from price that may be related to demand shift, e.g. time proximity in the electric grid industry Joe-Wong et al. (2012), type of promotional discount in restaurants (Susskind, Reynolds, & Tsuchiya, 2004), socioeconomic characteristics and purpose in the traffic sector (Burris & Pendyala, 2002), the level of being strategic or myopic of a customer (M. Chen & Chen, 2015). In this paper, we will be assuming homogenous customers and a generic type of system so the factors to include are price (i.e. the higher the price reduction, the stronger the demand shift), time proximity (i.e. the farther the time period, the weaker the demand shift), and relative demand difference (i.e. the higher the difference in demand between two time periods, the higher the demand shift) since if information is known a customer has the tendency to shift to the most unutilized time periods to achieve lowest lead time or waiting time in queue.
The behavior of demand shift can be characterized by a linear model, e.g. Joe-Wong et al. (2012), or by a discrete choice model wherein the choices are the time periods to which a customer may shift into or stay at. Strauss, Klein, and Steinhardt (2018) reviewed the parametric discrete choice models of random utility theory used in revenue

management literature: multinomial logit, finite-mixture logit, nested logit, Markov chain model, exponomial model. In this paper, we will be using the simplest models: linear and conditional logit, which is a variant of multinomial logit for homogenous customers but with alternative-variant factors. In this study, we will apply the models using data derived from a real-life system and then perform sensitivity analysis to get insights on the possible values of the parameters.
We summarize in Table 1 the variables and notations to be used in formulating the different demand shifting functions.

Table 1: Summary of variables and notations for the demand shifting functions.

Demand shifting function:	**Description:**
$s_{ki}(r_i)$	Proportion of demand that shifts from time period k to time period i as a function of decision variable, r_i
Decision variable:	**Description:**
r_i	Price reduction (or price discount)
Parameters and factors:	**Description:**
γ	Normalizing constant and parameter to adjust magnitude of shift for a linear model
α	Parameter to adjust effect of price reduction ri for a conditional logit model
β	Parameter to adjust effect of time difference $\|ik\|$ for a conditional logit model
μ	Scaling factor for conditional logit model (which can be set to 1 for numerical example)
D_k, D_i	Demand in time period k (or i)
$(D_k - D_i)^+$	Positive part of difference of D_k and D_i, i.e. $\text{Max}(D_k D_i, 0)$
k, i	Time period indices
$\|i - k\|$	Absolute value of difference between time index i and time index k

This paper will use the following three demand shifting functions:

$$s_{ki}(r_i) = \gamma r_i (D_k - D_i)^+ \tag{1}$$

Equation 1 is the simplest demand shifting function which is a linear model with both factors proportional to demand shift. The function, $ski(r_i)$, gives the proportion of demand that shifts from time period k to time period i and is a function of price reduction at time period i, r_i, which is the value to be determined in the profit optimization model, and of the demand difference between time periods k and i as given by $(D_k - D_i)^+$ such that there will only be a shift if time period i has lower demand (customers will not shift to a time period with higher demand). We add a parameter γ that acts both as normalizing constant to keep the proportion of demand shift from 0 to 1 and a parameter that can be adjusted depending on the magnitude of shift.

$$s_{ki}(r_i) = \gamma \frac{r_i}{|i - k|}, \quad k \neq i \tag{2}$$

Equation 2 is another simple demand shifting function which is a linear model with one factor, price reduction, r_i, proportional to the proportion of demand that shifts from time period k to time period i, $s_{ki}(r_i)$, and another factor, time difference as given by $|ik|$, which is inversely proportional since customers have less tendency to shift to farther time periods. Similarly, there is a parameter γ that acts both as normalizing constant and for characterizing the magnitude of shift.

$$s_{ki}(r_i) = \frac{e^{(\frac{1}{\mu})(\alpha r_i - \beta |i-k|)}}{\sum_{j=1}^{n} e^{(\frac{1}{\mu})(\alpha r_j - \beta |j-k|)}} \tag{3}$$

Equation 3 is a demand shifting function based on a conditional logit model. The proportion of demand at time period k that stays at time period k or shifts into any time period i is calculated as the proportion of the utility of that particular discrete choice over the probability of the sum of utilities of all discrete choices. The utility, as given by the conditional logit model, has a scaling factor, μ, that characterizes the covariance structure of the discrete choices, and two alternative-variant factors. These factors are price, r_i , which has a parameter α, and time difference as given by $|ik|$, which has a parameter β. In the conditional logit model, the factors are additive inside the exponential function and given a positive sign if positively related or negative sign if negatively related.

3.2 Profit optimization model

We then construct the profit optimization model for this problem. For a manufacturing or service system with fixed capacity, we assume a finite planning horizon that is cyclical in nature (e.g. day, week, any seasonal time interval) wherein demand forecast is available and decision-making happens beforehand such that optimal prices can be calculated and price reductions pre-announced (e.g. in the form of promotional discount) so that customers may shift accordingly. The finite planning horizon is divided into multiple time periods wherein demand is different per time period. These time periods are finite and equal in length (e.g. 30-minute time period, 1-hour time period, 1 day) depending on the time interval of interest. Thus, the pricing decisions (i.e. when to offer price discount, how much price discount to offer) can be done in such a way that the model determines the optimal price reduction per time period to maximize total profit. We allow continuous prices for a single product with a single default price. Additionally, we assume that the system is closed such that no additional demand goes in (i.e. customers not previously buying see the price reduction and now decide to buy). This is a conservative analysis since any additional demand will surely yield a higher profit. Note that we only allow price reductions since a number of papers in literature state that price discounts are more ethical and acceptable in the field of revenue management, e.g. Selmi (2010).

A comprehensive review on the performance evaluation of time-dependent queuing systems done by Schwarz, Selinka, and Stolletz (2016) provide different interpretations of capacity and demand. Thus, we construct two models based on two plausible approaches differing on how customers react upon seeing a fully utilized system (i.e. either they balk or wait). Interestingly, in the review by Schwarz et al. (2016), they mentioned that one good direction for future work in the study of time-dependent queuing systems is the integration of performance evaluation with optimization models.

We first summarize in Table 2 the variables and notations to be used in constructing the profit optimization models.

Table 2: Summary of variables and notations for the profit optimization models.

