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Optimization of Fuzzy Integrated Inventory Model with Ordering Cost Reduction Dependent on Lead Time

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Abstract: The intellectual and industrial design of a complex inventory system becomes a vital issue for the organization of responsiveness to uncertainties. The parameters involved in inventory model are likely to be varied due to the fluctuating business environment. Therefore, it will be more realistic apply fuzzy model rather than crisp model. This paper derives a single-vendor and a single-buyer integrated inventory model with ordering cost reduction dependent on lead time in a fuzzy environment. In this model, buyer and vendor cost parameters are uncertainties which necessitate the use of trapezoidal fuzzy numbers. The purpose of this model is to determine the minimum integrated total cost and optimal order quantity in the fuzzy scenario. There are two mathematical inventory models proposed in this paper. Initially, a crisp model is developed with fuzzy total inventory cost along with crisp optimal order quantity. Next, the fuzzy model is formulated with fuzzy total inventory cost and fuzzy optimal order quantity. Graded mean integration formula is employed to defuzzify the total inventory cost and the extension of the Lagrangian method is used to determine the optimal order quantity. An algorithm is developed to obtain the optimal order quantity and minimum integrated total cost. The comparison of a fuzzy inventory model with the conventional crisp inventory model is made through numerical examples. This proposed fuzzy model is also compared with some specific cases of the previous models. Finally, the graphical representation is presented to demonstrate the proposed model. The result illustrates that this fuzzy model can be quite useful in determining the optimal order quantity and minimum integrated total cost procedure when the lead time is analysed.
Keyword — Optimal integrated total cost, Optimal order quantity, Graded mean integration representation method, Fuzzy inventory system, Lagrangian method.

1. INTRODUCTION

Many authors handle inventory systems with various lead time cases where the cost components are considered as crisp values which do not represent the actual inventory system completely. In rare cases, the inventory cost components are considered as fuzzy values. In actual life, varying physical or synthetic features may cause an influence on the cost components and exact values of cost features as it becomes a risk to measure the exact amount of holding, order, and setup cost. Thus, in controlling the inventory system, it may allow some flexibility in the cost parameter values in order to treat the ambiguity which always fits the actual situations. Fuzzy set theory meets these prerequisites to some extent. In this paper, fuzziness is introduced by allowing the buyer and vendor ordering cost, inventory holding cost, setup cost and lead time crashing cost. It is suitable for the inventory system to fit the real situation and proves to be profitable.

The integrated inventory management organization is a common exercise in the global markets and provides economic benefits both for the vendor and the buyer. In recent years, most integrated inventory management organizations have focused on the integration between vendor and buyer. Once they form a tactical alliance to minimize their own cost or maximize their own income, trading parties can cooperate and share information to achieve enhanced benefits. Currently, companies can no longer contribute solely as individual entities in the constantly varying business world. Globalization of marketplace and increased competition force organizations to depend on effective supply chains to progress their overall performance. Lead time management is an important issue in manufacture and operation management. In many practical circumstances, lead time can be compacted using crashing cost.

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Inventory models considering lead time as a determinate variable have been developed by several researchers recently. Initially, Liao and Shyu (1991) allowed an inventory model in which lead time is a sole decision variable and the order quantity is fixed. Annadurai and Uthayakumar (2010a) developed a combination inventory model with backorders and lost sales in which the order quantity, reorder point, lead time and setup cost are ruling variables. It is assumed that an appearance order lot may contain some defective products and the number of defective products is a random variable. There are two inventory models scheduled in this paper, one with normally distributed demand and the another with distribution-free demand. Again Annadurai and Uthayakumar (2010b) proposed (T, R, L) inventory model to analyze the effects of increasing two different kinds of investments to lessen the lost-sales rate, in which the review period, lead time and lost-sales rate are preserved as decision variables. Billington (1987) addressed to solve situations where setup cost varies exponentially and linearly as a function of capital expenditure. Decision rules are expressed to indicate specific situations in which setup cost reduction reduces total cost. Optimal manufacturing batch size with rework in a single-stage production system is dealt by Cárdenas-Barrón (2008). C. K. Chen, Chang, and Ouyang (2001a) considered minimizing the total related cost by concurrently improving order quantity, reorder point, and lead time. The lead time demand is supposed to be normally distributed.

F. Chen, Federgruen, and Zheng (2001b) proposed an optimal policy, maximizing total organization-wide profits in a centralized system. Economic production quantity models concerning lead time as a determinate variable are constructed by Chiu (1998). Goyal (1985) extended an economic order quantity under conditions of permissible delay in payments. Jaggi, Goyal, and Geol (2008) presented a retailer's optimal replenishment decisions with credit linked demand under permissible delay in payments. Credit financing in economic ordering policies of deteriorating items is developed by Jaggi and Aggarwal (1994). Li, Xu, Zhao, Yeung, and Ye (2012) demonstrated a supply chain containing a vendor and a buyer with controllable lead time. They measured two situations such as complete information and incomplete information about the buyer. Ouyang, Chuang, and Lin (2007) designed minimized lead time and ordering price that are inter-dependent in an inventory system with a backorder cost discount. The objective is to minimize the total related cost by instantaneously optimizing the review period, lead time and backorder price discount. A single-vendor single-buyer combined production inventory model under the hypothesis that the lead time is stochastic and lead time is decision variable are investigated by Ouyang, Wu, and Ho (2004). Pan and Yang (2002) considered delivering a lower total cost and tinier lead time compared to previous inventory problems.

Render (1994) elaborated the nonlinear programming methods. Taha (1997) gave the Lagrangian method used to solve uncertainty problems as mentioned in operations research. Vijayashree and Uthayakumar (2015) designed an integrated inventory model to determine optimizing the optimal order quantity, process quality, lead time and the number of deliveries. Vijayashree and Uthayakumar (2016) constructed an optimizing integrated inventory model with deals for quality improvement and setup cost reduction. Vijayashree and Uthayakumar (2017) focused on the minimized integrated total cost by adopting linear and logarithmic ordering costs that decrease depending on lead time. An integrated inventory model to minimize the total cost by optimizing order quantity, lead time, and number of deliveries is offered by Yang and Pan (2004).

The enhancement of the above inventory models depends on the terms lead time, controllable lead time, stochastic demand in controllable lead time, controlling the setup cost, ordering cost and permissible delay in payments. The specified crisp models are designed to get economic order quantity, economic production quantity, optimal total cost using differential calculus methods and algorithms. The above mentioned papers provide quantitative analysis for comparisons of the previous inventory models.

In the present scenario, it becomes extremely difficult to determine the exact value of the parameters. One way of managing this vagueness is through fuzzy numbers. It is pertinent to discuss the work done in this area before the formulation of the proposed fuzzy economic model. Fuzzy set theory introduced by Zimmerman (1983) focused on fuzzy sets in operational research which are more flexible to solve different kind of problem structures and improvised the model for human estimation and decision-making processes, than traditional mathematics. S. H. Chen, Wang, and Arthur Ramer (1996) explored the median rule to find the optimal economic order quantity (EOQ) and shortage quantity. S. H. Chen (1985) discussed arithmetic operations on fuzzy numbers with function principle, which might be used as the fuzzy arithmetic operations with generalized fuzzy numbers. S. H. Chen and Hsieh (2000) designed a generalized L-R type fuzzy number using graded mean integration representation method to prove some relationships. S. H. Chen and Hsieh (1999) elaborated a graded mean integration representation of the generalized fuzzy number. El-Wakeel and Al-yazidi (2016) presented a fuzzy constrained probabilistic inventory model depending on trapezoidal fuzzy numbers. Hsieh (2002) developed optimization of fuzzy production inventory model. Jaggi, Yadavalli, Anuj Sharma, and Sunil Tiwari (2016) discussed a fuzzy EOQ model with admissible shortage under different trade credit terms. Optimization of fuzzy production inventory model with repairable defective products under crisp or fuzzy production quantity is estimated by S. H. Chen, Wang, and Chang (2005).

After the introduction of fuzzy set theory, operations on fuzzy numbers under the function principle, graded mean integration representation, ranking, signed distance and similarity of L-R type fuzzy number concepts are the great turning points of many application fields. The above concepts are applied in the inventory models to find the optimization of production lot size, ordering lot size and total cost, which give better results compared to previous crisp models.

This paper aims at developing a fuzzy inventory model allowing ordering cost, holding cost, setup cost and lead time crashing cost under lead time. Order quantity is assumed to be a crisp and fuzzy form of the integrated inventory model where cost parameters are fuzzified. Due to modification in the integrated vendor buyer sector, the parameters involved in the inventory system may deviate more or less from their actual value. To deal with such type of situation, trapezoidal fuzzy numbers have been employed in the current study. Graded mean integration method is used for defuzzification of the integrated total cost and the extension of Lagrangian method is used to determine optimal order quantity. An algorithm is developed to obtain the optimal order quantity and integrated total cost. The integrated inventory model is demonstrated using numerical illustrations and graphical representations. This model identifies the most suitable inventory model which has a larger impact on enhancing the profit of the business organisation.

The rest of the paper is organized as follows: Section 2 introduces the notations and assumptions. Section 3 deals with a mathematical model to minimize integrated total cost and optimal order quantity. In Section 4, the methodology of the Lagrangian method is discussed. Section 5 describes fuzzy inventory models and an algorithm is designed to find the optimal order quantity and minimum integrated total cost. In Section 6, numerical examples and graphical representation are presented to illustrate crisp and fuzzy sense. In Section 7, comparative study is presented. This is followed by the conclusion.

2. NOTATIONS AND ASSUMPTIONS

To develop the proposed model, the following notations and assumptions which are similar to those used in (Pan and Yang (2002)) are adopted. Besides, additional notations and assumptions will be applied based on requirement.

2.1 Notations

Q - Order quantity for the buyer,
L - Length of lead time for the buyer,
A - Buyer's ordering cost per order,
m - The number of lots in which the product is delivered from the vendor to the buyer in one production cycle,
D - Average demand per unit time on the buyer,
P - Production rate of the vendor $P > D$,
S - Vendor's setup cost per setup,
C_v - Unit production cost paid by the vendor $C_v < C_b$,
C_b - Unit purchase cost paid by the buyer,
r - Annual inventory holding cost per dollar invested in stocks,
R - Reorder point of the buyer,
ITC - Integrated total cost for the single vendor and the single buyer.

