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Generalized Sen-Coherence and Existence of Preferred With Probability At Least Half Winners

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Abstract: In this paper, we provide two sufficient (though not necessary) conditions under which for a given profile of state-dependent preferences (linear orders) on a given and fixed set of alternatives, the set of alternatives which are individually preferred to all alternatives other than itself with probability at least half, is non-empty. We thereby extend the domain of decision making under uncertainty from state-dependent utility functions to the domain of state-dependent rankings whose informational requirements are considerably more parsimonious. The first sufficient condition-valid under a mild asymmetry condition- is the well-known Sen Coherence. The second sufficient condition which is valid unconditionally is a generalisation of Sen Coherence. It says that there exists at least one alternative which is never at the end of a particular structure (sigma) associated with a preference profile. The scientific contribution of this second property-apart from its novelty- as we show by means of two examples is that it is applicable even when Sen Coherence is not.
Keyword — state-dependent preferences, preferred with probability at least half, fixed set of alternatives

1. INTRODUCTION

In this paper, we are concerned with choosing an alternative from a given non-empty finite set of alternatives, when the material consequence of the chosen alternative is realized only after the uncertain future state of nature has been revealed. The preferences of the decision maker are given by state dependent rankings of alternatives, and information about the future that is known to the decision maker is contained in a probability distribution over the non-empty finite set of states of nature, that is revealed only after the choice has been made. This framework of analysis is discussed in Lahiri (2019). This framework extends the Arrowian model of multi criteria decision theory which along with significant and original theoretical contributions is available in Sen (1970). For the classical theory of decision making under uncertainty in the state dependent case - which is the other and major motivation behind this paper- one may refer to Karni (1985). Karni (1985) and Sen (1970) comfortably surpass the prerequisites related to decision making that is required to be able to understand the framework of analysis and results presented here. State dependent utility functions from a more advanced perspective is discussed in chapter 8 of Gilboa (2013). However, the classical theory of decision making under uncertainty that rests on the assumption of maximization of expected utility (state-dependent or not) has two important limitations. First, expected utility maximization has often failed to be consistent with observed human behaviour in situations involving risk (i.e. uncertainty with probabilistic information about all states of nature available to or plausibly attributable by the decision maker) as was shown in the seminal work of Maurice Allais- also known as Allais paradox (see Allais (1953)). The second reason is that the decision maker's preferences may not be available in the form of cardinal utility functions, but only as rankings. That leads to a departure from the classical theory and opens up the possibility of decision makers using other algorithms (decision aids) for the purpose of decision making under risk. One such procedure is what we are concerned with in the paper. The reasons for our interest in state-dependent preferences are precisely the same as the ones discussed in Karni (1985), i.e. it is so obviously true that it does not need justification beyond citing trivial day-to-day examples as Karni has done in his book.

A very simple and mundane example could be used to illustrate what we are really concerned with here. Consider a person who before going out to work in the morning has to choose between a cap and a rain-coat to carry with himself. Before going out to work the sky appears to be clear but the weather could change when he has to return from work in the evening. There are three different possible states of nature that he may have to reckon with while returning

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from work: sunshine, rain or cloudy skies without rain. In the first and third states of nature, he prefers a cap to a raincoat, and in the second state of nature his preferences are opposite.

The first thing to notice regarding this example, is that since the only information that is available are rankings, expected utility as a solution concept is completely useless, regardless of whether the person has probabilistic information about the weather in the evening or not. In case the person does have probabilistic information about the weather that he may have to confront in the evening, a reasonable rule for him to apply would be the "probabilistic plurality" or what may also be called "the most likely best alternative" rule, i.e. choose the alternative that is ranked first with the highest probability. If all three states of nature are equally likely, then the probability that he would prefer a cap to a rain-coat is greater than the probability that he would prefer a raincoat to a cap and hence would make his choice accordingly. However, is there any justification to assume that all three states of nature are equally likely through-out the year? During the peak of monsoon the chances of a day being rainy far exceeds the probability of the weather being otherwise and then the right thing to do for the decision maker who wants to use the probabilistic plurality rule is to take the rain-coat and not the cap, to work.