Variables in both models:	**Description:**
Z	Total profit to be maximized
n	Number of time periods
P	Price of one unit of capacity
r_i	Price reduction (or price discount)
a_i	Revenue per demand fulfilled at time period i
$s_{ki}(r_i)$	Proportion of demand that shifts from time period k to time period i
Newsboy-based model:	**Description:**
Z_i	Profit for time period i, such that Z = summation of Z_i (from i = 1 to n)
B	Penalty per demand unfulfilled (assume same at any time period)
C	Capacity (fixed at all time periods)
d_i	Demand at time period i, which becomes the sum of three components: original demand, plus summation of demand that shifts into, minus summation of demand that shifts out
D_i, D_k	Original demand in time period i (or k)
t_i	Auxiliary variable to transform Min function representing demand shortage
Queuing-based model:	**Description:**
α_i	Arrival rate at time period i, which becomes the sum of three components: original arrivals, plus summation of arrivals that shift into, minus summation of arrivals that shift out
$\wedge_i, \wedge_k$	Original arrival rate at time period i (or k)
μ_i	Service rate at time period i
s	Number of servers
$g(\lambda_i, \mu_i, s)$	Waiting cost at time period i as a function of arrival rate, service rate, and number of servers
K_q	Cost of waiting per time unit of waiting
$W_q(\lambda_i, \mu_i, s)$	Average waiting time per customer at time period i as a function of arrival rate, service rate, and number of servers

3.2.1 Newsboy-based model

The first model is motivated by two papers (Carr & Lovejoy, 2000; Matsuyama, 2006) that extend the classical newsboy problem into a profit optimization model for a manufacturing or service system. This model is applicable if we assume that customers seeing a fully utilized system will balk (i.e. exit the system immediately) and thus incur shortage cost. Defining a_i as revenue per demand fulfilled at time period i, B as penalty per demand unfulfilled (assume same at any time period), C as capacity (fixed at all time periods), d_i as demand at time period i, n as the number of time periods, and Z_i as profit for time period i, we have these two equations to compute for the profit in one time period.

$$Z_i = a_i C - B(d_i - C), \text{ if } d_i > C \tag{4}$$

$$Z_i = a_i d_i, \text{ if } d_i \leq C \tag{5}$$

Using a MIN function to compress the equations and summing profit for all time periods n, we get total profit:

$$Z = \sum_{i=1}^{n} [a_i d_i + (a_i + B)(\text{MIN}[C - d_i, 0])] \tag{6}$$

Revenue per time period, a_i, differs per time period because of the price reduction, r_i, which is the decision variable in our profit maximization model. Hence, we change a_i to Pr_i where P is the default price and r_i is the price reduction to be determined. Then incorporating demand shifting, d_i at each time period i becomes the sum of three components: the current original demand D_i, plus the sum of all demand that shifts from other time periods as given by the summation of $D_{ks_{ki}}(r_i)$, minus the sum of all demand that shifts into other time periods as given by the summation of $D_{is_{ik}}(r_k)$. Knowing that r_i is constrained from zero to P, we have our profit maximization model:

$$\begin{aligned} \text{Max}Z = \sum_{i=1}^{n} \big[&(P - r_i)\big(D_i + \sum_{k \neq i} D_k S_{ki}(r_i) - \sum_{i \neq k} D_i S_{ik}(r_k) bigr) \\ &+ (P - r_i + B)(\text{MIN}[C - (D_i + \sum_{k \neq i}) D_k S_{ki}(r_i) - \sum_{i \neq k} D_i S_{ik}(r_k), 0]) bigr] \\ &\text{s.t. } 0 \leq r_i \leq P \ \forall \ i = 1, 2, \cdots, n \end{aligned} \tag{7}$$

For this model, we apply the interpretation of capacity and demand as in the infinite-server approximation approach mentioned in Schwarz et al. (2016) and applied by Jennings, Mandelbaum, Massey, and Whitt (1996). We compute capacity in terms of server-hours and demand also in terms of server-hours by dividing arrival rate with service rate. For example, in a manufacturing system, if we are provided the number of resources, average arrival rate of orders (e.g. forecast from historical demand data), average service rate to fulfill those orders (e.g. production rate), the values of capacity and demand can be computed as deterministic inputs; or similarly if in a service system we are provided the number of servers, arrival rate of customers, and service rate of servers, we can compute deterministic inputs for C and D_i. Note that for this model, we assume that each time period is independent of each other. Hence, we also do not need to know the distribution of arrival rate and service rate but only the average (as in the infinite-server model). Further, we can show that if we use the demand shifting function of Equation 1 or Equation 2 (i.e. linear functions), we can transform the profit optimization model into a quadratic program with linear constraints by adding an auxiliary variable t_i to reduce the MIN function as in the transformation of maximin functions. We get:

$$\begin{aligned} \text{Max}Z = \sum_{i=1}^{n} \big[(P - r_i)\big(D_i + \sum_{k \neq i} D_k S_{ki}(r_i) - \sum_{i \neq k} D_i S_{ik}(r_k)\big) + (P - r_i + B)(t_i) \big] \\ \text{s.t. } 0 \leq r_i \leq P \ \forall \ i = 1, 2, \cdots, n \end{aligned} \tag{8}$$

$$\begin{aligned} t_i \leq C - \big(D_i + \sum_{k \neq i} D_k S_{ki}(r_i) - \sum_{i \neq k} D_i S_{ik}(r_k)\big) \ \forall \ i = 1, 2, \cdots, n \\ T_i \leq 0 \ \forall \ i = 1, 2, \cdots, n \end{aligned} \tag{9}$$

We see that if the demand shifting function, $s_{ki}(r_i)$ or $s_{ik}(r_k)$, is linear with respect to r_i and t_i, the profit optimization model is a quadratic program with $4n$ linear constraints. In general, quadratic programming is NP-hard unless we can determine the convexity of the search space. While we will not prove this mathematically, we will get some insights regarding the structure of the model in the numerical experiments that follow. It is also obvious that if we use the demand shifting function of Equation 3, the model does not become a quadratic program and is in fact quite complex.