2.2 Assumptions

To develop the model, following assumptions are adopted.

1. The system consists of single-vendor and single-buyer for a single product in this model.
2. The buyer orders a lot of size Q and the vendor manufactures mQ with a finite production rate $P(P > D)$ at one setup but ships quantity Q to the buyer over m times. The vendor incurs a set up cost S for each production run and the buyer incurs an ordering cost A for each order of quantity Q.
3. The demand X during lead time L follows a normal distribution with mean μL and standard deviation $\sigma\sqrt{L}$.
4. The inventory is continuously reviewed. The buyer places the order when inventory reaches the reorder point R.
5. The reorder point (ROP) equals the sum of the expected demand during lead time and safety stock. The reorder point R = expected demand during lead time + safety stock, that is $R = DL + k\sigma\sqrt{L}$ where k is safety factor.

6. The lead time L consists of n mutually independent components. The i-th component has a normal duration b_i, minimum duration a_i, and crashing cost per unit time c_i. For convenience, c_i is arranged such that $c_1 < c_2 < c_3 < \cdots < c_n$.

7. The components of lead time are crashed one at a time starting from the first component because it has the minimum unit crashing cost and then the second component, and so on.

8. Let $L_0 = \sum_{i=1}^{n} b_i$, and L_i be the length of lead time with components $1, 2, 3, \ldots i$ crashed to their minimum duration, then L_i can be expressed as $L_i = L_0 - \sum_{j=1}^{n}(b_j - a_i)$, $i = 1, 2, 3, \ldots, n$; and the lead time crashing cost per cycle $R(L)$ is given by $R(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_i)$, $L \in [L_i, L_{i-1}]$. In addition, the length of lead time is equal for all shipping cycles, and the lead time crashing cost occurs in each shipping cycle. The relationship between lead time and crashing cost is shown in Figure 1.

9. The reduction of lead time L accompanies reduced ordering cost A and A is firmly the concave function of L, i.e., $A'(L) > 0$ and $A''(L) < 0$ (Ouyang et al. (2007), C. K. Chen et al. (2001a)).

10. If extra costs are incurred by the vendor, it will be fully transferred to the buyer when shortened lead time is required (Pan and Yang (2002)).

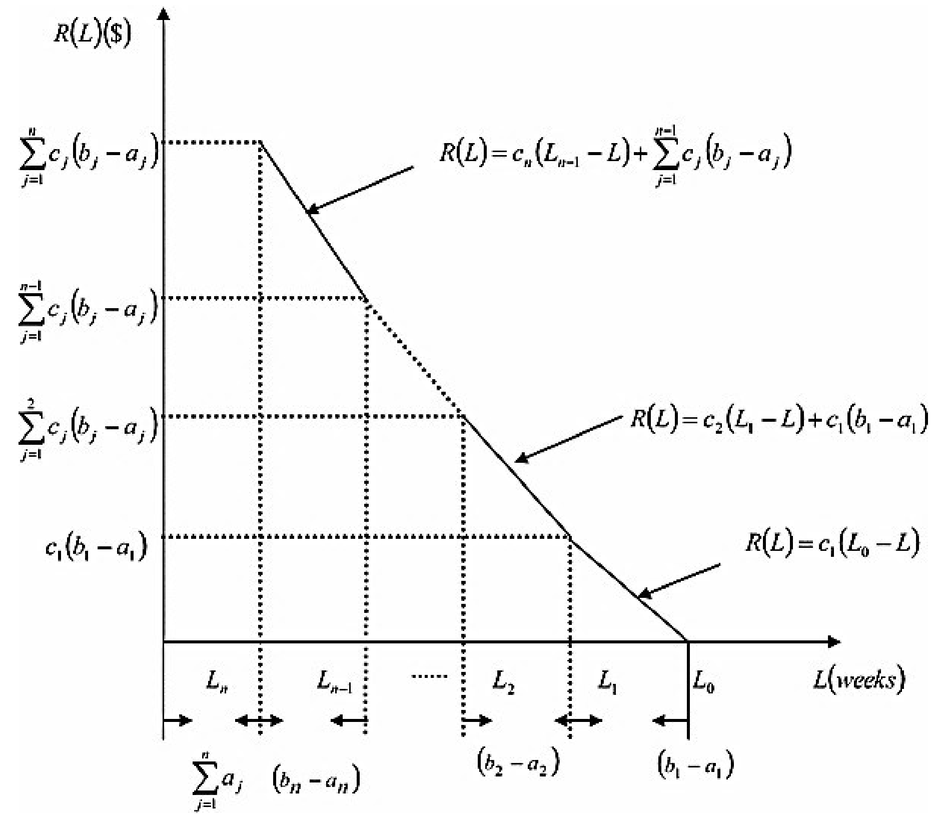

Figure 1: The relationship between lead time and crashing cost

3. MATHEMATICAL MODEL

3.1 Conventional Crisp Inventory Model

In this model, the integrated total cost consists of buyer and vendor ordering cost, inventory holding cost, setup cost and lead time crashing cost.

Integrated total cost (ITC)
The joint total expected cost per unit time derived in Pan and Yang (2002) is the sum of the following elements,

$$\text{Ordering cost per unit time } = \frac{A}{Q/D} = \frac{AD}{Q}, \tag{1}$$

$$\text{Buyer's holding cost per unit time is } = \left(\frac{Q}{2} + k\sigma\sqrt{L}\right) rC_b, \tag{2}$$

$$\text{Lead time crashing cost per unit time } = \left(\frac{D}{Q}\right) R(L), \tag{3}$$

$$\text{Vendor setup cost per year } = \left(\frac{D}{mQ}\right) S, \tag{4}$$

Vendor's holding cost per unit time is obtained from vendor's average inventory is evaluated as the difference of the vendor's accumulated inventory and the buyer's accumulated inventory (see Figure 2).
Hence, vendor's average inventory cost

$$\begin{aligned}
&= \left\{\left[mQ\left(\frac{Q}{P} + (m-1)\frac{Q}{D}\right) - \frac{m^2Q^2}{2P}\right] - \left[\frac{Q^2}{D}(1+2+\cdots+(m-1))\right]\right\}\frac{D}{mQ}, \\
&= \frac{Q}{2}\left[m\left(1-\frac{D}{P}\right) - 1 + \frac{2D}{P}\right].
\end{aligned}$$

$$\text{So the vendor's holding cost per unit time is } = \frac{Q}{2}\left[m\left(1-\frac{D}{P}\right) - 1 + \frac{2D}{P}\right] rC_v \tag{5}$$

According to the assumptions (1) to (5) and the equations (1) to (5) described above, (Pan and Yang (2002)) the integrated total cost per unit time for the single vendor and the single buyer integrated inventory system which is composed of buyer and vendor ordering cost, inventory holding cost, setup cost and lead time crashing cost, is expressed by

$$ITC(Q,L,m) = \frac{D}{Q}\left(A + \frac{S}{m} + R(L)\right) - \frac{QrC_v}{2}\left(\frac{mD}{P} + 1\right) + \frac{Qr}{2}\left(\left(m + \frac{2D}{P}\right)C_v + C_b\right) + rC_bk\sigma\sqrt{L}. \tag{6}$$

If a particular value of m and L the integrated total cost is $ITC(Q,L,m)$ optimal order quantity Q is determined when integrated total cost $ITC(Q,L,m)$ is minimum. In order to find the minimization of $ITC(Q,L,m)$ we find the partial derivative of $ITC(Q,L,m)$ with Q and equate to zero, then we have

$$-\frac{D}{Q^2}\left(A + \frac{S}{m} + R(L)\right) - \frac{rC_v}{2}\left(\frac{mD}{P} + 1\right) + \frac{r}{2}\left(\left(m + \frac{2D}{P}\right)C_v + C_b\right) = 0. \tag{7}$$

For a fixed m and L, the integrated total cost $ITC(Q,L,m)$ is positive definite at point Q. By examining the sufficient conditions for a minimum value of $ITC(Q,L,m)$, second order partial derivatives of $ITC(Q,L,m)$ with respect to Q and obtain

$$\frac{\partial^2 ITC}{\partial Q^2} = \frac{2D}{Q^3}\left(A + \frac{S}{m} + R(L)\right) > 0. \tag{8}$$

