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Some Results on Fermat-Weber Location Problem in Frechet Spaces

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Abstract: We capture the Fermat-Weber location problem for Frechet spaces, where the Frechet space is determined by the inverse limit of the projective system of Banach spaces. Countable collections of continuous seminorms on the Frechet space are used as gauges to define the Fermat-Weber location problem. Sufficient conditions on the existence of the set of solutions for the functions are obtained via reflexivity of the space and the Hahn-Banach theorem, while convexity structure is used to establish the uniqueness of the solution. Ouresults extend and generalize the corresponding results available for Banach spaces and others in that direction.
Keyword -- Fermat-Weber problem; Frechet spaces; strict convexity; collinear points; continuous seminorm.

1. INTRODUCTION

The location problem deals with the optimal placement of new facilities among existing facilities. In the 17th century, Fermat posed a problem to Torricelli on the determination of the point that is a minimizer of the sum of distances from that point to the given three points in the plane. Torricelli, on his own part, solved the problem in several ways.

Later in the year 1909, a German economist, Weber (1929), generalized the location problem by Fermat with the inclusion of a cost function which takes the functional to the whole set of real numbers. The generalized version was named after Weber and coined the Weber or Fermat-Weber problem. There has been a great deal of work on the generalized version as a result of its wide applications in practical studies, which include those that are connected with network optimization and wireless communications. However, the following authors studied Fermat-Torricelli and Fermat-Weber problems focusing on the space of the domain of definition of the functional.

Brimberg and Love (1999) worked on the Weber problem on Eucleadian space with the norm as the distance function. Martini *et al.* (2002) extended the problem to normed planes, and this yielded mathematically interesting results. Vesley (1993) looked at the Fermat-Torricelli problem in reflexive normed linear spaces, and Papini and Puerto (2004) based their work on the Fermat-Torricelli problem in Banach Spaces as they considered a finite subset of a Banach space by minimizing the sum of distances from k furthest points of the domain. Dragomir and Comanescu (2008) considered the Fermat-Torricelli problem for inner product spaces, and their work was based on the problem of minimizing the sum of distances from a point to n distinct fixed points in an inner product space. Dragomir *et al.* (2013) published some results on the Fermat-Torricelli problem and focused on the existence of a set of points that minimizes the sum of distances to the n distinct points in a normed linear space. Radulescu *et al.* (2015) considered the existence and uniqueness of the set of minimizers for the modified Weber problem, where they looked at the set of linear operators between normed linear spaces. The work of Ayinde and Osinuga (2019) was based on location problems in pre-inner product spaces. Others in this direction include Jahn *et al.* (2014), Nguyen (2013), Nguyen (2018), and Bo and Ruriko (2018). The results in Dragomir *et al.* (2013) and Radulescu *et al.* (2015) lead to useful extensions and serve as motivations behind this work. Our goal is to complement the results on existence and also the uniqueness of the solution of the Fermat-Weber location problem from Radulescu *et al.* (2015) point of view by formulating them in the more general setting of Frechet spaces.

2. PRELIMINARIES

We give some definitions and notions that are required for subsequent development. For more details, see Kelley and Namioka (1963), Raymond and Yol (2001), Treves (1967), and Tsotuiashivili and Zuernadze (2006).

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A family $\mathcal{P}=\{p_i\}_{i\in I}$ of continuous seminorms on a topological linear space X is called a fundamental system of seminorms if the sets $U_i=\{x\in X : p_i(x)<1\}(i\in I)$ form a fundamental system of zero neighborhoods.

A locally convex space (LCS) is a topological linear space with the fundamental system of 0-neighborhoods comprising convex sets.

A metrizable LCS is an LCS whose topology is given by a countable system of continuous seminorms. The completion of a metrizable LCS is a Frechet Space.

A normed linear space (NLS) X is a topological vector space whose topology is determined by the norm $\|\cdot\|$. The completion of an NLS is a Banach space.

The Frechet spaces considered in this work are graded. If the Frechet space X is graded, this implies that it is equipped with a fixed fundamental system of seminorms whereby its topology is given by an increasing sequence of norms $p_n=\|.\|_n$, $n>0$ for all $x\in X$, $\|x\|_n\le\|x\|_{n+1}$, where $X_n=X/kerp_n$ is a Banach space.

Hence, this suggests the following projective system.

$\to X_{n+1}\to X_n\to X_{n-1}\to\ldots\to X_0$

Each identity map is continuous and injective.

Definition 2.1 (Taylor, 1995)

Let X be a Frechet space and $\{p_n\}_{n\in\mathbb{N}}=P$ a system of seminorms defining the topology of X and define $kerp_n=\{x\in X \mid p_n(x)=0\}$ as a linear subspace of X, then a norm is defined on $X/kerp_n$ by $\|x+kerp_n\|=\hat{p}_n(x)=p_n(x)$. Hence we call $X_n:=\{X/kerp_n \mid \hat{p}_n(x)\}$ the Banach space for the seminorm $\hat{p}_n$. Therefore, for the canonical map $f_n: X\to X_n$, $f_n(x):=x+kerp_n$ and $\|f_n(x)\|_n=\hat{p}_n\circ f_n(x)=\hat{p}_n(f_n(x))=p_n(x)$ for all $x\in X$ and for all p_n.

We remark that $X=\lim_{\leftarrow}X_n=\bigcap X_n$ is a dense subspace of X_n. Therefore, $U_n=\{x\in X : p_n(x)\le\alpha\}$ $(n\in N)$ is a fixed fundamental system of neighborhoods with the fixed fundamental system $\{p_n\}_{n\in\mathbb{N}}=P$ of seminorms/norms. We let $X_n=X/kerp_n$ be a Banach space which is of finite dimension.