3.2.2 Queuing-based model

The second model is based from a simple queuing optimization model such that arriving jobs/customers are served upon arrival if there is an available server or waits in queue if there are no available servers. Defining a_i as revenue per demand fulfilled at time period i, λ_i as demand at time period i, $g(\lambda_i, \mu_i, s)$ as the waiting cost which is a function of λ_i (arrival rate at time period i), μ_i (service rate at time period i), and s (number of servers which is the same at all time periods), n as the number of time periods, and Z as total profit across all time periods, we get total profit:

$$Z = \sum_{i=1}^{n} \left[a_i \lambda_i - g(\lambda_i, \mu_i, s) \right] \tag{10}$$

In this model, we similarly assume that time periods are independent of each other. This means that we assume that each time period is an independent and stationary M/M/s queuing system with arrival rate λ_i, service rate μ_i, and number of servers s. This is consistent with the stationary independent period-by-period approximation (SIPP) approach as discussed by Schwarz et al. (2016) under piecewise stationary (independent periods) approaches and applied by Green, Kolesar, and Soares (2001). This is however restricted to the case of no overlap, i.e. no queuing customers carry over to the next period and all service gets finished at the time period where it started, and no overload, i.e. the independent and stationary M/M/s queuing systems achieve steady-state so that limiting probabilities and queuing performance measures can be computed. Some approaches in literature use an Erlang loss model wherein the loss function translates to shortage cost, but in this model, we will use an M/M/s queuing system with infinite queue wherein the waiting time of customers if servers are busy translate to waiting cost. If we desire to relax these assumptions such that we allow overlap and overload, another approach discussed by Schwarz et al. (2016) under piecewise stationary (linked periods) approaches can be used stationary backlog-carryover approximation (SBCA), which is used by Stolletz (2008). This approach however is already outside the scope of this paper. For this model, we are essentially using the SIPP approach in a profit optimization model that incorporates demand shifting.
Defining a_i as Pr_i where P is the default price and r_i is the price reduction to be determined, incorporating the phenomenon of demand shifting such that λ_i becomes the sum of three components: current original demand $\wedge_i$, plus the sum of all demand that shifts from other time periods as given by the summation of $\wedge_{k S_{ki}}(r_i)$, minus the sum of all demand that shifts into other time periods as given by the summation of $\wedge_{i S_{ik}}(r_k)$, then using the linear function of waiting cost Hillier and Lieberman (1995) wherein a cost of waiting K_q is multiplied to each time unit of waiting $W_q(\lambda_i, \mu_i, s)$ and multiplied to the number of customers arriving λ_i, and finally adding the price constraint of r_i, we have our profit maximization model:

$$\begin{aligned} &\text{MaxZ} = \sum_{i=1}^{n} \left[(P - r_i)(\lambda_i) - K_q W_q(\lambda_i, \mu_i, s)\lambda_i \right] \\ &\text{s.t. } \; 0 \le r_i \le P \;\; \forall \;\; i = 1, 2, \cdots, n \end{aligned} \tag{11}$$

We define λ_i as (summation of three components as a result of demand shifting):

$$\lambda_i = \wedge_i + \sum_{k \neq i} \wedge_{k S_{ki}}(r_i) - \sum_{i \neq k} \wedge_{i S_{ik}}(r_k) \tag{12}$$

Following Hillier and Lieberman (1995), the waiting time in queue $W_q(\lambda_i, \mu_i, s)$ for an M/M/s queuing system can be computed as:

$$W_q(\lambda_i, \mu_i, s) = \frac{1}{\sum_{n=0}^{s-1} \frac{(\frac{\lambda_i}{\mu_i})^n}{n!} + \frac{(\frac{\lambda_i}{\mu_i})^s}{s!} \frac{1}{1 - \frac{\lambda_i}{s\mu_i}}} \frac{(\frac{\lambda_i}{\mu_i})^s (\frac{\lambda_i}{s\mu_i})}{s!(1 - \frac{\lambda_i}{s\mu_i})^2} \frac{1}{\lambda_i} \tag{13}$$

If we use a demand shifting function $s_{ki}(r_i)$ that is linear as in Equation 1 and Equation 2, the resulting objective function is an operation of polynomial functions depending on the number of servers s. Obviously, the same cannot be concluded if we use the multinomial function as in Equation 3. Nevertheless, we can guarantee a smooth function (except for the asymptotes) since the resulting function is just an expression of linear and exponential functions. We can get more insights regarding the structure of these models in the next section.

4. NUMERICAL EXAMPLES

In this section, we use values derived from a real-life service system. We use this set of values because it has features that describe situations of interest (i.e. equal demand in two proximal time periods, a significantly overutilized time period, a significantly underutilized time period, moderately utilized time periods). We do not have specific values however for the parameters of the demand shifting functions (e.g. γ, α, β). Thus, we are interested in the effect of these parameters in the magnitude of demand shift and ultimately in the profit. Partially knowing the mathematical structure of our models, we are also interested in the computation times and whether the found solution is a local or global maximum. We will do analysis on each combination of model (i.e. newsboy-based, queuing-based) and demand shifting function (i.e. linear with demand difference, linear with time difference, conditional logit) by using different values of the parameters. We run the models using Lingo 18.0 and use the nonlinear solver (which gives a local optimum in a relatively fast time) and the global solver (which attempts to find a global optimum) with multiple multi-start attempts. According to Lindo Systems Inc., the developer of Lingo 18.0, they have found that five multi-start attempts usually give adequate results for most models, although possibly more for larger models (Inc., 2019).