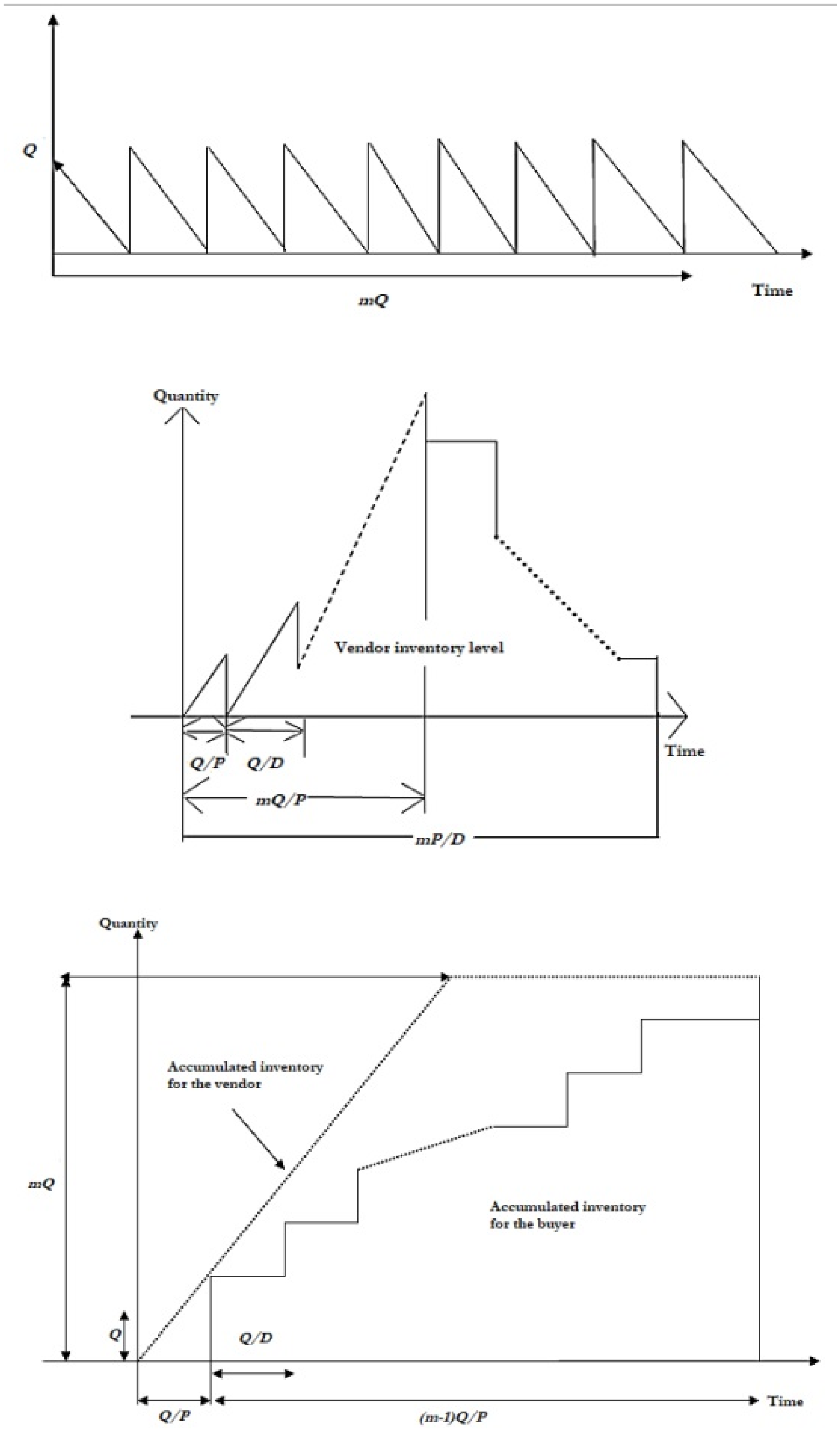

Figure 2: The inventory pattern for the buyer and vendor

Therefore, $ITC(Q, L, m)$ is convex in Q, for a fixed m and L. As a result, examine for the optimal derivatives, Q^* is reduce to find a local minimum. Hence, we obtain the optimal order quantity Q^* by the above equation (7) is,

$$Q^* = Q = \sqrt{\frac{2D\left(A + \frac{S}{m} + R(L)\right)}{r\left(\left(m\left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P}\right)C_v + C_b\right)}}. \tag{9}$$

4. METHODOLOGY

In this paper, Graded mean integration method and function principle are used to find the optimal order quantity with a fuzzy inventory model. When the quantities are fuzzy numbers Lagrangian method is used to solve the model.

4.1 Graded Mean Integration Representation Method

S. H. Chen and Hsieh (1999) introduced Graded mean integration representation method based on the integral value of graded mean h-level of generalized fuzzy number to defuzzify the same. Here, generalized fuzzy number is described as follows:

Suppose $\tilde{A}$ is a generalized fuzzy number as shown in Figure 3. It is described as any fuzzy subset of the real line R, whose membership function $\mu_{\tilde{A}}$ satisfies the following conditions.

1. $\mu_{\tilde{A}}(x)$ is a continuous mapping from R to $[0, 1]$,
2. $\mu_{\tilde{A}}(x) = 0, -\infty < x \leq a_1$,
3. $\mu_{\tilde{A}}(x) = L(x)$ is strictly increasing on $[a_1, a_2]$,
4. $\mu_{\tilde{A}}(x) = w_A, a_2 \leq x \leq a_3$,
5. $\mu_{\tilde{A}}(x) = R(x)$ is strictly decreasing on $[a_3, a_4]$,
6. $\mu_{\tilde{A}}(x) = 0, a_4 \leq x < \infty$,

where $0 < w_A \leq 1$ and a_1, a_2, a_3 and a_4 are real numbers. This type of generalized fuzzy number is also denoted as $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)_{LR}$. When $w_A = 1$, it can be simplified as $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$.

Now Graded mean integration representation method, L^{-1} and R^{-1} are the inverse functions of L and R respectively, and the graded mean h-level value of generalized fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)_{LR}$ is $\frac{h}{2}\left(L^{-1}(h) + R^{-1}(h)\right)$ as in Figure 3. Then, the Graded mean integration representation of $P(\tilde{A})$ with grade w_A is

$$P(\tilde{A}) = \frac{\int_0^{w_A} \frac{h}{2}\left(L^{-1}(h) + R^{-1}(h)\right)dh}{\int_0^{w_A} h dh}, \tag{10}$$

where $0 < h \leq w_A$ and $0 < w_A \leq 1$.

In the proposed fuzzy inventory models, popular trapezoidal fuzzy number is used as the type of all fuzzy parameters. Let $\tilde{B}$ be a trapezoidal fuzzy number and is denoted by $\tilde{B} = (b_1, b_2, b_3, b_4)$. Graded mean integration representation of $\tilde{B}$ by the equation (10) is

$$P(\tilde{B}) = \frac{\int_0^1 \frac{h}{2}\left[(b_1 + b_4) + h(b_2 - b_1 - b_4 + b_3)\right]dh}{\int_0^1 h dh},$$

$$P(\tilde{B}) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}. \tag{11}$$

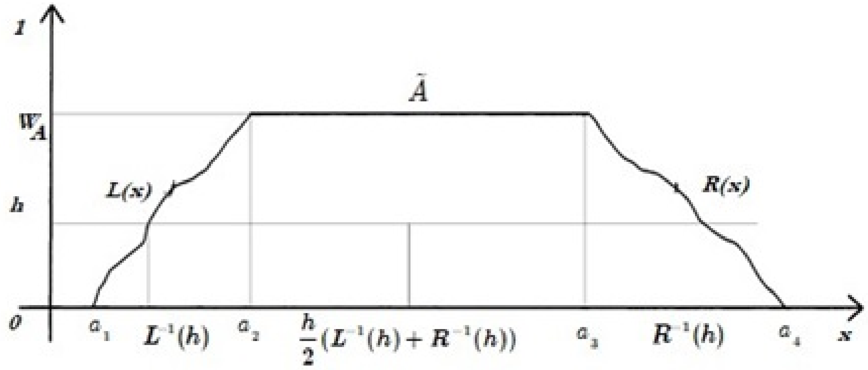

Figure 3: The graded mean h-level value of generalized fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$.

4.2 Fuzzy Arithmetical Operations under Function Principle

In S. H. Chen (1985), function principle is proposed to the fuzzy arithmetical operations by trapezoidal fuzzy numbers. Some fuzzy arithmetical operations under function principle are described as follows:

Suppose $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are two trapezoidal fuzzy numbers. Then,

1. The addition of $\tilde{A}$ and $\tilde{B}$ is $\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are any real numbers.
2. The multiplication of $\tilde{A}$ and $\tilde{B}$ is $\tilde{A} \otimes \tilde{B} = (c_1, c_2, c_3, c_4)$, where $T = \{a_1b_1, a_1b_4, a_4b_1, a_4b_4\}$, $T_1 = \{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}$, $c_1 = \min T$, $c_2 = \min T_1$, $c_3 = \max T$, $c_4 = \max T_1$. If $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are all nonzero positive real numbers, $\tilde{A} \otimes \tilde{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4)$.
3. $-\tilde{B} = (-b_4, -b_3, -b_2, -b_1)$ and the subtraction of $\tilde{A}$ and $\tilde{B}$ is $\tilde{A} \ominus \tilde{B} = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are any one of the real numbers.
4. $\frac{1}{\tilde{B}} = \tilde{B}^{-1} = \left(\frac{1}{b_4}, \frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1}\right)$, where b_1, b_2, b_3 and b_4 are all positive real numbers. If $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are all nonzero positive real numbers, then the division of $\tilde{A}$ and $\tilde{B}$ is $\tilde{A} \oslash \tilde{B} = \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right)$.
5. For any $\alpha \in R$,

   a) If $\alpha \geq 0$, then $\alpha \otimes \tilde{A} = (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4)$,

   b) If $\alpha < 0$, then $\alpha \otimes \tilde{A} = (\alpha a_4, \alpha a_3, \alpha a_2, \alpha a_1)$.

4.3 Extension of Lagrangian Method

Taha (1997) discussed to solve nonlinear programming problem with equality constraints by using Lagrangian Method to find an optimum solution, and showed that Lagrangian method may be extended to solve inequality constraints. The general idea of extending the Lagrangian procedure is that if the unconstrained optimum of the problem does not satisfy all constraints, the constrained optimum must occur at a boundary point of the solution space. The problem is

Minimize $y = f(x)$, subject to $g_i(x) \geq 0$, $i = 1, 2, 3, \ldots, m$.

The non-negativity constraints $x \geq 0$, if any are included in the m constraints. Then, the procedure of Extension of the Lagrangian method involves the following steps.

Step 1 : Solve the unconstrained problem

Minimize $y = f(x)$.

If the resulting optimum satisfies all the constraints, stop because all constraints are redundant. Otherwise, set $k = 1$ and go to Step 2.

Step 2 : Activate any k constraints (i.e., convert them into equality) and optimize $f(x)$ subject to k active constraints by the Lagrangian method. If the resulting solution is feasible with respect to the remaining constraints, stop; it is a local optimum. Otherwise, activate another set of k constraints and repeat the step. If all sets of active constraints taken k at a time are considered without encountering a feasible solution, go to Step 3.

Step 3 : If $k = m$, stop; no feasible solution exists. Otherwise, set $k = k + 1$ and go to Step 2.

5. FUZZY INVENTORY MODELS

5.1 Fuzzy Integrated Inventory Model for Crisp Order Quantity

Throughout this paper, following parameters are used in order to simplify the treatment of the fuzzy inventory models. Let $\tilde{D}, \tilde{A}, \tilde{S}, \tilde{r}, \tilde{P}, \tilde{C}_v$ and $\tilde{C}_b$ be fuzzy parameters. Now, fuzzy integrated inventory model is introduced with fuzzy parameters for crisp order quantity Q as follows.