The interesting thing about the probabilistic plurality or the most likely best alternative rule in the above example is that it leads to a choice which is at least as likely to be preferred to an alternative than the latter is to it, for all other alternatives, i.e. "*preferred with probability at least half*" to any other alternative. This happens simply because there are only two alternatives to choose from. If there are more than two alternatives, then the consequences of using "the most likely best alternative" rule may be very uncomfortable. This can be seen if we consider a modified version of the above example by including the third option of taking neither of the two items mentioned above. If the weather is sunny the decision maker ranks cap first, neither cap nor raincoat second and raincoat last. If the weather is cloudy then the decision maker ranks neither cap nor raincoat first, cap second and raincoat last. If the weather is rainy then the decision maker ranks the raincoat first, cap second and neither cap nor raincoat last.

Now suppose the weather forecast for the day predicts a 34% chance of rain, 33% chance of cloudy skies and 33% chance of a sunny day. If the decision maker used "the most likely best alternative" rule, he would be taking a raincoat to work which regretfully has 66% chance of being his least preferred alternative for the day. The alternative that is preferred with probability at least half to any other alternative is the cap, which has a 66% chance of being preferred to a raincoat and 67% chance of being preferred to taking neither a cap nor a raincoat. Makes sense too! After all, a cap may protect one's head from being drenched in the rain, which is better than not having either a raincoat or a cap. If we do agree that choosing an alternative that is preferred with probability at least half to all other alternatives – a "*preferred with probability at least half winner*" - is a desirable decision making procedure, then the question that arises is that does such an alternative always exist? The answer to this question is in the negative and the well known Condorcet paradox with three states of nature, where all states of nature are equally likely, provides the disagreeable answer to the question. In our case with the three alternatives mentioned above, if for some reason the decision maker's preferences on a cloudy day was neither a cap nor a raincoat first, raincoat second (it may rain?) and a cap last, with the rankings in the other states of nature as before, then with all three states of nature being equally likely, the probability that the decision maker prefers a raincoat to a cap is two-thirds, the probability that a cap is preferred to neither a raincoat nor a cap is two-thirds and the probability that neither a cap nor a raincoat is preferred to a raincoat is two thirds. Hence there is no alternative that is a "preferred with probability at least half winner".

Note that in this case, we do not appeal to empirical observations which may or may not be easy to explain by logical reasoning. In decision making under uncertainty with state dependent rankings of alternatives, choosing an alternative that is ranked higher than another alternative with probability at least half where the latter can be any other alternative, appears to be a plausible algorithm that the decision maker may use.

The main result here, provides a sufficient (though not necessary) condition under which for a given profile of linear orders (strict rankings- one for each state of nature) on a given (and fixed) set of alternatives, there exists a "preferred with probability at least half winner", regardless of the probability distribution with which the various states of nature occur. So far all research on this topic has been restricted to the situation where all states of nature are equally likely (equiprobable states of nature). We extend the investigation to situations, where the probabilities with which the states of nature are realized may differ across states of nature.

The starting point of this line of research with equiprobable states of nature is the work of Black (1948). Subsequently, Inada (1964) provided an analysis similar in spirit but different in content to the one provided in Black (1948). The results in both the papers were substantially generalized by Sen (1966) for the case where the number of states of nature is odd. All three are concerned with sufficient conditions that lead to a transitive "preferred with probability at least half" relation. Our first result here shows that the property invoked by Sen in his 1966 paper to show the transitivity of the "preferred with probability at least half" relation when all states of nature are equiprobable, continues to imply the transitivity of the relation for arbitrary probability distributions and without any restriction on the cardinality of the finite number of states of nature, provided the said relation is asymmetric. Asymmetry of the "preferred with probability at least half" relation does not require that the number of states of nature be odd and that the probabilities of occurrence the states nature be equal- it can be asymmetric even with two states of nature with one having higher probability of occurrence than the other.

A significantly more contemporary investigation of this line of research, carried out in the framework of judgment aggregation of which preference aggregation with equiprobable states of nature is a special case, as well as citations to research concerned with the transitivity of the "preferred with probability at least half" relation but which approaches the problem differently from what Black, Inada and Sen did, is reported in Dietrich and List (2010). We will have occasion to discuss the related literature at relevant points as our discussion unfolds. In a related investigation due to Kramer (1973) it is shown that when all states of nature are equiprobable, the set of possible alternatives is some appropriately defined multi-dimensional commodity or policy space and the state-dependent preferences "can be represented by quasi-concave, differentiable utility functions, the various equilibrium conditions for" the existence of a preferred with probability at least half winner" are incompatible with even a very modest degree of heterogeneity of tastes across states of nature, and for most purposes are probably not significantly less restrictive than the extreme condition of complete unanimity" of state-dependent preferences.