4.1 Newsboy-based model linear with demand difference demand shifting function

We use the newsboy-based model in Equation 9 to 12 and the demand shifting function in Equation 1. The inputs are $n = 7, D_1 = 25, D_2 = 25, D_3 = 11, D_4 = 7, D_5 = 28, D_6 = 52, D_7 = 2, P = 200, B = 20, C = 25$. We use different values of γ starting from the highest possible (i.e. γ is a normalizing constant so the highest value is $1/[r_{max}(D_{max} D_{min})] = 0.0001$ then in equal decrements down to zero (e.g. 0.0001, 0.000095, 0.00009, etc.) for a total of 21 points.
Using the nonlinear solver, the model is identified as a quadratic program and a local optimum is found in all runs with a small amount of iterations (i.e. < 103 iterations) and a very low run time (i.e. < 0.26 seconds). Using the global solver with 10 multi-start attempts, a global optimum is found in all runs with a larger amount of iterations in a decreasing trend (i.e. 242102 iterations for the largest γ down to 56 iterations for the smallest γ) and a longer run time also in a decreasing trend (i.e. 19.29 seconds for the largest γ down to 1.23 seconds for the smallest γ). This decreasing computational effort is explained by the lessening complexity of the profit optimization model because as γ goes down, the magnitude of demand shift weakens and thus the model does not need to explore higher values of r_i. The results of the nonlinear solver (i.e. local optima) and global solver (i.e. global optima) yield similar values for each value of γ except for five specific values which yield an insignificant discrepancy of 0.0044% on average in the objective function value. This tells us that the local optima found are practically global optima and thus for this model, there may be no need to use a global solver that requires higher computational effort.
The results of profit optimization are summarized in Figure 1 with the y-axis being percent increase in profit compared to when no price reductions are introduced and x-axis being the values of γ. As an example, if we use the highest possible value of $\gamma = 0.0001$, the model returns $r_1 = 3.33629$, $r_2 = 3.33629$, $r_3 = 32.48156$, $r_4 = 36.63501$, $r_5 = 0$, $r_6 = 0$, $r_7 = 40.63657$ as the optimal solution with a maximum profit $Z = 27562.27$ which is a 17.79% increase from the profit $Z = 23400$ if there is no price reduction and demand shifting that occur.

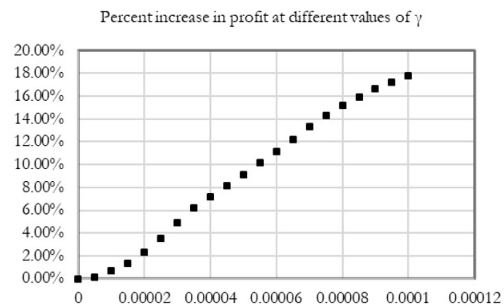

Figure 1: Sensitivity analysis for Model 4.1 (different values of γ)

From the results, we see that percent increase in profit is monotonic increasing relative to γ. Starting from the highest possible value (rightmost point) then slowly decreasing it, we find that there is a value wherein there is no increase in profit and the model returns 0 for all r_i. In our experiments, this occurs at around $\gamma = 0.0000025$ which is 2.50% of the highest possible value. At this point, the increase in profit (i.e. increase in revenue plus decrease in shortage cost) due to the demand shift does not compensate the lost revenue due to reduction in price and it is therefore better to set no price reduction.

4.2 Newsboy-based model linear with time difference demand shifting function

Similarly, we use the newsboy-based model in Equation 9 to 12 but now the demand shifting function in Equation 2. The inputs are $n = 7, D_1 = 25, D_2 = 25, D_3 = 11, D_4 = 7, D_5 = 28, D_6 = 52, D_7 = 2, P = 200, B = 20, C = 25$.

We use different values of γ starting from the highest possible (i.e. $\gamma = |ik|\min/r_{max} = 0.005$) then in equal decrements down to zero (e.g. 0.005, 0.0048, 0.0046, etc.) for a total of 26 points.
Similar to Section 4.1, using the nonlinear solver, the model is identified as a quadratic program and a local optimum is found in all runs with a small amount of iterations (i.e. < 71 iterations) and a very low run time (i.e. < 0.26 seconds). Using the global solver with 10 multi-start attempts, a global optimum is found in all runs with a larger amount of iterations also in a decreasing trend (i.e. 1025977 iterations for the largest γ down to 56 iterations for the smallest γ) and a longer run time still also in a decreasing trend (i.e. 36.52 seconds for the largest γ down to 1.36 seconds for the smallest γ). This decreasing computational effort is also explained by the lessening complexity of the profit optimization model. Also, the results of the nonlinear solver (i.e. local optima) and global solver (i.e. global optima) yield similar values for each value of γ except for two specific values which yield an insignificant discrepancy of 0.014% on average in the objective function value. This tells us that the local optima found are practically global optima and there may be no need to use a global solver that requires higher computational effort. The results of profit optimization are summarized in Figure 2 with the y-axis being percent increase in profit compared to when no price reductions are introduced and x-axis being the values of γ. As an example, if we use the highest possible value of $\gamma = 0.005$, the model returns $r_1 = 0, r_2 = 0.90701, r_3 = 21.41166, r_4 = 37.62434, r_5 = 19.65375, r_6 = 0, r_7 = 58.01627$ as the optimal solution with a maximum profit $Z = 26909.99$ which is a 15.00% increase from the profit $Z = 23400$ if there is no price reduction and demand shifting that occur.

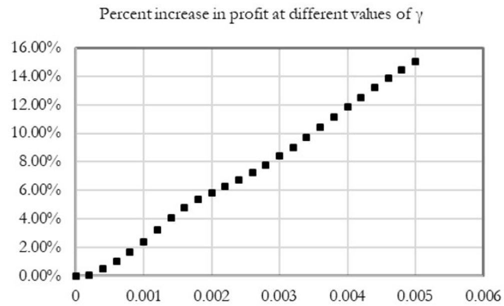

Figure 2: Sensitivity analysis for Model 4.2 (different values of γ)

From the results, we also see that percent increase in profit is monotonic increasing relative to γ. Starting from the highest possible value (rightmost point), we also find that there is a value of γ wherein there is no increase in profit and the model returns 0 for all r_i. In our experiments, this occurs at around $\gamma = 0.00012$ which is 2.40% of the highest possible value.