In this model, the fuzzy integrated total cost consists of buyer and vendor fuzzy ordering cost, fuzzy inventory holding cost, fuzzy setup cost and fuzzy lead time crashing cost. That is fuzzy ordering cost per unit time $= (\tilde{A} \otimes \tilde{D}) \oslash Q$, buyer's fuzzy holding cost per unit time is $= ((Q \oslash 2) \oplus (k \otimes \sigma \otimes \sqrt{L})) \otimes (\tilde{r} \otimes \tilde{C}_b)$, fuzzy lead time crashing cost per unit time $= (\tilde{D} \oslash Q) \otimes R(L)$, vendor fuzzy setup cost per year $= (\tilde{D} \oslash (m \otimes Q)) \otimes \tilde{S}$ and vendor's fuzzy holding cost per unit time is $= (Q \oslash 2) \otimes [(m \otimes (1 - (\tilde{D} \oslash \tilde{P})) \ominus 1 \oplus ((2 \otimes \tilde{D}) \oslash \tilde{P}))] \otimes (\tilde{r} \otimes \tilde{C}_v)$. Then, the fuzzy integrated total cost is

$$\begin{aligned} I\tilde{T}C(Q,L,m) =& [(\tilde{D} \oslash Q) \otimes (\tilde{A} \oplus (\tilde{S} \oslash m) \oplus R(L))] \ominus [[(Q \otimes \tilde{r} \otimes \tilde{C}_v) \oslash 2] \otimes [(m \otimes \tilde{D} \oslash \tilde{P}) + 1]] \\ & \oplus [[(Q \otimes \tilde{r}) \oslash 2] \otimes [[m \oplus ((2 \otimes \tilde{D}) \oslash \tilde{P})] \otimes \tilde{C}_v \oplus \tilde{C}_b]] \oplus [\tilde{r} \otimes \tilde{C}_b \otimes k \otimes \sigma \otimes \sqrt{L}], \end{aligned} \tag{12}$$

where $\oplus, \otimes, \oslash$ and $\ominus$ are the fuzzy arithmetical operators under function principle.

Suppose $\tilde{D} = (D_1, D_2, D_3, D_4)$, $\tilde{A} = (A_1, A_2, A_3, A_4)$, $\tilde{r} = (r_1, r_2, r_3, r_4)$, $\tilde{S} = (S_1, S_2, S_3, S_4)$, $\tilde{P} = (P_1, P_2, P_3, P_4)$, $\tilde{C}_v = (C_{v1}, C_{v2}, C_{v3}, C_{v4})$ and $\tilde{C}_b = (C_{b1}, C_{b2}, C_{b3}, C_{b4})$ are non-negative trapezoidal fuzzy numbers. Then the optimal order quantity of equation (12) is solved as in the following steps.

To begin with, fuzzy integrated total cost $I\tilde{T}C(Q,L,m)$ by equation (12) is

$$\begin{aligned} & I\tilde{T}C(Q,L,m) = \\ & \left[\left(\frac{D_1}{Q}\left(A_1 + \frac{S_1}{m} + R(L)\right) - \frac{Qr_4C_{v4}}{2}\left(\frac{mD_4}{P_1} + 1\right) + \frac{Qr_1}{2}\left(\left(m + \frac{2D_1}{P_4}\right)C_{v1} + C_{b1}\right) + r_1C_{b1}k\sigma\sqrt{L}\right),\right. \\ & \left(\frac{D_2}{Q}\left(A_2 + \frac{S_2}{m} + R(L)\right) - \frac{Qr_3C_{v3}}{2}\left(\frac{mD_3}{P_2} + 1\right) + \frac{Qr_2}{2}\left(\left(m + \frac{2D_2}{P_3}\right)C_{v2} + C_{b2}\right) + r_2C_{b2}k\sigma\sqrt{L}\right), \\ & \left(\frac{D_3}{Q}\left(A_3 + \frac{S_3}{m} + R(L)\right) - \frac{Qr_2C_{v2}}{2}\left(\frac{mD_2}{P_3} + 1\right) + \frac{Qr_3}{2}\left(\left(m + \frac{2D_3}{P_2}\right)C_{v3} + C_{b3}\right) + r_3C_{b3}k\sigma\sqrt{L}\right), \\ & \left.\left(\frac{D_4}{Q}\left(A_4 + \frac{S_4}{m} + R(L)\right) - \frac{Qr_1C_{v1}}{2}\left(\frac{mD_1}{P_4} + 1\right) + \frac{Qr_4}{2}\left(\left(m + \frac{2D_4}{P_1}\right)C_{v4} + C_{b4}\right) + r_4C_{b4}k\sigma\sqrt{L}\right)\right]. \end{aligned} \tag{13}$$

Next, fuzzy integrated total cost in equation (13) is defuzzified by Graded mean integration representation method in equation (11). The result is

$$\begin{aligned} & P(I\tilde{T}C(Q,L,m)) = \\ & \frac{1}{6}\left[\left(\frac{D_1}{Q}\left(A_1 + \frac{S_1}{m} + R(L)\right) - \frac{Qr_4C_{v4}}{2}\left(\frac{mD_4}{P_1} + 1\right) + \frac{Qr_1}{2}\left(\left(m + \frac{2D_1}{P_4}\right)C_{v1} + C_{b1}\right) + r_1C_{b1}k\sigma\sqrt{L}\right)\right. \\ & + 2\left(\frac{D_2}{Q}\left(A_2 + \frac{S_2}{m} + R(L)\right) - \frac{Qr_3C_{v3}}{2}\left(\frac{mD_3}{P_2} + 1\right) + \frac{Qr_2}{2}\left(\left(m + \frac{2D_2}{P_3}\right)C_{v2} + C_{b2}\right) + r_2C_{b2}k\sigma\sqrt{L}\right) \\ & + 2\left(\frac{D_3}{Q}\left(A_3 + \frac{S_3}{m} + R(L)\right) - \frac{Qr_2C_{v2}}{2}\left(\frac{mD_2}{P_3} + 1\right) + \frac{Qr_3}{2}\left(\left(m + \frac{2D_3}{P_2}\right)C_{v3} + C_{b3}\right) + r_3C_{b3}k\sigma\sqrt{L}\right) \\ & \left. + \left(\frac{D_4}{Q}\left(A_4 + \frac{S_4}{m} + R(L)\right) - \frac{Qr_1C_{v1}}{2}\left(\frac{mD_1}{P_4} + 1\right) + \frac{Qr_4}{2}\left(\left(m + \frac{2D_4}{P_1}\right)C_{v4} + C_{b4}\right) + r_4C_{b4}k\sigma\sqrt{L}\right)\right]. \end{aligned} \tag{14}$$

Then, we get the optimal order quantity Q when defuzzified fuzzy integrated total cost $P[I\tilde{T}C(Q,L,m)]$ is minimized. In order to find the minimization of $P[I\tilde{T}C(Q,L,m)]$, Q is solved by taking the partial derivative of $P[I\tilde{T}C(Q,L,m)]$ with respect to Q and equated to zero. Now $\frac{\partial P[I\tilde{T}C(Q,L,m)]}{\partial Q} = 0$ becomes,

$$\begin{aligned} & \frac{1}{6}\left[\left(-\frac{D_1}{Q^2}\left(A_1 + \frac{S_1}{m} + R(L)\right) - \frac{r_4C_{v4}}{2}\left(\frac{mD_4}{P_1} + 1\right) + \frac{r_1}{2}\left(\left(m + \frac{2D_1}{P_4}\right)C_{v1} + C_{b1}\right)\right)\right. \\ & + 2\left(-\frac{D_2}{Q^2}\left(A_2 + \frac{S_2}{m} + R(L)\right) - \frac{r_3C_{v3}}{2}\left(\frac{mD_3}{P_2} + 1\right) + \frac{r_2}{2}\left(\left(m + \frac{2D_2}{P_3}\right)C_{v2} + C_{b2}\right)\right) \\ & + 2\left(-\frac{D_3}{Q^2}\left(A_3 + \frac{S_3}{m} + R(L)\right) - \frac{r_2C_{v2}}{2}\left(\frac{mD_2}{P_3} + 1\right) + \frac{r_3}{2}\left(\left(m + \frac{2D_3}{P_2}\right)C_{v3} + C_{b3}\right)\right) \\ & \left. + \left(-\frac{D_4}{Q^2}\left(A_4 + \frac{S_4}{m} + R(L)\right) - \frac{r_1C_{v1}}{2}\left(\frac{mD_1}{P_4} + 1\right) + \frac{r_4}{2}\left(\left(m + \frac{2D_4}{P_1}\right)C_{v4} + C_{b4}\right)\right)\right] = 0. \end{aligned} \tag{15}$$

For a fixed m and L, the integrated total cost $P[I\tilde{T}C(Q,L,m)]$ is positive definite at point Q. By examining the sufficient conditions for a minimum value of $P[I\tilde{T}C(Q,L,m)]$, second order partial derivatives of $P[I\tilde{T}C(Q,L,m)]$ with respect to Q and obtain

$$\begin{aligned}\frac{\partial^2 P[I\tilde{T}C(Q,L,m)]}{\partial Q^2} =& \frac{1}{6}\left(\frac{2D_1}{Q^3}\left(A_1+\frac{S_1}{m}+R(L)\right)+\frac{4D_2}{Q^3}\left(A_2+\frac{S_2}{m}+R(L)\right)\right.\\ &\left.+\frac{4D_3}{Q^3}\left(A_3+\frac{S_3}{m}+R(L)\right)+\frac{2D_4}{Q^3}\left(A_4+\frac{S_4}{m}+R(L)\right)\right)>0.\end{aligned} \tag{16}$$

Therefore, $P[I\tilde{T}C(Q,L,m)]$ is convex in Q, for a fixed m and L. As a result, examine for the optimal derivatives, Q^* is reduce to find a local minimum. Hence, we get the optimal order quantity Q^* is obtained by solving the above equation (15),

$$\begin{aligned}Q_c^* &= Q^*\\ &= \sqrt{\frac{2D_1\left(A_1+\frac{S_1}{m}+R(L)\right)+4D_2\left(A_2+\frac{S_2}{m}+R(L)\right)+4D_3\left(A_3+\frac{S_3}{m}+R(L)\right)+2D_4\left(A_4+\frac{S_4}{m}+R(L)\right)}{\begin{array}{c}r_1\left(\left(m\left(1-\frac{D_1}{P_4}\right)-1+\frac{2D_1}{P_4}\right)C_{v1}+C_{b1}\right)+2r_2\left(\left(m\left(1-\frac{D_2}{P_3}\right)-1+\frac{2D_2}{P_3}\right)C_{v2}+C_{b2}\right)\\ +2r_3\left(\left(m\left(1-\frac{D_3}{P_2}\right)-1+\frac{2D_3}{P_2}\right)C_{v3}+C_{b3}\right)+r_4\left(\left(m\left(1-\frac{D_4}{P_1}\right)-1+\frac{2D_4}{P_1}\right)C_{v4}+C_{b4}\right)\end{array}}}.\end{aligned} \tag{17}$$

5.2 Fuzzy Integrated Inventory Model for Fuzzy Order Quantity

In this section, fuzzy integrated inventory model is introduced by changing the crisp order quantity into fuzzy order quantity. Suppose fuzzy order quantity $\tilde{Q}$ be a trapezoidal fuzzy number, $\tilde{Q}=(Q_1,Q_2,Q_3,Q_4)$ with the constraint $0<Q_1\le Q_2\le Q_3\le Q_4$.