Let X be a graded Frechet space with the Banach space $X_n=X/kerp_n$ finite dimensional. Given a convex and closed set $U_n\subset X$ with $0\in U_n$, the gauge function of the set U_n.

$p_{U_n}: X\to\mathbb{R}^+$ is defined as

$$p_{U_n}(x)=\begin{cases}+\infty & if \quad \{\alpha>0 : x\in\alpha U_n\}=\phi\\ inf\{\alpha>0:\ x\in\alpha U_n\} & otherwise\end{cases} \tag{1}$$

The gauge function $p_{U_n}=p_n$ is a seminorm. This implies that if $0\in U_n$, domain of p_n is X.

Suppose X and Y are Frechet spaces and if $T: X\to Y$ is continuous, linear and invertible then $T^{-1}: Y\to X$ is also continuous.

Let X be a locally convex space. A subset $U\subset X$, is called bornivorous if for every bounded set $B\subset X$. So there is $\lambda>0$ that $B\subset\lambda U$. X is called bornological if every absolutely convex bornivorous set is in the neighborhood of zero.

Definition 2.2

Let X be a nls. X is strictly convex if $\|v+t\|=\|v\|+\|t\|$, $v\ne t$, then $t=rv$ for some real number $r>0$, v,t in X. See Dragomir *et al.* (2013).

Definition 2.3

Distinct points $\{r_1,\cdots,r_n\}$ in an LCS X are referred to as collinear if for two distinct points $v,t\in X$ with $\{\gamma_i\}_{i=(1,\cdots,n)}\in\mathbb{R}$.

$r_i = \gamma_i(v) + (1-\gamma_i)t$ where $\gamma_1 < \gamma_2 < \cdots < \gamma_n$. (2)

Definition 2.4

Let X be a lcs with its topology given by $q_j \in Q = \{q_j\}_{i\in I}$. X is referred to as strictly convex if $q_j(v+t) = q_j(v) + q_j(t)$, $q_j(v) \neq q_j(t)$ then $q_j(t - rv) = 0$ for some real number $r > 0$ and $V_{v,t} \bigcap ker q_j = \{0\}$ for every $v, t \in X$,. $V_{(v,t)}$ spans v and t.

3. EXISTING RESULTS IN BANACH SPACES

A more generalized Fermat-Weber problem was considered and studied by Radulescu *et al.* (2015) for Banach spaces. We here present their existence and uniqueness results.

Suppose X and Y are normed linear spaces, $\{T_1, T_2, ..., T_n\}$ the set of continuous linear operators from X to Y and $u_i : \mathbb{R}^+ \to \mathbb{R}, i = 1, 2, ..., n, n \in \mathbb{N}$. Considering the distinct points $\{a_1, a_2, ..., a_n\} \in Y$, and $f : X \to \mathbb{R}$, then the Weber problem is given by $f(x) = \sum_i^n u_i \| (T_i x - a_i) \|$.

Let $x_0 \in X$, the solution set for the function $f(x)$ is defined as $M = \{x \in X \mid f(x) \leq f(x_0)\}$

The following results on the existence and uniqueness of solutions to the functions $f(x)$ are identified for Banach spaces.

Theorem 3.1. (Radulescu *et al.*, 2015)

Let X be a reflexive Banach space and let $u_i : \mathbb{R}^+ \to \mathbb{R}, i = 1, 2, ..., n, n \in \mathbb{N}$ be convex and increasing functions for which there exists $k \in \{1, 2, 3, ..., n\}$ such that $\lim_{t\to\infty} u_k(t) = \infty$ and T_k is invertible. Then $f(x) = \sum_i^n u_i \| (T_i x - a_i) \|$ admits at least one minimum point.

Theorem 3.2. (Radulescu *et al.*, 2015)

Let X be a reflexive Banach space, Y be a strictly convex Banach space, and $u_i : \mathbb{R}^+ \to \mathbb{R}$ $(i = 1, ..., n)$ be increasing convex functions. Assume that there exists $k \in \{1, ..., n\}$ such that u_k is strictly increasing, strictly convex and $Ker T_k = \{0\}$. Then f admits a point of global minimum, and this is unique.

4 EXISTENCE AND UNIQUENESS OF FERMAT-WEBER POINT IN FRECHET SPACES

Herein, our purpose is to find conditions on the Frechet spaces X and Y for which the Theorems (3.1 and 3.2) can be generalized.

4.1 Existence of minimum point

We begin with some preliminary results that give conditions on the Frechet spaces X and Y which will guarantee the existence of the minimizer . u_i represents weight function that is convex, continuous, and increasing (see (Radulescu *et al.*, 2015), Lemma 2.1).

Motivated by the definition of the Weber problem for Banach spaces given in section 3 and by [(Osinuga *et al.*, 2020) Theorem 4.1], we state the following problem.

Let X and Y be graded Frechet spaces, $\{T_1, T_2, ..., T_n\}$ set of continuous maps from X to Y and the topology of Y determined by fixed continuous gauges $q_j \in Q = \{q_j\}_{j\in\mathbb{N}}$, then for all j

$$W(v)=\sum_{i}^{n}u_i q_j(T_i v-r_i),\ r_i\in\{r_1,\cdots,r_n\}\subset Y \tag{3}$$

defines the Fermat-Weber problem, and its solution set is given as $R:=\{v\in X\,|\,W(v)\le W(t),t\in X\}$ where $u_i:[0,\infty)\to\mathbb{R}$. The following Theorem helps to establish the definition of the Fermat-Weber problem for Frechet spaces.