4.3 Newsboy-based model conditional logit demand shifting function

We again use the newsboy-based model in Equation 9 to 12 but now the demand shifting function in Equation 3. Note that in the conditional logit function, we allow shifting to the same time period (e.g. from time period 1 to 1, i.e. stay at time period 1), but we can use the same equation since the same term is produced in the positive summation and negative summation thus cancelling out. Note also that this demand shifting function has two parameters, α and β, that we cannot combine. The scale parameter μ however can be combined with these parameters for the sake of numerical example. The inputs are $n = 4, D_3 = 11, D_4 = 7, D_5 = 28, D_6 = 52, P = 200, B = 20, C = 25$. We first find realistic values of the parameters and then explore along these directions. In our experiments, we use values of α from 1 to 6 in increments of 1 and values of β from 0 to 11 in increments of 0.5 for a total of 138 points. Using the nonlinear solver, a local optimum is found in all runs with a small amount of iterations (i.e. < 173 iterations) and a very low run time (i.e. < 0.62 seconds). Using the global solver however, even with just 1 multi-start attempt, results in an impractically high number of iterations (exceeding 40 million iterations) and a very long run time (exceeding 1 hour and not yet converging) for just one point. This is explained by the complexity of using a multinomial function which tells us that it is extremely difficult to find a global optimum for this model. Thus, we instead resort to using the nonlinear solver and run each point in 5 and 10 multi-start attempts. We note that when using 5 multi-start attempts, the local optimum improves in 39 out of the 138 points, and when using 10 multi-start attempts, the local optimum improves in a total of 68 out of the 138 points (compared to when only one attempt was done). Observing that it takes a relatively small amount of iterations (i.e. < 9421 iterations) and a relatively short run time (i.e. < 7.10 seconds) to find a better local optimum when using the nonlinear solver with 10 multi-start attempts, we say that it is in fact better to use this approach for this model. The results of profit optimization are summarized in Figure 3 with the y-axis being percent increase in profit compared to when no price reductions are introduced and x-axis being the values of γ. As an example, the highest percent increase in profit attained from our experiments is achieved when $\alpha = 6, \beta = 6$, with the model returning $r_1 = 3.15398, r_2 = 3.26730, r_3 = 3.38082, r_4 = 2.47875, r_5 = 0.57595, r_6 = 0.85829, r_7 = 1.84439$ as the optimal solution with a maximum profit $Z = 34593.09$ which is a 47.59% increase from the profit $Z = 23438.64$ if there is no price reduction and demand shifting that occur. Note that when $r_i = 0$ for all time periods, the profit Z differs given different values of β due to the random utility effect of the conditional logit function such that even when there is no price reduction, every customer or unit of demand has a tendency to change their choice.

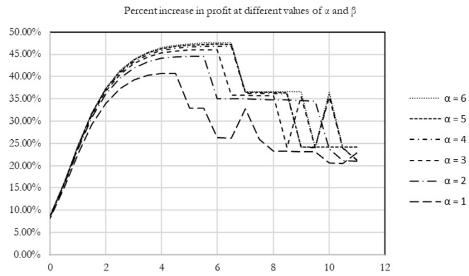

Figure 3: Sensitivity analysis for Model 4.3 (different values of α and β)

From the results of our experiments, we see that percent increase in profit is inverse parabolic with respect to β while increasing and asymptotic with respect to α. The conditional logit demand shifting function is different from the linear demand shifting function because all values of α and β are feasible, although only a certain range will produce realistic and significant results, such as the range used in our examples. Note that the values plotted are only local optima found using a nonlinear solver with 10 multi-start attempts and thus there are inconsistencies along the x-axis. While the inverse parabolic behavior with respect to β can be suspicious because there may be in fact better objective function values found if we allow a significantly longer computational time, we did a few experiments on very high values of β with excessive multi-start attempts which all find local optima that yield 0% increase in profit. But still, we cannot be sure unless we study the mathematical structure of the model. Similarly, for α, there is strong support for the monotonic increasing and asymptotic behavior as we did a few experiments in extremely high values of α in different values of β which all find local optima approaching a certain value (specifically, $Z = 35000$). Nevertheless, we have seen that it is impractically difficult to find a global optimum for this model. However, we have shown that there are values of α and β wherein profit increases as found by a nonlinear solver with multiple multi-start attempts in a relatively fast time. Also, we observe that there is a certain behavior of the objective function value depending on the values of α and β. Note however that if we will use this model in practice, we should be able to estimate the values for these parameters based on the real-life demand shifting behavior of customers.

4.4 Queuing-based model linear with demand difference demand shifting function

We now use the queuing-based model detailed in Equation 14 to 17 and the demand shifting function in Equation 1 replacing D_i with λ_i. Note however that we cannot use the same inputs because applying them to a queuing-based model with discrete-time periods will cause some queuing systems to have overload which cannot be handled by our model. Thus, we do reductions on the actual arrival rates such that all queuing systems are stationary (i.e. $\lambda/s\mu < 1$) but the overall demand pattern is retained and the resulting waiting costs due to queuing is comparable to the shortage costs if we use the newsboy-based model. This section cannot be directly compared to its counterpart in Section 4.1 because we use a different way to derive the inputs. Nevertheless, we can get insights on the structure and behavior of the profit optimization model. The inputs are $n = 7, \lambda_1 = 0.71429, \lambda_2 = 0.71429, \lambda_3 = 0.31429, \lambda_4 = 0.2, \lambda_5 = 0.8, \lambda_6 = 1.48571, \lambda_7 = 0.05714, P = 200, K_q = 120, s = 4, \mu = 0.5$. We use different values of γ starting from the highest possible then in equal decrements down to zero (e.g. 0.0035, 0.0034, 0.0033, etc.) for a total of 36 points.

Using the nonlinear solver, a local optimum is found in all runs with a small amount of iterations (i.e. < 54 iterations) and a very low run time (i.e. < 0.28 seconds). Using the global solver with 10 multi-start attempts, a global optimum is found in all runs with a larger amount of iterations (i.e. < 52644 iterations) and a longer run time (i.e. < 4.68 seconds) not seeing the decreasing trend as in Section 4.1. The results found by the nonlinear solver (i.e. local optima) and global solver (i.e. global optima) yield the same values for all values of γ. This tells us that in maximizing the objective function value, there is no need to use a global solver that requires higher computational effort.

The results of profit optimization are summarized in Figure 4 with the y-axis being percent increase in profit compared to when no price reductions are introduced and x-axis being the values of γ. As an example, if we use the highest possible value of $\gamma = 0.0035$, the model returns $r_1 = 0, r_2 = 0, r_3 = 17.67099, r_4 = 23.43266, r_5 = 0, r_6 = 0, r_7 = 28.89106$ as the optimal solution with a maximum profit $Z = 794.6131$ (which translates to 27811.46 if we revert the reduction) which is an 18.82% increase from the profit $Z = 668.7557$ (which similarly translates to 23406.45) if there is no price reduction and demand shifting that occur.