Contrary to issues related to the transitivity or for that matter acyclicity of the "preferred with probability at least half" relation, we show that the satisfaction of a property which we call Generalized Sen Coherence, guarantees the existence of a preferred with probability at least half winner. It says that there exists an alternative which is preferred with probability half to all other alternatives if there exists at least one alternative which is never at the end of a particular structure (sigma) associated with a preference profile. An alternative is said to be at the end of a sigma associated with a preference profile if it is the first alternative in a finite sequence of distinct alternatives with an equal number of state-dependent rankings (not necessarily distinct) such that the first three of the sequence are ranked in the reverse order by the first ranking, each subsequent ranking of a subsequent triplet reduces the ranks of the first and second alternatives in the previous ranking by one, and stops once there is a repetition of an alternative. We show by means of examples that Generalized Sen Coherence is satisfied and thus achieves the limited purpose of obtaining a global "preferred with probability at least half winner" even when the properties which lead to the transitivity or acyclicity of the "preferred with probability at least half" relation are not satisfied.

In comparison to Sen Coherence which is a requirement that all triplets of distinct alternatives have to satisfy, Generalized Sen Coherence is a requirement on at least one alternative, but in the context of all subsets of three or more alternatives containing it. It is for this reason, if not for any other that Generalized Sen Coherence is as plausible and perhaps more so in the present context than the existing properties which guarantee the existence of "preferred with probability at least half winners".

2. MODEL

The motivation for the following discussion and the proposition thereafter comes from Sen (1966). There is a lucid discussion of the same, in Taylor (1995).

Consider a decision maker (DM) faced with the problem of choosing one or more alternatives from a non-empty finite set of alternatives X containing at least three alternatives.

If R is a binary relation on X (i.e. $R \subset X \times X$), then whenever $(x, y) \in R$ where $x, y \in X$, we write this simply as xRy. Given a binary relation R on X, let P denote the asymmetric part of R (i.e. xPy if and only if xRy and not$[yRx]$) and I denote the symmetric part of R (i.e. xIy if and only if xRy and yRx). Further, let $G(X, R) = \{x \in X | xRy \text{ for all } y \in X\}$. $G(X, R)$ is said to be the set of **greatest** alternatives in the set X with respect to the binary relation R. It is quite possible that $G(X, R)$ is empty.

Let $B(X, R) = \{x \in X | xRy \text{ for all } y \in X \setminus \{x\}\}$. $B(X, R)$ is said to be the set of **best** alternatives in the set X with respect to the binary relation R. It is easy to see that $G(X, R) = B(X, R) \cup \{x \in B(X, R) | xRx\}$. Hence if R is reflexive, then $G(X, R) = B(X, R)$.

For a positive integer $n \geq 3$, let $N = \{1, 2, \cdots, n\}$ denote the set of sets of nature. The satisfaction from the chosen alternative is realized only after the state of nature reveals itself.

A **strict preference relation/strict ranking** on X is a reflexive, complete/connected/total, transitive and anti-symmetric binary relation (linear order) on X. If for $x, y \in X$, it is the case that xRy then we will say that x is **at least as good as** y. Similarly P is interpreted as x is **strictly preferred** to y, and xIy is interpreted as **there is indifference between** x and y. Given that a strict preference is a linear order, indifference between two alternatives is possible if and only if the two alternatives are identical.

Let $\mathcal{L}$ denote the set of all strict preference relations on X.

Note: Trivial but important observation that can be made at this point is that if R is a strict preference relation/strict ranking, then

(i) for all $x \in X$, $\{y \in X | yRx\} = \{x\} \cup \{y \in X | yPx\} = \{x\} \cup \{y \in X \setminus \{x\} | yRx\}$;

(ii) for all $x \in X$, $\{y \in X | xRy\} = \{x\} \cup \{y \in X | xPy\} = \{x\} \cup \{y \in X \setminus \{x\} | xRy\}$

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A **preference profile** denoted R_N is a function from N to $\mathcal{L}$. R_N is represented as the array $< R_i | i \in N >$, where R_i is the strict preference relation/strict ranking of the DM in state of nature i. The set of all preference profiles is denoted $\mathcal{L}^N$.