Theorem 4.1.1.

Let X and Y be Frechet spaces (projective limits of Banach space) with their topologies determined by continuous seminorms $p_j\in P=\{p_j\}_{j\in\mathbb{N}}$ and $q_j\in Q=\{q_j\}_{j\in\mathbb{N}}$, respectively. Given $A=\{r_1,r_2,\ldots,r_n\}$ a finite subset in Y of fixed points and suppose $u_i:[0,\infty)\to\mathbb{R}$ be increasing and continuous for $i=1,2,3,4,\cdots,n$. Then the following are equivalent.

(i) There is a Fermat-Weber problem defined on Y;
(ii) For each Banach space Y_i with a continuous map $g_i:Y\to Y_i$ there exists a Fermat-Weber problem defined on each Banach space Y_i.

Proof.

The proof follows the same pattern in [(Osinuga *et al.*, 2020) Theorem 4.1]. Moreover, the following proposition constructs a bounded set for the Fermat-Weber problem in Frechet space.

Proposition 4.1.2.

Suppose X and Y are Frechet spaces, $\{T_1,T_2,...,T_n\}$ the set of linear continuous and invertible operators from X to Y and $u_i:\mathbb{R}^+\to\mathbb{R},i=1,2,...,n,n\in\mathbb{N}$. Given a finite set $\{a_j\}_j^n$ in a Banach space Y_k, let B_k be a bounded set for the Fermat-Weber problem $\sum_{j=1}^{m}u_j\,\|(v'-T_{k_j}^{-1}a_j)\|$ on a Banach space X_k for each k, then given a finite set $\{r_j\}_j^n\subset Y$, the set B for the Fermat-Weber problem $\sum_{j=1}^{m}u_j\,p_k(v-T_j^{-1}r_j)$ in X is bounded for all k.

Proof.

The proof follows the same pattern in [Osinuga *et al.* (2020) Proposition 4.1.3]. Furthermore, the following lemma identifies the properties of the functional $W(v)$.

Lemma 4.1.3.

Let X and Y are Frechet spaces with their topologies determined by continuous seminorms $p_j\in P=\{p_j\}_{j\in\mathbb{N}}$ and $q_j\in Q=\{q_j\}_{j\in\mathbb{N}}$, respectively. Given $A=\{r_1,r_2,\ldots,r_n\}$ a finite subset in Y of fixed points and suppose $u_i:[0,\infty)\to\mathbb{R}$ be increasing and continuous for $i=1,2,3,4,\cdots,n$. Furthermore, let $T_i:X\to Y,i,k\in\{1,2,3,4,\cdots,n\}$ such that $\lim_{s\to\infty}u_k(s)=\infty$ and T_k be invertible. Then

(i) $W(v)=\sum_{i=1}^{n}u_i(q_j(T_i v-r_i))$ is continuous.
(ii) $\lim_{p(v)\to\infty}W(v)=\infty$.
(iii) $W(v)=\sum_{i=1}^{n}u_i(q_j(T_i v-r_i))$ is convex.

Proof.

(i) Given fixed and distinct points $\{r_1, r_2, \ldots, r_n\} \subset Y$, then $r_i + W_j$, $j \in \mathbb{N}$ is a neighborhood of r_i in Y with $W_j = \{T_i v \mid q_j(T_i v) \le \varepsilon\}$ for some $\varepsilon > 0$. Hence, $\{r_1, r_2, \ldots, r_n\} \subset \bigcup_{i=1}^{n}(r_i + \bigcap_j W_j)$ where $\varepsilon \bigcap_j W_j = \{T_i v \mid Sup\, q_j(T_i v) < \varepsilon\}$. Therefore, for $T_i v \in Y$ then, $T_i v \in r_i + W_j$. This shows that $T_i v - r_i \in W_j$ hence, $q_j(T_i v - r_i) \le \varepsilon$ for all j ; therefore, we have $\sum_i^n | q_j(T_i v) - q_j(r_i) | \le \sum_{i=1}^n q_j(T_i v - r_i) \le n\varepsilon$ and since u_i for all j is continuous, therefore, $W(v)$ is continuous.

(ii) We define $W(v) = \sum_{i=1}^{n} u_i(q_j(T_i v - r_i))$ where $q_j \in \{q_j\}_j$, $j \in \mathbb{N}$ is a seminorm that defines the topology of Y . By the property of seminorm $W(v) = \sum_{i=1}^{n} u_i(q_j(T_i v - r_i)) \ge \sum_{i=1}^{n} u_i(| q_j(T_i v) - q_j(r_i) |) \ge | \sum_{i=1}^{n} u_i q_j(T_i v) - u_i q_j(r_i) |$

Given that $T_i : X \to Y$ is continuous if and only if $q_j(T_i v) \le c p_j(v)$ for which $c > 0$ and $p_j \in \{p_j\}_j$, $j \in \mathbb{N}$.