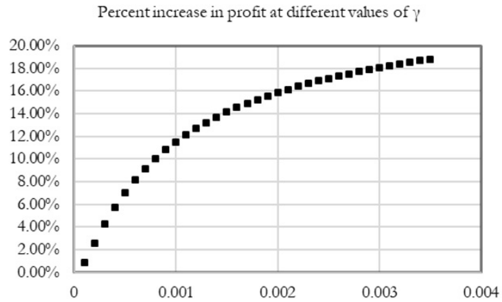

Figure 4: Sensitivity analysis for Model 4.4 (different values of γ)

From the results, we see that percent increase in profit is monotonic increasing relative to γ. Starting from the highest possible value (rightmost point) then slowly decreasing it, we also find a value of γ wherein there is no increase in profit and the model returns 0 for all r_i. In our experiments, this occurs at around $\gamma = 0.00001$ which is 0.29%of the highest possible value of γ. While we cannot prove mathematically, we may have hints that the local maxima found are actually global maxima since there are no inconsistencies that occur (i.e. the graph of sensitivity analysis is smooth) and the results found by the nonlinear solver and global solver are exactly the same.

4.5 Queuing-based model linear with time difference demand shifting function

Similarly, we use the queuing-based model detailed in Equation 14 to 17 but now the demand shifting function in Equation 2. As in Section 4.4, we do reductions on actual arrival rates to get queuing systems with no overload. Then, we do sensitivity analysis on the parameter γ. The inputs are $n = 7, \lambda_1 = 0.71429, \lambda_2 = 0.71429, \lambda_3 = 0.31429, \lambda_4 = 0.2, \lambda_5 = 0.8, \lambda_6 = 1.48571, \lambda_7 = 0.05714, P = 200, K_q = 120, s = 4, \mu = 0.5$. We use different values of γ starting from the highest possible then in equal decrements down to zero (e.g. 0.005, 0.0049, 0.0048, etc.) for a total of 51 points.

Using the nonlinear solver, a local optimum is found in all runs with a small amount of iterations (i.e. < 62 iterations) and a very low run time (i.e. < 0.21 seconds). Using the global solver with 10 multi-start attempts, a global optimum is found in all runs with a larger amount of iterations (i.e. < 1834104 iterations) and a longer run time (i.e. < 99.0 seconds) with a decreasing trend same as in Section 4.2 but more pronounced and abrupt. The results found by the nonlinear solver (i.e. local optima) and global solver (i.e. global optima) yield the same values for all values of γ and thus there is no need to use a global solver that requires higher computational effort.

The results of profit optimization are summarized in Figure 5 with the y-axis being percent increase in profit compared to when no price reductions are introduced and x-axis being the values of γ. As an example, if we use the highest possible value of $\gamma = 0.005$, the model returns $r_1 = 0, r_2 = 0, r_3 = 9.30721, r_4 = 18.51455, r_5 = 5.33952, r_6 = 0, r_7 = 39.47662$ as the optimal solution with a maximum profit $Z = 784.9902$ (which translates to 27474.66 if we revert the reduction) which is a 17.38% increase from the profit $Z = 668.7557$ (which similarly translates to 23406.45) if there is no price reduction and demand shifting that occur.

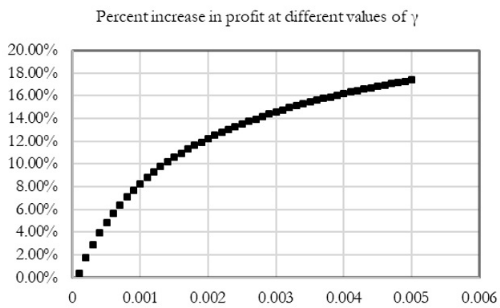

Figure 5: Sensitivity analysis for Model 4.5 (different values of γ)

From the results, we see that percent increase in profit is monotonic increasing relative to γ. We also find a value of γ wherein there is no increase in profit and the model returns 0 for all r_i. In our experiments, this occurs at around $\gamma = 0.00004$ which is 0.80% of the highest possible value of γ.

4.6 Queuing-based model conditional logit demand shifting function

Finally, we use the queuing-based model detailed in Equation 14 to 17 and the demand shifting function in Equation 3. We do the same reduction on actual arrival rates to get queuing systems with no overload. The inputs are $n = 7, \lambda_1 = 0.71429, \lambda_2 = 0.71429, \lambda_3 = 0.31429, \lambda_4 = 0.2, \lambda_5 = 0.8, \lambda_6 = 1.48571, \lambda_7 = 0.05714, P = 200, K_q = 120, s = 4, \mu = 0.5$. We first find realistic values of the parameters and then explore along these directions. In our experiments, we use values of α from 0.5 to 3 in increments of 0.5 and values of β from 0.1 to 3.1 in increments of 0.1 for a total of 186 points. Note however that a few values within the range either resulted in an endless number of iterations or yielded infeasibilities or unbounded solutions even with multiple multi-start attempts and were thus omitted. This might be because of the complex mathematical structure of an M/M/s waiting time function used with a multinomial function.

Using the nonlinear solver, a local optimum is found in all runs with a slightly larger amount of iterations (i.e. < 543 iterations) yet still a very low run time (i.e. < 0.60 seconds). Similar to Section 4.3, using the global solver results in an impractically high number of iterations and a very long run time for just one point. Thus, we resort to using the nonlinear solver and run each point in 5 and 10 multi-start attempts, noting also that better local optima are found in some of the points if we do more attempts. Observing that it takes a practically moderate amount of iterations (i.e. < 524554 iterations) and a practically short run time (i.e. < 22.69 seconds) to find a better local optimum when using the nonlinear solver with 10 multi-start attempts, we say that it is in fact better to use this approach for this model.

The results of profit optimization are summarized in Figure 6 with the y-axis being percent increase in profit compared to when no price reductions are introduced and x-axis being the values of γ. As an example, the highest percent increase in profit attained from our experiments is achieved when $\alpha = 3, \beta = 1.2$, with the model returning $r_1 = 1.85415, r_2 = 1.80536, r_3 = 1.99025, r_4 = 1.78914, r_5 = 0, r_6 = 1.38004, r_7 = 1.76030$ as the optimal solution with a maximum profit $Z = 1011.91$ (which translates to 35416.89 if we revert the reduction) which is a 20.01% increase from the profit $Z = 843.1775$ (which similarly translates to 29511.21) if there is no price reduction and demand shifting that occur.