The fuzzy integrated total cost consists of fuzzy buyer and vendor ordering cost, fuzzy inventory holding cost, fuzzy setup cost and fuzzy lead time crashing cost. That is fuzzy ordering cost per unit time $=(\tilde{A}\otimes\tilde{D})\oslash\tilde{Q}$, buyer's fuzzy holding cost per unit time is $=((\tilde{Q}\oslash 2)\oplus(k\otimes\sigma\otimes\sqrt{L}))\otimes(\tilde{r}\otimes\tilde{C}_b)$, fuzzy lead time crashing cost per unit time $=(\tilde{D}\oslash\tilde{Q})\otimes R(L)$, vendor fuzzy setup cost per year $=(\tilde{D}\oslash(m\otimes\tilde{Q}))\otimes\tilde{S}$ and vendor's fuzzy holding cost per unit time is $=(\tilde{Q}\oslash 2)\otimes[(m\otimes(1-(\tilde{D}\oslash\tilde{P}))\ominus 1\oplus((2\otimes\tilde{D})\oslash\tilde{P}))]\otimes(\tilde{r}\otimes\tilde{C}_v)$. Then, the fuzzy integrated total cost is

$$\begin{aligned}I\tilde{T}C(\tilde{Q},L,m) =& [(\tilde{D}\oslash\tilde{Q})\otimes(\tilde{A}\oplus(\tilde{S}\oslash m)\oplus R(L))]\ominus[[(\tilde{Q}\otimes\tilde{r}\otimes\tilde{C}_v)\oslash 2]\otimes[(m\otimes\tilde{D}\oslash\tilde{P})+1]]\\ &\oplus[[(\tilde{Q}\otimes\tilde{r})\oslash 2]\otimes[[m\oplus((2\otimes\tilde{D})\oslash\tilde{P})]\otimes\tilde{C}_v\oplus\tilde{C}_b]]\oplus[\tilde{r}\otimes\tilde{C}_b\otimes k\otimes\sigma\otimes\sqrt{L}],\end{aligned} \tag{18}$$

where $\oplus,\otimes,\oslash$, and $\ominus$ are the fuzzy arithmetical operators under function principle.

Suppose $\tilde{D}=(D_1,D_2,D_3,D_4)$, $\tilde{A}=(A_1,A_2,A_3,A_4)$, $\tilde{r}=(r_1,r_2,r_3,r_4)$, $\tilde{S}=(S_1,S_2,S_3,S_4)$, $\tilde{P}=(P_1,P_2,P_3,P_4)$, $\tilde{C}_v=(C_{v1},C_{v2},C_{v3},C_{v4})$ and $\tilde{C}_b=(C_{b1},C_{b2},C_{b3},C_{b4})$ are non-negative trapezoidal fuzzy numbers. Then the optimal order quantity of equation (18) is calculated as follows.

We start with, fuzzy integrated total cost $I\tilde{T}C(\tilde{Q},L,m)$ which is given by equation (18). Then,

$$\begin{aligned}&I\tilde{T}C(\tilde{Q},L,m)=\\ &\left[\left(\frac{D_1}{Q_4}\left(A_1+\frac{S_1}{m}+R(L)\right)-\frac{Q_4r_4C_{v4}}{2}\left(\frac{mD_4}{P_1}+1\right)+\frac{Q_1r_1}{2}\left(\left(m+\frac{2D_1}{P_4}\right)C_{v1}+C_{b1}\right)+r_1C_{b1}k\sigma\sqrt{L}\right),\right.\\ &\left(\frac{D_2}{Q_3}\left(A_2+\frac{S_2}{m}+R(L)\right)-\frac{Q_3r_3C_{v3}}{2}\left(\frac{mD_3}{P_2}+1\right)+\frac{Q_2r_2}{2}\left(\left(m+\frac{2D_2}{P_3}\right)C_{v2}+C_{b2}\right)+r_2C_{b2}k\sigma\sqrt{L}\right),\\ &\left(\frac{D_3}{Q_2}\left(A_3+\frac{S_3}{m}+R(L)\right)-\frac{Q_2r_2C_{v2}}{2}\left(\frac{mD_2}{P_3}+1\right)+\frac{Q_3r_3}{2}\left(\left(m+\frac{2D_3}{P_2}\right)C_{v3}+C_{b3}\right)+r_3C_{b3}k\sigma\sqrt{L}\right),\\ &\left.\left(\frac{D_4}{Q_1}\left(A_4+\frac{S_4}{m}+R(L)\right)-\frac{Q_1r_1C_{v1}}{2}\left(\frac{mD_1}{P_4}+1\right)+\frac{Q_4r_4}{2}\left(\left(m+\frac{2D_4}{P_1}\right)C_{v4}+C_{b4}\right)+r_4C_{b4}k\sigma\sqrt{L}\right)\right].\end{aligned} \tag{19}$$

Also, the Graded mean integration representation of $I\tilde{T}C(\tilde{Q},L,m)$ is obtained by equation (11) as

$$\begin{aligned}&P[I\tilde{T}C(\tilde{Q},L,m)]=\\ &\frac{1}{6}\left[\left(\frac{D_1}{Q_4}\left(A_1+\frac{S_1}{m}+R(L)\right)-\frac{Q_4r_4C_{v4}}{2}\left(\frac{mD_4}{P_1}+1\right)+\frac{Q_1r_1}{2}\left(\left(m+\frac{2D_1}{P_4}\right)C_{v1}+C_{b1}\right)+r_1C_{b1}k\sigma\sqrt{L}\right)\right.\\ &+2\left(\frac{D_2}{Q_3}\left(A_2+\frac{S_2}{m}+R(L)\right)-\frac{Q_3r_3C_{v3}}{2}\left(\frac{mD_3}{P_2}+1\right)+\frac{Q_2r_2}{2}\left(\left(m+\frac{2D_2}{P_3}\right)C_{v2}+C_{b2}\right)+r_2C_{b2}k\sigma\sqrt{L}\right)\\ &+2\left(\frac{D_3}{Q_2}\left(A_3+\frac{S_3}{m}+R(L)\right)-\frac{Q_2r_2C_{v2}}{2}\left(\frac{mD_2}{P_3}+1\right)+\frac{Q_3r_3}{2}\left(\left(m+\frac{2D_3}{P_2}\right)C_{v3}+C_{b3}\right)+r_3C_{b3}k\sigma\sqrt{L}\right)\\ &\left.+\left(\frac{D_4}{Q_1}\left(A_4+\frac{S_4}{m}+R(L)\right)-\frac{Q_1r_1C_{v1}}{2}\left(\frac{mD_1}{P_4}+1\right)+\frac{Q_4r_4}{2}\left(\left(m+\frac{2D_4}{P_1}\right)C_{v4}+C_{b4}\right)+r_4C_{b4}k\sigma\sqrt{L}\right)\right],\end{aligned} \tag{20}$$

with $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$. It will not change the meaning of equation (20) if inequality conditions are replaced $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$ into the inequality condition $Q_2 - Q_1 \geq 0$, $Q_3 - Q_2 \geq 0$, $Q_4 - Q_3 \geq 0$ and $Q_1 > 0$. In the following steps, extension of the Lagrangian method is used to find the solutions of Q_1, Q_2, Q_3 and Q_4 to minimize $P[I\tilde{TC}(\tilde{Q}, L, m)]$ in equation (20).