So also, since for $k \in \{1, 2, 3, 4, \cdots, n\}$ $\lim_{s\to\infty} u_k(s) = \infty$ and T_k be invertible. Hence, there is $v \in X$ where $p_j(T_k^{-1}(y)) \le c q_j(T_k v)$, for $y = T_k v \in Y$, we then have $p_j(v) \le c q_j(T_k v)$ for all j .
Let $s_0 = \max_{1\le k\le n} q_j(r_k)$.

$$W(v) = \sum_{k=1}^{n} u_k(q_j(T_k v - r_k)) \ge \sum_{k=1}^{n} u_k(| q_j(T_k v) - q_j(r_k) |) \ge | \sum_{i=k}^{n} u_k(p_j(v) - s_0) | . \tag{4}$$

Suppose $s = p_j(v) - s_0$. Now, as $p_j(v) \to \infty, s \to \infty$ and $\lim_{s\to\infty} u_k(s) = \infty$. Hence, this implies that as $p_j(v) \to \infty, W(v) \to \infty$

Therefore, $\lim_{p_j(v)\to\infty} W(v) = \infty$

(iii) For all v, t in X with $r \in [0,1]$ Hence,

$$W(rv + (1-r)t) = \sum_{i=1}^{n} u_i q_j((rT_i v + (1-r)T_i t - r_i) \le \sum_{i=1}^{n} u_i q_j(r(T_i v - r_i) + (1-r)(T_i t - r_i))$$

$$\le r\sum_{i=1}^{n} u_i q_j(T_i v - r_i) + (1-r)\sum_{i=1}^{n} u_i q_j(T_i t - r_i) = rW(v) + (1-r)W(t)$$

Hence, $W(v)$ is convex. Lemma 4.1.4 highlights the properties of the solution set for the functional $W(v)$.

Lemma 4.1.4.

Suppose a Frechet space X is reflexive and let Y be another Frechet space with its topology determined by continuous semi norms $Q = \{q_j\}_{j\in\mathbb{N}}$. Given a finite set $A = \{r_1, r_2, \ldots, r_n\} \subset Y$. Then, for $W(v)$ as given in Lemma 4.1.3, $R := \{v \in X \mid W(v) \le W(t), t \in X\}$ is convex, closed, and bounded subset of X .

Proof.

Suppose $v_1, v_2 \in R$, $t_1, t_2 \in X$ and $r \in [0,1]$ let consider $[v_1, v_2] \in rv_1 + (1-r)v_2$ and $[t_1, t_2] \in rt + (1-r)t_2$. From the definition of R , we have $W(rv_1 + (1-r)v_2) \le W(rt_1 + (1-r)t_2)$.

From Lemma 4.1.3(iii), W is convex. Therefore, the convexity of R follows from the convexity of $W(v)$

Next we show that R is closed. For $v \in R$ we have by definition $W(v) = \sum_{i=1}^{n} u_i q_j (T_i v - r_i)$.

Hence, the pullback of the gauge q_j for all j is the pre-image of a closed set, and by the continuity of the maps therefore, R is closed.

We finally show that $R = \{v \mid W(v) \le W(t), t \in X\}$ is bounded. From Proposition 4.1.2, we have $\sup_{v \in B} p_k(v) < \infty$. By definition $v \in R$, and since R is closed, it implies that $v \in B \subset R$. Hence, R as the closure of a bounded set B is bounded.

Our next result connects the functional $W(v)$ with lower semi-continuity.

Lemma 4.1.5.

Let X be a Frechet space. By Lemma 4.1.4, $R \subset X$ is a weakly compact set. Then $W : R \to [-\infty, \infty]$ on R is a weakly lower semi-continuous function.

Proof.

From Lemma 4.1.4, R is weakly compact in X. $W(R)$ is a continuous and bounded image in $[-\infty, \infty]$. Since R is weakly compact, we define a sequence $\{W_n\}_n$ finite in $W(R) \subset [-\infty, \infty]$ and hence, uniformly convergent. For $v \in R$, there exists $|W_n(v) - W(v)| < \delta$, $\delta > 0$, $n > N \in \mathbb{N}$. Hence, there exists a complete lattice formed by this expression in $[-\infty, \infty]$; therefore, we have :

$$k = \sup_{v \in R} \{ | W_n(v) - W(v) | \} \quad \textit{for all} \quad v \in R \tag{5}$$

However, R which is weakly compact in X means that we can have a convergent subsequence $\{v_{n_i}\}$ from the sequence $\{v_n\} \subset R$ $(n = 1, 2, \ldots,)$ which converges weakly to v say, for which $n_i > N \in \mathbb{N}$ $v_{n_i} \overset{weakly}{\to} v \textit{ in } R$ or $\lim v_{n_i} = v$ in R. Hence,

$$| W_n(\lim v_{n_i}) - W(\lim v_{n_i}) | \le k \tag{6}$$

that is

$$| W(\lim v_{n_i}) - W_n(\lim v_{n_i}) | \le k \tag{7}$$

which implies
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$$-k + W_n(\lim v_{n_i}) \le W(\lim v_{n_i}) \le k + W_n(\lim v_{n_i}) \tag{8}$$

and

$$W(\lim v_{n_i}) \le k + \liminf W_n(v_{n_i}) \le k + k + \liminf W(v_{n_i}) \le 2k + \liminf W(v_{n_i}) \tag{9}$$

Hence,

$$W(\lim v_{n_i}) \le 2\alpha + \liminf W(v_{n_i}) \tag{10}$$

For all $\alpha > 0$, which gives $W(\lim v_{n_i}) \le \liminf W(v_{n_i})$.

Hence,

$$W(\lim v_n) \leq \liminf W(v_n) \tag{11}$$

That is,

$$W(v) \leq \liminf W(v_n) \tag{12}$$

Therefore, W is weakly lower semi-continuous on R.

Theorem 4.1.6.

Let X be a Frechet space, $R \subset X$ a weakly compact set and by Lemma 4.1.5, $W : R \to [-\infty, \infty]$ is weakly lower semi-continuous on R. Then, $W : R \to [-\infty, \infty]$ is bounded on R.

Proof.

The proof of Theorem 4.1.6 is similar to Bector *et al.* (2007), p 16, Theorem 1.3.1.