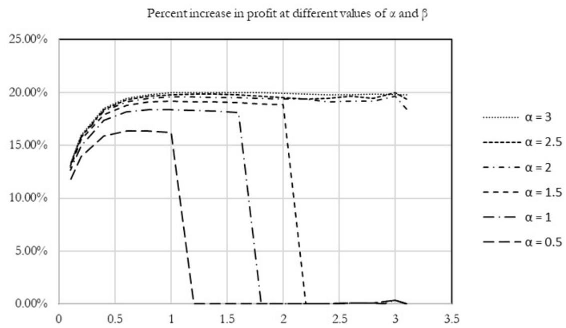

Figure 6: Sensitivity analysis for Model 4.6 (different values of α and β)

From the results of our experiments, we get a hint that percent increase in profit is also inverse parabolic with respect to β while increasing and asymptotic with respect to α. Note that for the sake of numerical experiments, we use this range of values because it lets us see the behavior of the function in practically manageable run times. It seems that the mathematical structure of this model is far more complex that there are a few values of α and β inside the range wherein even a local optimum cannot be found.

Nevertheless, we see similar trends to the effect of changing the values of α and β as in the previous models. We also see that it is impractically difficult to find a global optimum for this model and thus base only on local optima that seem to converge to the global optimum. Similarly, we have shown that there are values of α and β wherein profit increases as found by a nonlinear solver with multiple multi-start attempts in relatively short run times.

5. DISCUSSION

From the numerical experiments in Section 4.1, 4.2, 4.4, and 4.5, we find that percent increase in profit is always monotonic increasing with respect to the parameter γ. Thus, if we start from its highest possible value and slowly decrease, we always find a value of γ wherein the optimal solution is $r_i = 0$ for all time periods which implies that setting no price reductions is the best decision. This is explained by the fact that at a significantly low value of γ, the demand shift due to price reduction is weak such that the opportunity for more revenue and shortage cost reduced do not compensate for the loss in revenues due to lower prices. As for the parameters α and β, as seen in Section 4.3 and 4.6, percent increase in profit is monotonic increasing with respect to α while having an inverse parabolic relationship with β. Unlike the linear demand shifting functions, the values of α and β do not have limits (i.e. any value would give a realistic utility function), but objective function values found along different values of α seem to converge while those found along different values of β seem to have a best possible value. But while we know the behavior of these parameters, in real-life they are not chosen. They should be estimated given the real-life demand shifting behavior of customers. In this study, we performed numerical experiments for their different values since we do not know their actual values and to at least show that there are values wherein profit can be improved.

We created two models for profit optimization the newsboy-based model and the queuing-based model. In the comprehensive review of Schwarz et al. (2016), they are regarded as different approaches with different applications. Perhaps the best way to determine which model to use is by determining which model best approximates the situation in real life. While the newsboy-based model approximates backlog as shortage cost (i.e. customers balk upon seeing a fully utilized system) and the queuing-based model approximates waiting cost but without overload (i.e. customers wait in line upon seeing a fully utilized system), we should choose the more appropriate model based on the actual situation. There are other approaches that can be used such as using an Erlang loss model and the SBCA approach used by Stolletz (2008) but they shall be topics for another paper.

We used three demand shifting functions linear with demand difference, linear with time difference, and conditional logit with two factors. The major disadvantage of using a linear demand shifting function is that they can be limited in modelling the actual behavior of demand shift. Also, the values allowable for the parameter γ can have restrictions (i.e. they can return erroneous results if outside range) so the parameters must be determined carefully and model checked if the case still reflects real-life behavior. Comparing Model 4.1 and Model 4.2, we see that at the highest possible value of γ, the percent increase in profit is lower if we use the demand shifting function with time difference, obviously because of its negative effect. However, this can also be explained by the fact that we now use a demand shifting function that considers proximity of time periods, which is in fact shown by the different optimal values of r_1 and r_2 in Model 4.2 even though D_1 and D_2 have the same initial demand (i.e. r_2 is given a higher value than r_1 since shifting is stronger to a nearer time period than to a farther time period, while in Model 4.1 they have equal values). On the other hand, the conditional logit demand shifting function can be more realistic since it is derived from the well-known random utility theory. But due to this randomness, even when there is no price reduction introduced (i.e. $r_i = 0$ for all time periods), there is demand shifting that happens. This is not a big problem, but it may cause some inconsistencies if used in practical situations.

Models 4.1, 4.2, 4.4, and 4.5 may actually be finding the global optimum even if only a single-attempt nonlinear solver is used as shown in the numerical experiments because the local optima found are either very near (i.e. discrepancy is insignificant) or exactly the same as with the global optima found. While we do not prove it mathematically in this paper, we may infer about the possible convexity of these nonlinear programs. Model 4.3 and Model 4.6 obviously do not find the global optimum given the inconsistencies in the numerical experiments and the complex mathematical structure due to the multinomial logit function. All models however have found local optima (i.e. profit improvement) in practically fast run times.

When applied in a real-life manufacturing or service system, we must remember that r_i represent price discounts, usually seen as the percent reduction in default price. This is done so that customers seeing this price discount will change their behavior in our paper, their time of consumption along time periods. There are many papers in literature (e.g. those mentioned in Section 3.1) that assume and explain how this happens, but the difficulty is always in estimating the parameters. Without a huge amount of data, one approach that can be used to estimate the parameters is to implement the price discounts in real life and observe the magnitude of demand shifting that happens. Then, we can find the value of γ or α and β most suitable to represent this level of demand shifting.

6. CONCLUSION AND FURTHER STUDY

We presented six profit optimization models that incorporate demand shifting for the time-varying capacity-demand imbalance problem faced by manufacturing and service systems. We summarize the models in the table below:

Table 3: Summary of profit optimization models.