The DM's beliefs or assessments about the possibility of the various states of nature being realized is summarized by a probability distribution, i.e. $p \in \mathbb{R}^N_+$ such that $\sum_{i=1}^{N} p_i = 1$. Let P^N denote the set of all probability distributions on N.

Given a pair (R_N, p), define the **preferred with probability at least half (PPALH) relation** $R^\#(R_N, p)$ on X as follows: for all $x, y \in X$ with $x \neq y$, $[xR^\#(R_N, p)y$ if and only if $\sum_{\{i \in N | xR_iy\}} p_i \geq \sum_{\{i \in N | yR_ix\}} p_i$ i.e. $\sum_{\{i \in N | xR_iy\}} p_i \geq \frac{1}{2}]$. It is clear from the definition of $R^\#(R_N, p)$, that this binary relation is irreflexive. However, it is complete, i.e. for all $x, y \in X$, with $x \neq y$, either $xR^\#(R_N, p)y$ or $yR^\#(R_N, p)x$ (and possibly sometimes both!).

From the note above (immediately) after the definition of strict rankings it is clear that for all $i \in N$ and $x, y \in X$ with $x \neq y$: xR_iy if and only if $xP(R_i)y$.

If $xR^\#(R_N, p)y$, then we say that x is **preferred with probability at least half to** y. Since n may be even with $p_i = \frac{1}{n}$ for all $i \in N$, the inequality need not be strict.

When there is no scope for confusion we will write $R^\#$ instead of $R^\#(R_N, p)$.

The asymmetric part of $R^\#$ denoted $P(R^\#)$ is called the **with probability greater than half relation**.

When there is no scope for confusion we will write $P^\#$ instead of $P(R^\#)$.

Note that, $xP(R^\#)y$ if and only if $\sum_{\{i \in N | xR_i \in y}\}p_i > \sum_{\{i \in N | yR_ix\}} p_i$. The latter naturally implies that $x \neq y$.

$xP(R^\#)y$ then we say that x is **preferred with probability greater than half to** y.

We want to obtain condition(s) under which $B(X, R^\#)$ is non-empty. An alternative in $B(X, R^\#)$ is called a **preferred with probability greater than half winner (PPALHW)**.

Note: If in the definition of $R^\#(R_N, p)$, we had dropped the requirement that $x \neq y$, then by the reflexivity of strict rankings we would have got $xR^\#x$ and then as we pointed out earlier $B(X, R^\#)$ would be equal to $G(X, R^\#)$. However "an alternative is preferred to itself with probability at least half" is an absurd statement and linguistically jarring. Hence although our results would remain unaffected, we decided to introduce the caveat $x \neq y$.

3. SEN COHERENCE AND TRANSITIVITY OF PPALH RELATION

Transitivity of the PPALH relation clearly implies that there exists a PPALHW. In Taylor (1995) there is a property on preference profiles referred to as Sen Coherence, which Sen (1966) showed implies the transitivity of the PPALH relation when the number of states of nature is odd and all are equally likely. In Sen (1966), the property we call Sen Coherence is referred to as value restriction. In Elsholtz and List (2005) and Dietrich and List (2010), Sen Coherence is called triple-wise value restriction.

A preference profile R_N is said to satisfy **Sen Coherence** if given any three distinct alternatives $x, y, z \in X$, there exists an alternative $w \in \{x, y, z\}$ such that either

(i) w is never ranked first among $\{x, y, z\}$; or

(ii) w is never ranked second among $\{x, y, z\}$; or

(iii) w is never ranked third among $\{x, y, z\}$.

Here we show that if a preference profile R_N satisfies Sen Coherence then for all $p \in P^N$, the corresponding PPALH relation is transitive, provided it is asymmetric.

Proposition 1: Let R_N satisfy Sen Coherence and $p \in P^N$. Suppose $R^\#(R_N, p)$ is asymmetric. Then, $R^\#(R_N, p)$ is transitive.

Proof: Suppose R_N satisfies Sen Coherence, $p \in P^N$ and $R^\#(R_N, p)$ is asymmetric.

We know that $R^\#$ is complete. Hence towards a contradiction suppose it is not transitive so that, there exists distinct $x, y, z \in X$, such that $xR^\#y$, $yR^\#z$ but not $xR^\#z$. By completeness of $R^\#$ it must be that $zR^\#x$.