Theorem 4.1.7.

Let X be a reflexive Frechet space and $u_i : [0, \infty) \to \mathbb{R}$ $i = 1, 2, 3, 4, \cdots, n$ be convex and increasing. Let $\lim_{t\to\infty} u_k(t) = \infty$ and $T_k : X \to Y$ be invertible. Then the function $(W : X \to \mathbb{R}) := \sum_{i=1}^{n} u_i q_j (T_i v - r_i)$ has at least a minimum point in R for all j.

Proof.

By Lemma 4.1.4, R is weakly compact. W is weakly lower semi-continuous on R by Lemma 4.1.5. So also in theorem 4.1.6, W attains bound on R and this shows that W attains a minimum on R.

The following existence result relies on the Hahn-Banach theorem, and it is of independent interest.

Theorem 4.1.8.

Suppose X and Y are Frechet spaces and $T_k : X \to Y$ invertible and let the topology of Y be defined by the family of continuous seminorms $Q = \{q_j\}_{j\in N}$. If given a set $A = \{r_1, r_2, ..., r_n\}$ consisting of distinct points in Y and the subspace $S = span\, T_k^{-1}(A) = span\, \{T_1^{-1} r_1, T_2^{-1} r_2, ..., T_k^{-1} r_k\}$ of X is reflexive, then $R = \{v \in X \mid W(v) \leq W(t), t \in X\}$, a solution set for the Fermat-Weber problem $W(v) = \sum_{i=1}^{n} u_i q_j (T_i v - r_i)$ is non-empty in X for all q_j.

Proof.

Given $A = \{r_1, r_2, ..., r_n\}$ in Y we define $span\, A := span\, \{r_1, r_2, ..., r_n\}$ such that there exists $T_k^{-1}(A) = \{T_1^{-1} r_1, T_2^{-1} r_2, ..., T_k^{-1} r_k\}$, with $S = span\, T_k^{-1}(A) = span\, \{T_1^{-1} r_1, T_2^{-1} r_2, ..., T_k^{-1} r_k\}$ defined, which is a subspace of X. Since $S = span\, T_k^{-1}(A)$ is reflexive, then by Theorem 4.1.7 there exists a solution set $R_S = \{v \in S \mid W(v) \leq W(s), s \in S\}$ in S.

Given that the family of continuous seminorms $Q = \{q_j\}_{j\in N}$ defined the topology of Y, by Hahn-Banach theorem, there exists a linear functional q_m defined on $Span\ A$ such that $q_m(T_i s) \leq q_j(T_i s)$ for all q_j and for each $s \in S$.

Suppose $v \in T_k^{-1}(A)$, then it implies that $v \in S$. However, S as the intersection of all subspaces containing

$T_k^{-1}(A)$, contains weakly compact sets. Hence, let $v \in S / T_k^{-1}(A)$. For both cases, this implies that for an element $y \in X$, we have $q_m(T_i s + \lambda T_i v) \le q_j(T_i s + \lambda T_i y)$ for all q_j and $|\lambda| \le 1$. Since q_m is a linear functional, the following suffices $q_m(T_i s + \lambda T_i v) = q_m(T_i s) + \lambda q_m(T_i v) \le q_j(T_i s + \lambda T_i y)$.

For $r_i \in \{r_i\}_i$, $q_m(T_i s - r_i) + \lambda q_m(T_i v - r_i) \le q_j((T_i s - r_i) + \lambda(T_i y - r_i))$. Let $\lambda = 1$, we then have $q_m(T_i v - r_i) \le q_j((T_i s - r_i) + (T_i y - r_i)) - q_m(T_i s - r_i)$. For finite set $\{r_i\}_i$ and continuous and increasing weight function u_i, we can have $\sum_{i=1}^{n} u_i q_m(T_i v - r_i) \le \sum_{i=1}^{n} u_i q_j((T_i s - r_i) + (T_i y - r_i)) - u_i q_m(T_i s - r_i)$.

On the other hand, let $\lambda = -1$ and replace $(T_i s - r_i)$ by $-(T_i s - r_i)$.

Hence,

$$-q_m(T_i v - r_i) \le q_j(-(T_i s - r_i) - (T_i y - r_i)) - q_m(-(T_i s - r_i)). \tag{13}$$

That is:

$$q_m(T_i v - r_i) \ge -q_j(-(T_i s - r_i) - (T_i y - r_i)) - q_m(T_i s - r_i). \tag{14}$$

For finite set $\{r_i\}_i$ and continuous and increasing weight function u_i, we can have

$$\sum_{i=1}^{n} u_i q_m(T_i v - r_i) \ge \sum_{i=1}^{n} u_i(-q_j(-(T_i s - r_i) - (T_i y - r_i))) - u_i q_m(T_i s - r_i) \tag{15}$$

Hence

$$\sum_{i=1}^{n} u_i(-q_j(-(T_i s - r_i) - (T_i y - r_i))) - u_i q_m(T_i s - r_i) \le \sum_{i=1}^{n} u_i q_m(T_i v - r_i) \le \sum_{i=1}^{n} u_i q_j((T_i s - r_i) + (T_i y - r_i)) - q_m(T_i s - r_i)$$

by subadditivity. (16)

Therefore,

$$\bar{h} := sup_{s \in S} \sum_{i=1}^{n} u_i \,|(-q_j(-(T_i s - r_i) - (T_i y - r_i))) - u_i q_m(T_i s - r_i)| \le$$

$$inf_{s \in S} \sum_{i=1}^{n} u_i \,|q_j((T_i s - r_i) + (T_i y - r_i)) - u_i q_m(T_i s - r_i)| := \underline{h} \tag{17}$$

for all q_j and a linear function q_m.