Model name:	Approach (balk or wait):		Behavior of demand shift:		Factors that affect demand shift:		
	Newsboy based	Queuing based	Linear model	Conditional logit model	Price reduction	Demand difference	Time difference
Model 4.1	✓		✓		✓	✓	
Model 4.2	✓		✓		✓		✓
Model 4.3	✓			✓	✓		✓
Model 4.4		✓	✓		✓	✓	
Model 4.5		✓	✓		✓		✓
Model 4.6		✓		✓	✓		✓

The models vary in the demand shifting function used (i.e. linear or conditional logit; and the factors that affect demand shift: price, relative demand, time difference) and their application (i.e. newsboy-based if we assume customers are lost, queuing-based if we assume customers wait in line). Regarding practical use, we performed numerical experiments by using inputs derived from data of a real-life service system and changed the values of the parameters to see their effect. As a summary of our findings, we have:

- For all models, there are values of the parameters wherein profit improvement can be found and computed in practically short run times
- For the linear models, the profit improvement found are also the global maxima
- For the conditional logit models, the profit improvement found are not necessarily global maxima
- For the linear models, profit improvement is monotonic increasing with respect to γ, i.e. there is a low value of γ wherein there is no possible profit improvement and optimal price reduction is zero
- For the conditional logit models, profit improvement is seen to be monotonic increasing and converging with respect to α while inverse parabolic with respect to β

In general, when we are to use these models in a real-life setting, we should first choose the approach to use (i.e. newsboy-based or queuing-based) depending on the situation of the manufacturing or service system mainly based on whether customers balk/leave immediately or wait in line upon seeing a fully utilized system. Then, we should estimate the parameters based on actual customer shifting behavior while also determining if a linear or a conditional logit function fits this behavior better as well as decide which factors to include in the model. Finally, using data to fill in values for the inputs, we can run the appropriate profit optimization model and determine optimal price settings.

In the future, the mathematical structure of all models shall be further investigated and possibly prove the convexity of the quadratic programs of Models 4.1 and 4.2 as well as the nonlinear programs of Models 4.4 and 4.5 and find structural patterns in the complex nonlinear programs of Models 4.3 and 4.6. Superior techniques for searching better objective function values shall be done for the conditional logit models such as the use of metaheuristics or soft computing methods. This may also require studying other types of datasets such as those with higher variability of demand or more time periods and see the effects to profit improvement and computational effort required. We may as well include other factors of demand shifting that will make the functions more complex as we increase the number of parameters. Better yet, we shall try to gather real-life data with actual demand shifting in order to perform empirical estimation of these parameters. Aside from linear and conditional logit, we may also use other functions such as higher polynomials or those mentioned in Strauss et al. (2018), whichever is most appropriate for practical use. An interesting question raised by Stolletz (2008) is how large or small the time periods should be as divided from the planning horizon (e.g. 15-minutes, 5-minutes) which also warrants further investigation. Most interestingly, the stationary backlog-carryover approximation (SBCA) approach done by Stolletz (2008) may be a more accurate approximation of time-varying queuing systems since it considers overlap and overload and shall thus be incorporated in the future.

REFERENCES

Burris, M. W., & Pendyala, R. M. (2002). Discrete choice models of traveler participation in differential time of day pricing programs. *Transport Policy*, *9*(3), 241-251.

Carr, S., & Lovejoy, W. (2000). The inverse newsvendor problem: Choosing an optimal demand portfolio for capacitated resources. *Management Science*, *46*(7), 912-227.

Chao, X., & Zhou, S. X. (2006). Joint inventory-and-pricing strategy for a stochastic continuous-review systems. *IIE Transactions*, *38*(5), 401-408.

Chen, M., & Chen, Z.-L. (2015). Recent developments in dynamic pricing research: Multiple products, competition, and limited demand information. *Production and Operations Management*, *25*(4), 704-731.

Chen, X., & Simchi-Levi, D. (2004). Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Mathematics of Operations Research*, *29*(3), 698-723.

Faria, P., & Vale, Z. (2011). Demand response in electrical energy supply: An optimal real time pricing approach. *Energy*, *36*(8), 5374-5384.

Green, L. V., Kolesar, P. J., & Soares, J. (2001). Improving the sipp approach for staffing service systems that have cyclic demands. *Operations Research*, *49*(4), 549-564.

Hillier, F. S., & Lieberman, G. J. (1995). *Mcgraw-hill science, engineering and mathematics*. New York, United States: Introduction to Operations Research.

Inc., L. S. (2019). Multi-start solver. *LINGO 18.0 Users Manual*.

Ingold, M.-B. U., A., & Yeoman, L. E. (2000). *Thomson learning*. London, United Kingdom: Yield Management Strategies for the Service Industries Second Edition.

Jennings, O. B., Mandelbaum, A., Massey, W. A., & Whitt, W. (1996). Server staffing to meet time-varying demand. *Management Science*, *42*(10), 1383-1394.

Joe-Wong, C., Sen, S., Ha, S., & Chiang, M. (2012). Optimized day-ahead pricing for smart grids with device-specific scheduling flexibility. *IEEE Journal on Selected Areas in Communication*, *30*(6), 1075-1085.

Klein, R., Koch, S., Steinhardt, C., & Strauss, A. K. (2019). A review of revenue management: Recent generalizations and advances in industry applications. *European Journal of Operations Research.*.

Kobayashi, K., Maruta, I., Sakurama, K., & Azuma, S. (2014). Modeling and design of real-time pricing systems based on markov decision processes. *Applied Mathematics*, *5*, 1485-1495.

Matsuyama, K. (2006). The multi-period newsboy problem. *European Journal of Operations Research*, *171*(1), 170-188.

McAfee, R. P., & Te Velde, V. (2006). *Dynamic pricing in the airline industry*. Handbook on Economics and Information Systems, Ed: TJ Hendershott, Elsevier.

Schwarz, J. A., Selinka, G., & Stolletz, R. (2016). Performance analysis of time-dependent queuing systems: Survey and classification. *Omega*, *63*, 170-189.

Selmi, N. (2010). Effects of culture and service sector on customers perceptions of the practice of yield management. *International Journal of Marketing Studies*, *2*(1), 245-253.

Shen, Z.-J. M., & Su, X. (2007). Customer behavior modeling in revenue management and auctions: A review and new research opportunities. *Production and Operations Management*, *16*(6), 713-728.

Stolletz, R. (2008). Approximation of the non-stationary m(t)/m(t)/c(t)-queue using stationary queuing models: The stationary backlog-carryover approach. *European Journal of Operations Research*, *190*, 478-493.

Strauss, A. K., Klein, R., & Steinhardt, C. (2018). A review of choice-based revenue management: Theory and methods. *European Journal of Operations Research*, *271*(2), 375-387.

Susskind, A. M., Reynolds, D., & Tsuchiya, E. (2004). An evaluation of guests' preferred incentives to shift time-variable demand in restaurants. *Cornell Hotel and Restaurant Administration Quarterly*, *45*(1), 68-84.

Whitin, T. M. (n.d.). Inventory control and price theory. *Management Science*, *2*(1), 61-68.