Step 1 : Solve the unconstraint problem: In order to find the minimization of $P[I\tilde{TC}(\tilde{Q}, L, m)]$, we find the partial derivatives of $P[I\tilde{TC}(\tilde{Q}, L, m)]$ with respect to Q_1, Q_2, Q_3 and Q_4 then equate them to zero as follows.

$$\frac{1}{6}\left[-\frac{D_4}{Q_1^2}\left(A_4 + \frac{S_4}{m} + R(L)\right) - \frac{r_1 C_{v1}}{2}\left(\frac{mD_1}{P_4} + 1\right) + \frac{r_1}{2}\left(\left(m + \frac{2D_1}{P_4}\right)C_{v1} + C_{b1}\right)\right] = 0, \tag{21}$$

$$\frac{2}{6}\left[-\frac{D_3}{Q_2^2}\left(A_3 + \frac{S_3}{m} + R(L)\right) - \frac{r_2 C_{v2}}{2}\left(\frac{mD_2}{P_3} + 1\right) + \frac{r_2}{2}\left(\left(m + \frac{2D_2}{P_3}\right)C_{v2} + C_{b2}\right)\right] = 0, \tag{22}$$

$$\frac{2}{6}\left[-\frac{D_2}{Q_3^2}\left(A_2 + \frac{S_2}{m} + R(L)\right) - \frac{r_3 C_{v3}}{2}\left(\frac{mD_3}{P_2} + 1\right) + \frac{r_3}{2}\left(\left(m + \frac{2D_3}{P_2}\right)C_{v3} + C_{b3}\right)\right] = 0, \tag{23}$$

$$\frac{1}{6}\left[-\frac{D_1}{Q_4^2}\left(A_1 + \frac{S_1}{m} + R(L)\right) - \frac{r_4 C_{v4}}{2}\left(\frac{mD_4}{P_1} + 1\right) + \frac{r_4}{2}\left(\left(m + \frac{2D_4}{P_1}\right)C_{v4} + C_{b4}\right)\right] = 0, \tag{24}$$

Solving the above equations (21) to (24), we get the optimal order quantities which are given by

$$Q_1 = \sqrt{\frac{2D_4\left(A_4 + \frac{S_4}{m} + R(L)\right)}{r_1\left(\left(m\left(1 - \frac{D_1}{P_4}\right) - 1 + \frac{2D_1}{P_4}\right)C_{v1} + C_{b1}\right)}}, \tag{25}$$

$$Q_2 = \sqrt{\frac{4D_3\left(A_3 + \frac{S_3}{m} + R(L)\right)}{2r_2\left(\left(m\left(1 - \frac{D_2}{P_3}\right) - 1 + \frac{2D_2}{P_3}\right)C_{v2} + C_{b2}\right)}}, \tag{26}$$

$$Q_3 = \sqrt{\frac{4D_2\left(A_2 + \frac{S_2}{m} + R(L)\right)}{2r_3\left(\left(m\left(1 - \frac{D_3}{P_2}\right) - 1 + \frac{2D_3}{P_2}\right)C_{v3} + C_{b3}\right)}}, \tag{27}$$

$$Q_4 = \sqrt{\frac{2D_1\left(A_1 + \frac{S_1}{m} + R(L)\right)}{r_4\left(\left(m\left(1 - \frac{D_4}{P_1}\right) - 1 + \frac{2D_4}{P_1}\right)C_{v4} + C_{b4}\right)}}. \tag{28}$$

The above results show that $Q_1 > Q_2 > Q_3 > Q_4$ and they do not satisfy the constraint $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$. Therefore set $k = 1$ and go to Step 2.

Step 2 : Convert the inequality constraint $Q_2 - Q_1 \geq 0$ into equality constraint $Q_2 - Q_1 = 0$ and optimize $P[I\tilde{TC}(\tilde{Q}, L, m)]$ subject to $Q_2 - Q_1 = 0$ by the Lagrangian method.

The Lagrangian function is $L(Q_1, Q_2, Q_3, Q_4, \lambda) = P[I\tilde{TC}(\tilde{Q}, L, m)] - \lambda(Q_2 - Q_1)$. The partial derivatives of $L(Q_1, Q_2, Q_3, Q_4, \lambda)$ with respect to Q_1, Q_2, Q_3, Q_4 and λ are taken to find the minimization of $L(Q_1, Q_2, Q_3, Q_4, \lambda)$. Let all the partial derivatives be equal to zero to solve Q_1, Q_2, Q_3 and Q_4. Then,

$$Q_1 = Q_2 = \sqrt{\frac{4D_3\left(A_3 + \frac{S_3}{m} + R(L)\right) + 2D_4\left(A_4 + \frac{S_4}{m} + R(L)\right)}{r_1\left(\left(m\left(1 - \frac{D_1}{P_4}\right) - 1 + \frac{2D_1}{P_4}\right)C_{v1} + C_{b1}\right) + 2r_2\left(\left(m\left(1 - \frac{D_2}{P_3}\right) - 1 + \frac{2D_2}{P_3}\right)C_{v2} + C_{b2}\right)}}, \tag{29}$$

$$Q_3 = \sqrt{\frac{4D_2\left(A_2 + \frac{S_2}{m} + R(L)\right)}{2r_3\left(\left(m\left(1 - \frac{D_3}{P_2}\right) - 1 + \frac{2D_3}{P_2}\right)C_{v3} + C_{b3}\right)}}, \tag{30}$$

$$Q_4 = \sqrt{\frac{2D_1\left(A_1 + \frac{S_1}{m} + R(L)\right)}{r_4\left(\left(m\left(1 - \frac{D_4}{P_1}\right) - 1 + \frac{2D_4}{P_1}\right)C_{v4} + C_{b4}\right)}}. \tag{31}$$

Again, the above results show that $Q_3 > Q_4$ and it does not satisfy the constraint $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$. Therefore it is not a local optimum. Similarly, the same result can be achieved by considering any other inequality constraint as equality constraint. Therefore set $k = 2$ and go to Step 3.

Step 3 : Convert the inequality constraint $Q_2 - Q_1 \geq 0$, $Q_3 - Q_2 \geq 0$ into equality constraints $Q_2 - Q_1 = 0$ and $Q_3 - Q_2 = 0$. $P[I\tilde{TC}(\tilde{Q}, L, m)]$ is optimized subject to $Q_2 - Q_1 = 0$ and $Q_3 - Q_2 = 0$ by the Lagrangian method.

Then the Lagrangian function is $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2) = P(I\tilde{TC}(\tilde{Q}, L, m)) - \lambda_1(Q_2 - Q_1) - \lambda_2(Q_3 - Q_2)$. In order to find the minimization of $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2)$, we take the partial derivatives of $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2)$ with respect to $Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2$ and let all the partial derivatives equal to zero and solve Q_1, Q_2, Q_3 and Q_4 then we get,

$$Q_1 = Q_2 = Q_3 = \sqrt{\frac{4D_2\left(A_2 + \frac{S_2}{m} + R(L)\right) + 4D_3\left(A_3 + \frac{S_3}{m} + R(L)\right) + 2D_4\left(A_4 + \frac{S_4}{m} + R(L)\right)}{\begin{array}{c} r_1\left(\left(m\left(1 - \frac{D_1}{P_4}\right) - 1 + \frac{2D_1}{P_4}\right)C_{v1} + C_{b1}\right) + 2r_2\left(\left(m\left(1 - \frac{D_2}{P_3}\right) - 1 + \frac{2D_2}{P_3}\right)C_{v2} + C_{b2}\right) \\ +2r_3\left(\left(m\left(1 - \frac{D_3}{P_2}\right) - 1 + \frac{2D_3}{P_2}\right)C_{v3} + C_{b3}\right)\end{array}}}, \tag{32}$$

$$Q_4 = \sqrt{\frac{2D_1\left(A_1 + \frac{S_1}{m} + R(L)\right)}{r_4\left(\left(m\left(1 - \frac{D_4}{P_1}\right) - 1 + \frac{2D_4}{P_1}\right)C_{v4} + C_{b4}\right)}}. \tag{33}$$

From the above results it is clear that $Q_1 > Q_4$ and does not satisfy the constraint $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$. Therefore it is not a local optimum. Similarly, the same result can be achieved by considering any two inequality constraint as equality constraint. Therefore set $k = 3$ and go to Step 4.

Step 4 : Convert the inequality constraint $Q_2 - Q_1 \geq 0$, $Q_3 - Q_2 \geq 0$ and $Q_4 - Q_3 \geq 0$ into equality constraints $Q_2 - Q_1 = 0$, $Q_3 - Q_2 = 0$ and $Q_4 - Q_3 = 0$. We optimize $P[I\tilde{TC}(\tilde{Q}, m, L)]$ subject to $Q_2 - Q_1 = 0$, $Q_3 - Q_2 = 0$ and $Q_4 - Q_3 = 0$ by the Lagrangian method.

The Lagrangian function is given by

$$L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3) = P(I\tilde{TC}(\tilde{Q}, m, L)) - \lambda_1(Q_2 - Q_1) - \lambda_2(Q_3 - Q_2) - \lambda_3(Q_4 - Q_3).$$

In order to find the minimization of $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3)$, we take the partial derivatives of $L(Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3)$ with respect to $Q_1, Q_2, Q_3, Q_4, \lambda_1, \lambda_2, \lambda_3$ and let all the partial derivatives equal to zero and solve Q_1, Q_2, Q_3 and Q_4 then we get,

$$Q_1 = Q_2 = Q_3 = Q_4 = \sqrt{\frac{2D_1\left(A_1 + \frac{S_1}{m} + R(L)\right) + 4D_2\left(A_2 + \frac{S_2}{m} + R(L)\right) + 4D_3\left(A_3 + \frac{S_3}{m} + R(L)\right) + 2D_4\left(A_4 + \frac{S_4}{m} + R(L)\right)}{\begin{array}{c} r_1\left(\left(m\left(1 - \frac{D_1}{P_4}\right) - 1 + \frac{2D_1}{P_4}\right)C_{v1} + C_{b1}\right) + 2r_2\left(\left(m\left(1 - \frac{D_2}{P_3}\right) - 1 + \frac{2D_2}{P_3}\right)C_{v2} + C_{b2}\right) \\ +2r_3\left(\left(m\left(1 - \frac{D_3}{P_2}\right) - 1 + \frac{2D_3}{P_2}\right)C_{v3} + C_{b3}\right) + r_4\left(\left(m\left(1 - \frac{D_4}{P_1}\right) - 1 + \frac{2D_4}{P_1}\right)C_{v4} + C_{b4}\right)\end{array}}}. \tag{34}$$

As the above solution $\tilde{Q} = (Q_1, Q_2, Q_3, Q_4)$ satisfies all inequality constraints, the procedure terminates with $\tilde{Q}$ as a local optimum solution to the problem. Since the above local optimum solution is the only feasible solution of equation (20), it is an optimum solution of the inventory model with fuzzy order quantity according to Extension of the Lagrangian Method. Let $Q_1 = Q_2 = Q_3 = Q_4 = \tilde{Q}^*$. The optimal fuzzy order quantity is $\tilde{Q}^* = (Q^*, Q^*, Q^*, Q^*)$, where

$$\tilde{Q}^* = \sqrt{\frac{2D_1\left(A_1 + \frac{S_1}{m} + R(L)\right) + 4D_2\left(A_2 + \frac{S_2}{m} + R(L)\right) + 4D_3\left(A_3 + \frac{S_3}{m} + R(L)\right) + 2D_4\left(A_4 + \frac{S_4}{m} + R(L)\right)}{\begin{array}{c} r_1\left(\left(m\left(1 - \frac{D_1}{P_4}\right) - 1 + \frac{2D_1}{P_4}\right)C_{v1} + C_{b1}\right) + 2r_2\left(\left(m\left(1 - \frac{D_2}{P_3}\right) - 1 + \frac{2D_2}{P_3}\right)C_{v2} + C_{b2}\right) \\ +2r_3\left(\left(m\left(1 - \frac{D_3}{P_2}\right) - 1 + \frac{2D_3}{P_2}\right)C_{v3} + C_{b3}\right) + r_4\left(\left(m\left(1 - \frac{D_4}{P_1}\right) - 1 + \frac{2D_4}{P_1}\right)C_{v4} + C_{b4}\right)\end{array}}}. \tag{35}$$

5.3 Algorithm for Inventory Models

The following algorithm is designed to find the optimal order quantity and integrated total cost. In the crisp sense using equations (9) and (6) respectively, we get the optimal order quantity Q^* and minimum integrated total cost $ITC(Q, L, m)$. In the fuzzy sense using equations (35) and (20) respectively, the optimal fuzzy order quantity $\tilde{Q}^*$ and minimum fuzzy integrated total cost $P[I\tilde{T}C(\tilde{Q}, L, m)]$ are obtained. Further, the comparisons are given for both conventional crisp model and the fuzzy model. This is used to determine the suitable model for finding the optimal order quantity and the integrated total cost with their savings.