Any $\sum_{i=1}^{n} u_i q_m(T_i v - r_i)$ between $\bar{h}$ and $\underline{h}$ we do. By the Hahn-Banach theorem, the following hold.

$$|\sum_{i=1}^{n} u_i q_m(T_i v - r_i)| \le \sum_{i=1}^{n} u_i q_j(T_i v - r_i), \forall j. \tag{18}$$

$$|q_m(T_i s - r_i)| \le q_j(T_i s - r_i), \forall j. \tag{19}$$

$$|q_m(T_i s - r_i)| = min_{s \in S}\ q_j(T_i s - r_i), \forall j \tag{20}$$

Hence,

$$\sum_{i=1}^{n} u_i q_j((T_i s - r_i) + (T_i y - r_i)) - min_{s \in S}\ u_i q_j(T_i s - r_i) \le \sum_{i=1}^{n} u_i q_j(T_i s - r_i) + u_i q_j(T_i y - r_i) - min_{s \in S}\ u_i q_j(T_i s - r_i),\ \forall j \tag{21}$$

and for some $t \in X$ we finally have

$$W(v) = \sum_{i=1}^{n} u_i q_j (T_i v - r_i) \le \sum_{i=1}^{n} u_i q_j (T_i t - r_i) = \sum_{i=1}^{n} u_i q_j (T_i s - r_i) + u_i q_j (T_i y - r_i) - min_{s\in S}\, u_i q_j (T_i s - r_i) = W(t) \tag{22}$$

for $q_j \in Q = \{q_j\}_{j\in\mathbb{N}}$. The implication is that $R_S \subset R$. Therefore, $R = \{v \,|\, W(v) \le W(t), t \in X\}$ is non-empty.

4.2 Uniqueness of minimum point

Our results here complement the importance of convexity and reflexivity of the space in optimization. The results show that $W(v)$ has a global minimum on X . **Theorem 4.2.1.**

If Y is an (a Frechet space) inverse limit of $\{Y_j\}_{j\in N}$ where Y_j is a Banach space. Then, the following are equivalent:
(i) Y is strictly convex.
(ii) Y_j is strictly convex for each j

Proof.

Given $Q = \{q_j\}_{j\in N}$ as the system of seminorms defining the topology on Y . Set N_j as a set in the Banach space Y_j $(j = 1,2,3,...)$ with $V_{v,t} = g_j^{-1}(N_j)$ where $V_{v,t}$ spans v and t in the projective limit (Frechet space) Y and $g_j : Y \to Y_j$.

$(i) \Rightarrow (ii)$ considering the composition map $Y \to Y_j \to \mathbb{R}^+$ and map $q_j : Y \to \mathbb{R}^+$ where $q_j \in Q$ such that $\hat{q}_j \circ g_j = q_j$ by Definition 2.1, i.e. $\hat{q}_j(g_j(v)) = q_j(v)$ for all j . With this, we have $v_j \in N_j \subset Y_j$ such that $g_j(v) = v_j$. Therefore, $\hat{q}_j(g_j(v)) = \hat{q}_j(v_j)$ is $\|v_j\|$ which is a norm on Y_j.

From strict convexity of Y , by Definition 2.4 $q_j(v+t) = q_j(v) + q_j(t)$ which gives

$$\hat{q}_j(g_j(v+t)) = \hat{q}_j(g_j(v)) + \hat{q}_j(g_j(t)) \tag{23}$$

i.e.,

$$\hat{q}_j(v_j + t_j) = \hat{q}_j(v_j) + \hat{q}_j(t_j) \tag{24}$$

which is

$$\|v_j + t_j\| = \|v_j\| + \|t_j\| . \tag{25}$$

So also, by strict convexity of Y i.e. by Definition 2.4

$$q_j(v) \ne q_j(t) \tag{26}$$

Implies:

$$\hat{q}_j(g_j(v)) \ne \hat{q}_j(g_j(t)) \tag{27}$$

Which further implies

$$\hat{q}_j(v_j) \ne \hat{q}_j(t_j) . \tag{28}$$

Therefore,

$\|v_j\| \neq \|t_j\|$ implies $v_j \neq t_j$. (29)

Similarly, from strict convexity of

$Y, V_{v,t} \bigcap ker q_j = \{0\}$ by Definition 2.4 (30)

Given

$q_j(t - rv) = 0$ (31)

for some $r > 0$.

$V_{v,t}$ as the span of v and t implies that $t - rv \in V_{v,t}$. Also $q_j(t - rv) = 0$ implies that $t - rv \in ker q_j$ and with Y being Hausdorff

$t - rv \in V_{v,t} \bigcap ker q_j = \{0\}$. (32)

Hence,

$t - rv \in \cap g_j^{-1}(t_j - rv_j) = \{0\}$ (33)

by [(Robertson, A. and Robertson, W., 1973) pp.84-85, Propositions 11 and 13] from

$V_{r,t} = g_j^{-1}(N_j)$ (34)

Since g_j is continuous and by [(Robertson, A. and Robertson, W., 1973), p.84, Proposition 11]

$\cap g_j^{-1}(0) = \{0\}$. (35)

Therefore,

$t_j - rv_j = 0$ (36)

or

$t_j = rv_j$. (37)

This shows that Y_j is strictly convex.