Hence we have $xP^\#y$, $yP^\#z$ and $zP^\#x$. $[xP^\#y$ and $yP^\#z]$ if and only if $[\sum_{\{i \in N | xR_iy\}} p_i > \frac{1}{2}$ and $\sum_{\{i \in N | yR_iz\}} p_i > \frac{1}{2}]$. If whenever xP_hy, it is the case that zP_hy, then, $\sum_{\{i \in N | zR_iy\}} p_i > \frac{1}{2} > \sum_{\{i \in N | yR_iz\}} p_i$ leading to a contradiction.

Hence there exists $h \in N$ such that xP_hyP_hz. $[yP^\#z$ and $zP^\#x]$ if and only if $[\sum_{\{i \in N | yR_iz\}} p_i > \frac{1}{2}$ and $\sum_{\{i \in N | zR_ix\}} p_i > \frac{1}{2}]$.

Hence there exists $j \in N$ such that yP_jzP_jx.

$[zP^\#x$ and $xP^\#y]$ if and only if $[\sum_{\{i \in N | zR_ix\}} p_i > \frac{1}{2}$ and $\sum_{\{i \in N | xR_iy\}} p_i > \frac{1}{2}]$. Hence there exists $k \in N$ such that zP_kxP_ky. The existence of xP_hyP_hz, $yPjzPjx$ and zP_kxP_ky, contradicts Sen Coherence and proves the theorem. Q.E.D.

4. GENERALIZED SEN COHERENCE AND THE EXISTENCE OF AN PPALHW

The purpose of this section is to obtain a sufficient condition the satisfaction of which does not imply the transitivity of $R^{\#}(R_N, p)$ but yet ensures that $B(X, R^{\#}(R_N, p))$ is non-empty.

The following example illustrates that $B(X, R^{\#})$ may be non-empty even though $R^{\#}$ is not transitive.
Example 1: $X = \{y_1, y_2, y_3, y_4\}$ and $n = 3$. Suppose $p_k = \frac{1}{3}$ for all $k \in \{1, 2, 3\}$. Let $y_4P(R_1)y_1P(R_1)y_2P(R_1)y_3$, $y_4P(R_2)y_3P(R_2)y_1P(R_2)y_2$, $y_4P(R_3)y_2P(R_3)y_3P(R_3)y_1$. Here we have $y_1P(R^{\#})y_2P(R^{\#})y_3P(R^{\#})y_1$ and hence $R^{\#}$ is not transitive. In fact it is not even acyclic. However, $B(X, R^{\#}) = \{y_4\}$.

An $(R1, \cdots, Rn)$**-sigma** is a list of $K \geq 3$ distinct alternatives $< x_1, \cdots, x_K >$, (not necessarily distinct) states of nature $i_1, \cdots, i_K \in N$ and $k \in \{1, \cdots, K-2\}$ such that:

(a) $x_{j+2}P(R_{i_j})x_{j+1}P(R_{i_j})x_j$ for all $j \in \{1, \cdots, K-2\}$;

(b) $x_kP(R_{i_{K-1}})x_KP(R_{i_{K-1}})x_{K-1}$ and $x_{k+1}P(R_{i_K})x_kP(R_{i_K})x_K$.

More formally:
$An(R_1, \cdots, R_n)$**-sigma** is a triplet $(< x_1, \cdots, x_K >, (i_1, \cdots, i_K), k)$ for some $K \geq 3$ satisfying the following properties:

(i) for $i, j \in \{1, \cdots, K\} : x_i \neq x_j$;

(ii) $i_j \in N$ for all $j \in \{1, \cdots, K\}$;

(iii) $k \in \{1, \cdots, K-2\}$;

(iv) $x_{j+2}P(R_{i_j})x_{j+1}P(R_{i_j})x_j$ for all $j \in \{1, \cdots, K-2\}$;

(v) $x_kP(R_{i_{K-1}})x_KP(R_{i_{K-1}})x_{K-1}$;

(vi) $x_{k+1}P(R_{i_K})x_kP(R_{i_K})x_K$.

The reason why we refer to the above as an $(R_1, \cdots, R_n)$-sigma is because while depicting expressions like $x_{j+2}P(R_{i_j})x_{j+1}$ on a piece of paper, if one begins at $x_kP(R_{i_{K-1}})x_K$ and moves clock-wise through $x_K, x_{K-1}, \cdots, x_{k-1}$ and returns to x_k then one has to keep moving right through x_{k-1}, x_{k-2} etc. to stop at x_1. The resulting figure will then look like the lower case Greek letter sigma (σ) which ends with $x_2P(R_{i_1})x_1$. In particular it is possible that $k = 1$, and in that case the $(R_1, \cdots, R_n)$-sigma $< x_1, \cdots, x_K >$ reduces to a $(R_1, \cdots, R_n)$**-cycle**.