Algorithm

Step 1 : Calculate optimal order quantity and integrated total cost in the conventional crisp model for the given crisp values of D, P, k, C_v , r, A, S, σ , L, $R(L)$, m and C_b. Then crisp optimal order quantity Q^* and crisp integrated total cost $ITC(Q, L, m)$ are obtained.

Step 2 : Determine fuzzy integrated total cost using fuzzy arithmetic operations on fuzzy buyer and vendor ordering cost, fuzzy inventory holding cost, fuzzy setup cost and fuzzy lead time crashing cost which is taken as trapezoidal fuzzy number.

Step 3 : Use graded mean integration method to defuzzify integrated total cost $I\tilde{T}C(Q, L, m)$, in order to find the order quantity Q_c^* which can be obtained by putting the first derivative of $P[I\tilde{T}C(Q, L, m)]$ is equal to zero.

Step 4 : Use extension of the Lagrangian method to find the fuzzy optimal order quantity $\tilde{Q}^* = (Q^*, Q^*, Q^*, Q^*)$, which is the special method for trapezoidal fuzzy number. The optimal fuzzy order quantity $\tilde{Q}^* = (Q^*, Q^*, Q^*, Q^*)$, is obtained by substituting the first derivative of $P[I\tilde{T}C(\tilde{Q}, L, m)]$ that is equal to zero.

Step 5 : To check whether the optimal order quantity Q_c^* obtained by Graded Mean Integration is the same as the fuzzy optimal order quantity $\tilde{Q}^*$.

Step 6 : Compare the integrated total cost and the optimal order quantity in conventional crisp model and fuzzy model. If $Q^* > \tilde{Q}^*$ and $ITC(Q, L, m) > P[I\tilde{T}C(\tilde{Q}, L, m)]$ then the proposed fuzzy model is finest to find the optimal order quantity and integrated total cost, else $Q^* < \tilde{Q}^*$ and $ITC(Q, L, m) < P[I\tilde{T}C(\tilde{Q}, L, m)]$ then the conventional crisp model is the finest to find the optimal order quantity and integrated total cost.

Step 7 : Compare the optimal order quantity, integrated total cost obtained from both conventional crisp model and the fuzzy model with their savings.

6. NUMERICAL EXAMPLE

In this section, numerical examples are given to demonstrate the above solution procedure using the proposed algorithm. After applying the algorithm, the best inventory model is identified. The solutions to these examples are obtained by using computer Matlab software. The proposed vendor-buyer fuzzy model can be used in industries such as aircraft, healthcare, printers, cars, computers, textiles, clothing, refrigerators, cell phones, televisions, washing machines, tyres, air conditioners and huge items such as printed circuit boards, etc. Integrated inventory model is useful especially for Just-in-time (JIT) inventory systems where seller and buyer form a strategic partnership for profit sharing. The proposed integrated inventory model is more valid for the supply chain manufacturing process and seller and buyer management.

6.1 Crisp Model

Example. 1

To illustrate the solution procedure for crisp model, let us consider the system with the data used in Pan and Yang (2002), $D = 1000$ units/year, $P = 3200$ units/year, $k = 2.33$, $r = 0.2$, $S = \$400$/setup, $C_v = 20$/units, $C_b = 25$/units, $\sigma = 7$ units/week. In addition, we take $L = 3, 4, 6$ and 8 weeks, $R(L) = \$53.2, \$18.2, \$1.4$ and \$0, $m = 3, 4, 5$ and 5, $A = \$21.87$/order, \$22.50/order, \$23.75/order and \$25.00/order. Using equations (9) and (6) respectively, optimal order quantity Q^* and minimum integrated total cost $ITC(Q, L, m)$ are attained. The results are tabulated in Table 1.

Table 1: Optimal solution for crisp model

D	P	k	C_v	r	A	S	C_b	σ	L	$R(L)$	m	Q^*	$ITC(Q,L,m)$
1000	3200	2.33	20	0.2	21.87	400	25	7	3	53.2	3	188.34	2354.3
1000	3200	2.33	20	0.2	22.5	400	25	7	4	18.2	4	139.31	2183.1
1000	3200	2.33	20	0.2	23.75	400	25	7	6	1.4	5	110.41	2104.4
1000	3200	2.33	20	0.2	25	400	25	7	8	0	5	110.34	2133.9

6.2 Fuzzy Model

Example. 2
The data is the same as in Example. 1, except that the fuzzy parameters $\tilde{D}$, $\tilde{P}$, $\tilde{C_v}$, $\tilde{r}$, $\tilde{A}$, $\tilde{S}$ and $\tilde{C_b}$. The order quantity in Example. 1 is transferred as fuzzy order quantity $\tilde{Q} = (Q_1, Q_2, Q_3, Q_4)$ with $0 < Q_1 \leq Q_2 \leq Q_3 \leq Q_4$. The proposed algorithm yields the result as shown in Table 2. Using equations (35) and (20) respectively, optimal fuzzy order quantity $\tilde{Q}^*$ and minimum fuzzy integrated total cost $P[I\tilde{TC}(\tilde{Q}, L, m)]$ are obtained.

Table 2: Optimal solution for fuzzy model

$\tilde{D}$	$\tilde{P}$	k	$\tilde{C_v}$	$\tilde{r}$	$\tilde{A}$	$\tilde{S}$	$\tilde{C_b}$	σ	L	$R(L)$	m	$Q^* = \tilde{Q}^*$	$P[I\tilde{TC}(\tilde{Q},L,m)]$
(950, 975, 1025, 1050)	(3100, 3150, 3250, 3300)	2.33	(15, 17, 22, 27)	(0.12, 0.14, 0.2, 0.4)	(18, 19, 23.5, 28.22)	(380, 390, 410, 420)	(20, 22, 28, 30)	7	3	53.2	3	181.99	2317.0
(950, 975, 1025, 1050)	(3100, 3150, 3250, 3300)	2.33	(15, 17, 22, 27)	(0.12, 0.14, 0.2, 0.4)	(18.5, 20.6, 23.8, 27.7)	(380, 390, 410, 420)	(20, 22, 28, 30)	7	4	18.2	4	134.63	2148.3
(950, 975, 1025, 1050)	(3100, 3150, 3250, 3300)	2.33	(15, 17, 22, 27)	(0.12, 0.14, 0.2, 0.4)	(17, 21, 27.1, 29.3)	(380, 390, 410, 420)	(20, 22, 28, 30)	7	6	1.4	5	106.75	2067.2
(950, 975, 1025, 1050)	(3100, 3150, 3250, 3300)	2.33	(15, 17, 22, 27)	(0.12, 0.14, 0.2, 0.4)	(16, 24, 27, 32)	(380, 390, 410, 420)	(20, 22, 28, 30)	7	8	0	5	106.68	2096.8

Table 3: Summary of Optimal solution

L	Q^*	$ITC(Q,L,m)$	$\tilde{Q}^*$	$P[I\tilde{TC}(\tilde{Q},L,m)]$	Savings (%) Optimal order quantity	Savings (%) Integrated total cost
3	188.34	2354.3	181.99	2317.0	3.37	1.58
4	139.31	2183.1	134.63	2148.3	3.36	1.59
6	110.41	2104.4	106.75	2067.2	3.31	1.77
8	110.34	2133.9	106.68	2096.8	3.32	1.74

Table 1 shows in crisp environment, when lead time $L = 3, 4, 6$ and 8 weeks, we get the optimal order quantity Q^* and minimum integrated total cost $ITC(Q, L, m)$ which range from 110.34 units to 188.34 units and from $\$2133.9$ to $\$2354.3$ respectively. The results of Table 1 illustrate that when lead time increases optimal order quantity Q^* decreases and minimum integrated total cost $ITC(Q, L, m)$ initially decreases and increases later. Similarly, Table 2 illustrates fuzzy environment. When lead time $L = 3, 4, 6$ and 8 weeks we get the optimal fuzzy order quantity $\tilde{Q}^*$

decreases and minimum fuzzy integrated total cost $P[I\tilde{T}C(\tilde{Q}, L, m)]$ which range from 106.68 units to 181.99 units and from \$2096.8 to \$2317.0 respectively. The results of Table 2 demonstrate that the optimal fuzzy order quantity $\tilde{Q}^*$ and minimum fuzzy integrated total cost $P[I\tilde{T}C(\tilde{Q}, L, m)]$ initially decreases and increases later when the lead time increases. Summarized optimal order quantity and minimum integrated total cost are tabulated in Table 3. It is observed that optimal order quantity and minimum integrated total cost savings while using the fuzzy model range from 3.31% to 3.37% and from 1.58% to 1.74% respectively. It is also indicated that the optimal order quantity and minimum integrated total cost solutions of fuzzy case considerably fluctuate from the solutions of the crisp case.