$(ii) \Rightarrow (i)$. With $V_{v,t} \subset Y$ spanning v and t . The subspace Y_j^o of Y_j which represents the image of g_j is strictly convex as a subspace of a strictly convex Banach space Y_j . Hence, there is a set $N_j \subset Y_j^o$ with $t_j - rv_j \in N_j$. Since Y_j^o is strictly convex

$t_j - rv_j = 0.$ (38)

Since g_j is continuous on Y onto Y_j^o , we can set

$g_j(V_{v,t}) = N_j$, and (39)

$$V_{v,t} = g_j^{-1}(N_j) \ . \tag{40}$$

From strict convexity of Y_j^o and by Definition 2.2

$$\|v_j + t_j\| = \|v_j\| + \|t_j\| \tag{41}$$

or

$$\hat{q}_j(v_j + t_j) = \hat{q}_j(v_j) + \hat{q}_j(t_j) \tag{42}$$

which by Definition 2.2 gives

$$\hat{q}_j(g_j(v+t)) = \hat{q}_j(g_j(v)) + \hat{q}_j(g_j(v)), \forall j. \tag{43}$$

with

$$v, t \in V_{v,t} \ . \tag{44}$$

This implies that

$$q_j(v+t) = q_j(v) + q(t) \tag{45}$$

So also by strict convexity of Y_j^o and by Definition 2.2.

$$v_j \neq t_j \tag{46}$$

implies

$$\|v_j\| \neq \|t_j\| \tag{47}$$

or

$$\hat{q}_j(v_j) \neq \hat{q}_j(t_j) \tag{48}$$

which gives

$$q_j(v) \neq q_j(t) \ . \tag{49}$$

For

$$t_j - rv_j \in N_j \tag{50}$$

which implies that, for $t - rv \in V_{v,t}$, we have by the continuity of g_j and the property of projective system (see([16], p.85, Proposition 13))

$$t - rv \in \cap g_j^{-1}(t_j - rv_j) \subset V_{v,t} \tag{51}$$

Strict convexity of Y_j^o also implies $t_j - rv_j = 0$. Since g_j is continuous and for each

$$j \in \mathbb{N} \quad \cap g_j^{-1}(0) = \{0\} \tag{52}$$

by [(Robertson, A. and Robertson, W., 1973), p.84, Proposition 11].

Then,

$$t-rv \in \cap g_j^{-1}(t_j - rv_j) \subset V_{v,t} \bigcap ker\, q_j = \{0\}, \forall\, q_j \tag{53}$$

and each $j \in \mathbb{N}$ for

$$q_j \in Q,\ \forall q_j. \tag{54}$$

Therefore, $q_j(t-rv) = 0, \forall` q_j$. Hence, Y is strictly convex.

Proposition 4.2.2.

Suppose X and Y are Frechet spaces with Y strictly convex. Let $P = \{p_j\}_{j\in N}$ and $Q = \{q_j\}_{j\in N}$ be continuous seminorms determining the topologies of X and Y respectively. Let $u_i(i = 1, 2, ..., n)$ be continuous, increasing and convex, $T_i : X \to Y$ be continuous linear operators $i = 1, 2, 3, 4, \cdots, n$ and $\{r_1, \cdots, r_n\} \subset Y$ be distinct points with $W(v) = \sum_{i=1}^{n} u_i q_j (T_i v - r_i), v \in X$ and $\forall q_j$. If $k \in \{1, 2, 3, 4, \cdots, n\}$ such that u_k is strictly convex and strictly increasing, and $V_{(v,t)} \cap \ker T_k = \{0\}$, then W is strictly convex.

Proof.

Since $\{u_i\}_{i=\overline{1,n}}$ is convex and by Lemma 4.1.3, $W(v) = \sum u_i q_j (T_i v - r_i), \quad v \in X$ and for all q_j is convex, therefore $W(rv + (1-r)t) \le rW(v) + (1-r)W(t)$ with $r \in [0,1]$ and $v, t \in X$.

This is shown by contradiction. Suppose $W(v)$ is not strictly convex, that is suppose v and $t \in X$ and $0 < r < 1$ with $p_j(v) \ne p_j(t)$

This implies that

$$W(rv + (1-r)t) = rW(v) + (1-r)W(t) \tag{55}$$

That is,

$$\sum_{i=1}^{n} u_i q_j (T_i(rv + (1-r)t) - r_i) = r \sum_{i=1}^{n} u_i q_j (T_i v - r_i) + (1-r) \sum_{i=1}^{n} u_i q_j (T_i t - r_i),\ \forall\, q_j \tag{56}$$

Since all the terms are non-negative for all $i \in \{1, 2, 3, 4, \cdots, n\}$. Then,

$$u_i q_j (T_i(rv + (1-r)t) - r_i) = r u_i q_j (T_i v - r_i) + (1-r) u_i q_j (T_i t - r_i), \forall` q_j \tag{57}$$

We also have for $k \in \{1, 2, 3, 4, \cdots, n\}$

$$u_k q_j (T_k(rv + (1-r)t) - r_k) = r u_k q_j (T_k v - r_k) + (1-r) u_k q_j (T_k t - r_k),\ \text{for all}\ \ q_j \tag{58}$$

We imply that u_k be injective and since it is strictly increasing. Therefore,

$$q_j(T_k(rv + (1-r)t) - r_k) = r q_j (T_k v - r_k) + (1-r) q_j (T_k t - r_k) = q_j(r(T_k v - r_k)) + q_j((1-r)(T_k t - r_k)) \tag{59}$$

Strict convexity of Y implies that for

$$v_1 = r(T_k v - r_k) \ \text{ and }\ v_2 = (1-r)(T_k t - r_k) \tag{60}$$

$q_j(v_1 - \beta v_2) = 0$ for some $\beta > 0$ by Definition 2.4. (61)