$x \in X$ is said to be **at the end of an** $(R_1, \cdots, R_n)$-sigma if there exists a $(R_1, \cdots, R_n)$-sigma $< x_1, \cdots, x_K >$ with $x_1 = x$.

Due to its intuitive similarity with Sen Coherence, we call the next property Generalized Sen Coherence.

A preference profile R_N is said to satisfy **Generalized Sen Coherence (GSC)** if $\{x \in X | x$ is not at the end of any $(R_1, \cdots, R_n)$-sigma$\}$ is non-empty.
Note: In Example 1, $B(X, R^{\#}) = \{y_4\} = \{x \in X | x$ is not at the end of any (R_1, R_2, R_3)-sigma$\}$ and so GSC is satisfied, although $R^{\#}$ is not transitive- in fact not even acyclic.
Proposition 2: Suppose R_N satisfies GSC and $p \in P^N$. Then $B(X, R^{\#}(R_N, p))$ is non-empty. However, the converse is not true, i.e. there exists $p \in P^N$ such that $B(X, R^{\#}(R_N, p))$ is non-empty and yet R_N does not satisfy GSC.
Proof: First let us show that the converse is not true. Let $X = \{x_1, x_2, x_3\}, n = 4$ and $p \in P_N$ with $p_i = \frac{1}{4}$ for all $i \in N$. Let $x_1P(R_1)x_2P(R_1)x_3, x_3P(R_2)x_1P(R_2)x_2, x_2P(R_3)x_3P(R_1)x_1$ and $x_1P(R_4)x_2P(R_4)x_3$. Since $< x_1, x_2, x_3 >$ form a (R_1, R_2, R_3)-cycle GSC is violated. However $B(X, R^{\#}) = \{x_1\}$.

Now let us show that if R_N satisfies GSC and $p \in P_N$, then $B(X, R^{\#}(R_N, p))$ is non-empty.

Hence suppose R_N satisfies GSC, i.e. $\{x \in X | x$ is not at the end of any $(R_1, \cdots, R_n)$-sigma$\}$ is non-empty and towards a contradiction suppose $B(X, R^{\#}(R_N, p))$ is empty for some $p \in P_N$.

Let $x_1 \in X$. Since $x_1 \notin B(X, R^{\#})$, there exists $x_2 \in X \setminus \{x_1\}$, such that $\sum_{\{i \in N | x_2R_ix_1\}} p_i > \frac{1}{2}$. Since $x_2 \notin B(X, R^{\#})$, there exists $x_3 \in X \setminus \{x_1\}$, such that $\sum_{\{i \in N | x_3R_ix_2\}} p_i > \frac{1}{2}$.

Clearly, $x_3 \neq x_1$. Thus there exists an individual i_1 such that $x_3P(R_{i_1})x_2P(R_{i_1})x_1$.

Having found $x_1, x_2, \cdots, x_j, x_{j+1} \in X$ all distinct such that $x_{h+2}P(R_{i_h})x_{h+1}P(R_{i_h})x_h$ for $h \in \{1, \cdots, j-1\}$, there are two possibilities.

Case 1: $X = \{x_1, x_2, \cdots, x_j, x_{j+1}\}$. Since $B(X, R^{\#})$ is empty, there exists $k \in \{1, \cdots, j-1\}$ such that $x_k P^{\#} x_{j+1}$. Further, $x_{j+1}P^{\#}x_j$ and $x_{k+1}P^{\#}x_k$.

Let $K = j+1$. Then, there exists individuals i_{K-1} and i_K such that $x_k P(R_{i_{K-1}})x_K P(R_{i_{K-1}})x_{K-1}$ and $x_{k+1}P(R_{i_K}))x_k P(R_{i_K})x_K$.

Thus, x_1 is at the end of an $(R_1, \cdots, R_n)$-sigma.

Case 2: $\{x_1, x_2, \cdots, x_j, x_{j+1}\}$ is a proper subset of X. Since $B(X, R^{\#})$ is empty, there exists $x \in X$ such that $xP^{\#}x_{j+1}$.