6.3 Graphical Representation

Graphical representation of the optimal order quantity for the different numbers of lead time is compared both in the crisp and fuzzy models as depicted in Figure 4. It is clear that the optimal lot-size Q^* and $\tilde{Q}^*$ decrease when the lead time L increases. It is observed that the optimal order quantity is productively optimized in the fuzzy model when compared to the crisp model. The next graphical representation of integrated total cost along with the lead time is compared both in the crisp and fuzzy models as shown in Figure 5. The integrated total cost $ITC(Q, L, m)$ and fuzzy integrated total cost $P[I\tilde{T}C(\tilde{Q}, L, m)]$ decrease initially and then start to increase later when the lead time increases. It is perceived that the integrated total cost is effectively minimized in the fuzzy model when compared to the crisp model. Another graphical representation of optimal order quantity and integrated total cost along with the lead time L is compared both in the crisp and fuzzy models as exhibited in Figure 6. The integrated total cost $ITC(Q, L, m)$, fuzzy integrated total cost $P[I\tilde{T}C(\tilde{Q}, L, m)]$ initially decreases and increases later then optimal lot-size Q^*, fuzzy optimal lot-size $\tilde{Q}^*$ decrease correspondingly when the lead time L increases. It is observed that the optimal order quantity and integrated total cost are constructively optimized in the fuzzy model compared to the crisp model.

7. COMPARATIVE STUDY

A comparative study of the proposed model with that of optimal order quantity and integrated total cost are shown in Table 4. It is observed that as lead time increases, the percentage variation of the performance measures between the models decreases in optimal order quantity and increases then decreases in integrated total cost. The model with fuzzy optimal order quantity and integrated total cost has higher utilisation than the model with crisp optimal order quantity and integrated total cost. It is also observed that the assumption of fuzzy optimal order quantity and integrated total cost has a significant influence on all the performance measures of the model. Lead time has an important effect on the system performance measures, and this fuzzy inventory model can predict the performance measures more efficiently. This fuzzy inventory model is also compared with some specific cases of the previous models.

In Pan and Yang (2002) model they found the optimal order quantity, lead time and delivering number when the probability distribution of the lead time demand was normal. The aim of Vijayashree and Uthayakumar (2017) mathematical model was to reduce the ordering cost dependent on lead time. The proposed model followed procedures to find optimal order quantity and minimum integrated total cost in fuzzy sense. We compare the proposed model with the previous models and the results are tabulated in Table 4. From this an efficient result for the proposed fuzzy model is attained.

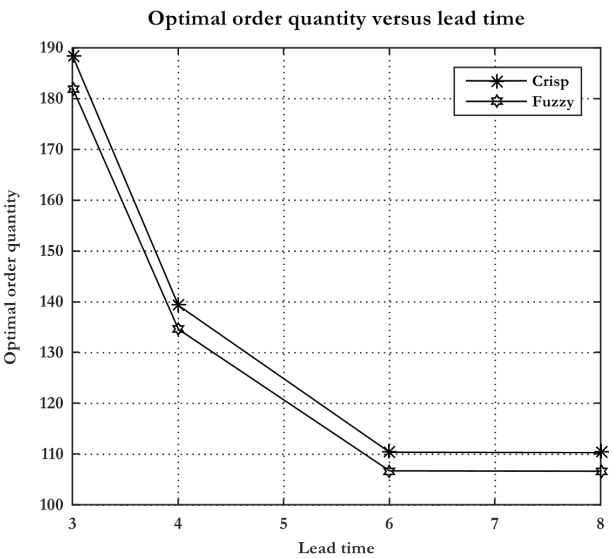

Figure 4: Graphical representation of optimal order quantity versus lead time.

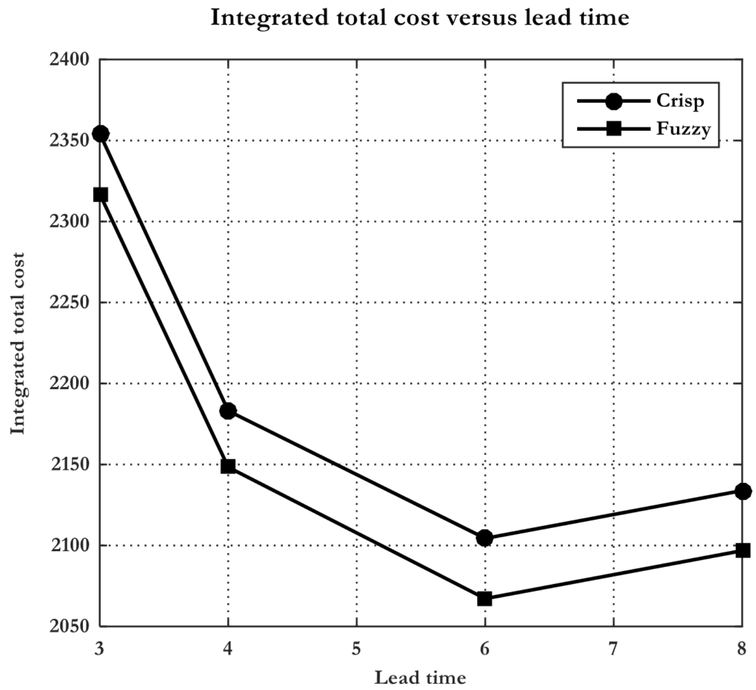

Figure 5: Graphical representation of integrated total cost versus lead time.

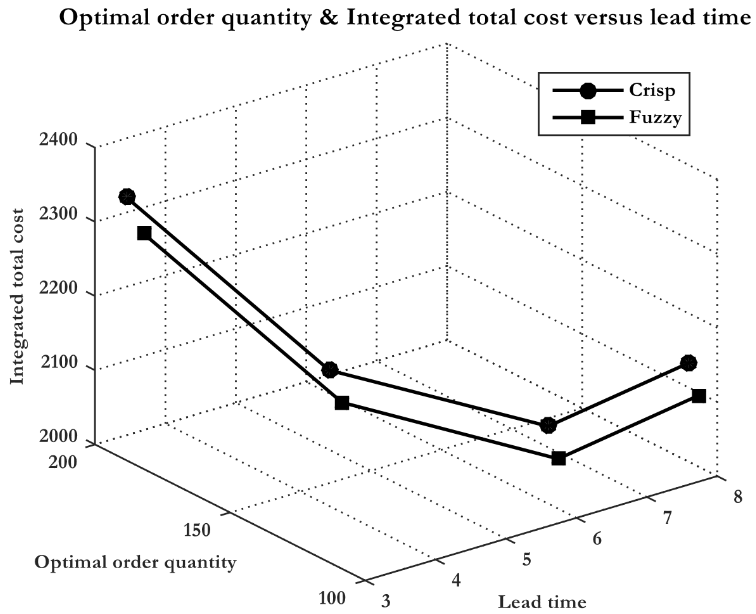

Figure 6: Graphical representation of optimal order quantity and integrated total cost versus lead time.

Table 4: Summary of the comparisons

L	$R(L)$	m	A	Pan and. Yang model (2002)		Vijayashree and Uthayakumar model (2017) Ordering cost considered as Linear case		This fuzzy inventory model with fuzzy order quantity		Savings (%)	
				Optimal order quantity	Integrated total cost	Optimal order quantity	Integrated total cost	Optimal order quantity	Integrated total cost	Optimal order quantity	Integrated total cost
3	53.2	3	25	190	2370.8	189.75	2370.8	183.37	2330.5	3.36	1.70
4	18.2	4		141	2200.9	140.54	2200.9	135.84	2161.4	3.34	1.79
6	1.4	5		111	2115.7	111.07	2115.7	107.38	2076.7	3.32	1.84
8	0	5		110	2134.0	110.34	2133.9	106.68	2096.8	3.32	1.74

Table 4 shows in crisp environment results of Pan and Yang (2002) & Vijayashree and Uthayakumar (2017) inventory models. They used lead time $L = 3, 4, 6$ and 8 weeks and got the optimal order quantity and integrated total cost which range from 110.34 units to 188.34 units and from $\$2133.9$ to $\$2354.3$ respectively. In our proposed fuzzy inventory model for lead time $L = 3, 4, 6$ and 8 weeks, we get the optimal order quantity and integrated total cost which range from 106.68 units to 183.37 units and from $\$2076.7$ to $\$2330.5$ respectively. It is observed that the considerable variations in optimal order quantity and integrated total cost in the application of two inventory models.

Hence, our fuzzy integrated inventory model helps the organization to handle uncertain inventory cost parameters. It is observed that uncertain cost parameters give 3.32% to 3.36% and 1.70% to 1.84% of the savings in the optimal order quantity and integrated total cost respectively. Then uncertain cost parameters are positive as a predictor statistically different from zero. Also, they have a significant and direct effect on inventory. Therefore, organizations are able to find optimal order quantity and minimum integrated total cost in profitable manner. From this, we conclude that if the models are dealt with crisp sense, it cannot properly embed with real situations. But taking the same models with fuzzy sense, it is possible to embed properly in real situations. Hence the fuzzy sense is the best to handle the inventory models.

8. CONCLUSION

A single-vendor and a single-buyer integrated inventory model with ordering cost reduction dependent on lead time is formulated and developed in both crisp and fuzzy environments. In fuzzy environment, all related inventory parameters are assumed to be trapezoidal fuzzy numbers. For defuzzification, graded mean integration method is employed to evaluate the optimal integrated total cost. Extension of Lagrangian method is used to determine the optimal order quantity. A computational algorithm is framed to investigate the effects of fuzzy parameters on the optimal order quantity and minimum integrated total cost of the proposed integrated vendor-buyer system. Graphical representation of numerical examples show that by using the proposed fuzzy model, one can obtain a significant amount of savings in integrated inventory model. By comparing the proposed model with the previous models, an efficient result for the proposed fuzzy model is attained. After comparing both the conventional crisp and fuzzy models, it is observed that the fuzzy model is better than the conventional crisp model.

Future researches on this problem can deal with inventory constraints, ordering constraints, etc. Further, various type of multi-echelon supply chain models can be considered in crisp sense, fuzzy sense or both.

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