If there exists $k \in \{1, \cdots, j-1\}$ such that $x_k P^{\#} x_{j+1}$, then we are back in Case 1. If no such k exists then $x \in X \setminus \{x_1, x_2, \cdots, x_j, x_{j+1}\}$. Since X is finite, this cannot go on endlessly and so we have to eventually come to a situation where there exists $k \in \{1, \cdots, j-1\}$ such that $x_k P^{\#} x_{j+1}$.

Thus, x_1 is at the end of an $(R_1, \cdots, R_n)$-sigma. Since x_1 is an arbitrary element of X, we get $\{x \in X | x$ is not at the end of any $(R_1, \cdots, R_n)$-sigma$\}$is empty, violating GSC and leading to a contradiction.

Thus, $B(X, R^{\#})$ is non-empty. Q.E.D.

The following is a non-trivial example of $R^{\#}$ not being transitive in spite of Sen Coherence as well as GSC being satisfied. The latter implies $B(X, R^{\#})$ is non-empty. However $R^{\#}$ is not transitive since it is not asymmetric. Hence Proposition 1 fails to hold if we drop the asymmetry assumption of $R^{\#}$.

Example 2: Let $n = 4, X = \{y_1, y_2, y_3\}$and $p \in P^N$ with $p_i = \frac{1}{4}$ for all $i \in N$. Suppose $y_1P(R_1)y_2P(R_1)y_3$, $y_2P(R_4)y_1P(R_4)y_3$ and $y_3P(R_j)y_1P(R_j)y_2$ for $j = 2, 3$. Here $B(X, R^{\#}) = \{y_1, y_3\}$. Sen-Coherence is satisfied since y_1 is never ranked last and so is GSC, since for three alternatives an R_N-sigma would require every alternative to be ranked last at least once and in this example, y_1 is never ranked last.

In our example $y_1P(R^{\#})y_2, y_2I(R^{\#})y_3$ and yet $y_1I(R^{\#})y_3$. Thus $R^{\#}$ is not transitive. Neither is $R^{\#}$ asymmetric, since $y_2I(R^{\#})y_3$ and $y_1I(R^{\#})y_3$ with $y_1 \neq y_3 \neq y_2 \neq y_1$.

5. CONCLUSION

In the special case of equal probability across states of nature, is preferred with probability at least half to y if and only if x is preferred to y in a weak majority of states of nature. In this case, we can say that the preferred with probability at least half relation is the pair-wise weak majority relation. Dietrich and List (2010) focuses primarily on conditions under which consistent judgment aggregation is possible. The aggregation of preference relations with equal probabilities across states of nature can be viewed as a special case of judgment aggregation. In this context, the preference aggregation problem that is being translated into the judgment aggregation framework is one where each profile of state-dependent preference relations is aggregated into an overall weak preference relation, not just into a set of winning options. Dietrich and List (2010) consider two types of axioms. Their first category of axioms based on an ordering of states of nature in the spirit of similar axioms for preference aggregation problems considered by Grandmont (1978) and Rothstein (1990). The conditions identified in the paper by Dietrich and List (2010) and applied to the special case of preference aggregation are sufficient for the acyclicity of the pair-wise weak majority binary relation, not just for the existence of pair-wise majority winners for a fixed set of alternatives. Our (rather trivial) Example 1 shows that the satisfaction of GSC and existence of pair-wise majority winners is perfectly compatible with the violation of acyclicity of the weak pair-wise majority relation $R^{\#}$. Hence as an axiom (for the purpose we are interested here), GSC is weaker than the conditions suggested by Dietrich and List (2010) when applied to preference aggregation. In section 6 of their paper, Dietrich and List (2010) discuss a condition for judgment aggregation that becomes equivalent to Sen Coherence when applied to the special case of preference aggregation. Once again Example 1 provides an example where GSC is satisfied leading to the existence of pair-wise weak majority winners but which violates the requirement of Sen Coherence. Further, Example 2 in our paper provides an example with $n = 4$, where GSC is satisfied leading to the existence of pair-wise majority winners. However although Sen Coherence (and hence the related assumption in Dietrich and List (2010) is satisfied in the example, the weak pair-wise majority relation is not transitive. Thus, in the context of preference aggregation the assumption equivalent to Sen Coherence available in Dietrich and List (2010) implies that the weak pair-wise majority relation is transitive only if number of states of nature is odd and not necessarily so if this number is even.

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