Therefore,

$$q_j(r(T_k v - r_k) - \beta(1-r)(T_k t - r_k)) = 0 \quad (62)$$

Let $\beta = \dfrac{r}{1-r}$. Hence,

$$q_j(rT_k v - rr_k - rT_k t + rr_k) = 0 \quad (63)$$

That is,

$$q_j(rT_k v - rT_k t) = 0 \quad (64)$$

or

$$rq_j(T_k v - T_k t) = 0 \quad (65)$$

which eventually gives

$$q_j(T_k v - T_k t) = 0 \quad (66)$$

Hence, there exists by definition,

$$U_{(T_k v, T_k t)} \quad (67)$$

the linear span of $T_k v$ and $T_k t$ in Y such that

$$T_k v - T_k t \in U_{(T_k v, T_k t)} \bigcap \ker q_j = \{0\} \quad (68)$$

Therefore,

$$T_k v - T_k t = 0 \quad (69)$$

This further implies that $T_k(v-t) = 0$ by linearity of the operator T_k. Hence, this means

$$v - t \in V_{(v,t)} \bigcap \ker T_k = \{0\} \quad (70)$$

based on the condition in the theorem.
This implies that $p_j(v-t) = 0$ and $| p_j(v) - p_j(t) | \le p_j(v-t) = 0$. Hence, this shows that $p_j(v) = p_j(t)$. This is a contradiction to the earlier assumption we made. Hence, W is strictly convex.

Theorem 4.2.3.

Suppose X and Y are Frechet spaces with (Y,τ) strictly convex and topology τ defined by continous seminorm $Q = \{q_j\}_{j\in N}$ where seminorm $P = \{p_j\}_{j\in N}$ defines the topology of X. If u_i is convex, strictly increasing function, $T_i : X \to Y$ invertible for all $i = 1,2,3,4,\cdots,n$ and $\{(T_i^{-1} r_i)\}_i^n$ is a set of non collinear fixed points in X where $\{r_1,\cdots,r_n\} \subset Y$ are fixed and distinct points, then, the function

$$W(v)=\sum_{i=1}^{n}u_i q_j(T_i v-r_i)\ \forall q_j \tag{71}$$

is strictly convex.

Proof.

By Lemma 4.1.3 and u_i being convex

$$W(v)=\sum_{i=1}^{n}u_i q_j(T_i v-r_i)\ \forall q_j \tag{72}$$

is convex. Hence,

$$W(rv+(1-r)t)\le rW(v)+(1-r)W(t)\text{ with } r\in[0,1] \text{ and } v,t\in X. \tag{73}$$

We shall show the strict convexity of $W(v)$ by assuming that $W(v)$ is not strictly convex, that is, suppose v and $t\in X$ and $0<r<1$ with $p_j(v)\ne p_j(t)$ and

$$W(rv+(1-r)t)=rW(v)+(1-r)W(t)=W(rv)+W((1-r)t) \tag{74}$$

which means

$$\sum_{i=1}^{n}u_i q_j(T_i(rv+(1-r)t)-r_i)=r\sum_{i=1}^{n}u_i q_j(T_i v-r_i)+(1-r)\sum_{i=1}^{n}u_i q_j(T_i t-r_i)\ \forall q_j. \tag{75}$$

Since these involve non negative terms. Then,

$$u_i q_j(T_i(rv+(1-r)t)-r_i)=ru_i q_j(T_i v-r_i)+(1-r)u_i q_j(T_i t-r_i)\ \forall q_j. \tag{76}$$

Since u_i is strictly increasing, then we have

$$q_j(T_i(rv+(1-r)t)-r_i)=q_j(r(T_i v-r_i))+q_j((1-r)(T_i t-r_i))\ \forall q_j. \tag{77}$$

Since Y is strictly convex, this implies that for any

$$v_1=(r(T_i v-r_i))\quad\text{and}\quad v_2=(1-r)(T_i t-r_i) \tag{78}$$

and for some $\beta>0$ $q_j(v_1-\beta v_2)=0$. (79)

This implies that $v_1-\beta v_2\in ker q_j$ and $v_1-\beta v_2\in U_{(v_1,v_2)}$

Hence,

$$v_1-\beta v_2\in U_{(v_1,v_2)}\bigcap\ker q_j=\{0\}\text{ by Definition 2.4 and} \tag{80}$$

$$v_1-\beta v_2=(r(T_i v-r_i)-\beta((1-r)(T_i t-r_i))=0 \tag{81}$$

$$(rT_i v-rr_i)-\beta(1-r)(T_i t-r_i)=0 \tag{82}$$

i.e.

$$rT_i v-rr_i-(\beta-\beta r)(T_i t-r_i)=0 \tag{83}$$

$$rT_iv - rr_i - \beta T_it + \beta r_i + \beta rT_it - \beta rr_i = 0 \tag{84}$$

$$-rr_i + \beta r_i - \beta rr_i = \beta T_it - rT_iv - \beta rT_it \tag{85}$$

$$r_i(\beta - r - \beta r) = (\beta - \beta r)T_it - rT_iv. \tag{86}$$

$$r_i = \frac{\beta(1-r)T_it}{\beta - r(1+\beta)} - \frac{r}{\beta - r(1+\beta)}T_iv \tag{87}$$

$$r_i = -\frac{r}{\beta - r(1+\beta)}T_iv + \frac{\beta(1-r)}{\beta - r(1+\beta)}T_it \tag{88}$$

Since T_i is invertible

$$T_i^{-1}T_i = I \tag{89}$$

we have

$$T_i^{-1}r_i = -\frac{rv}{\beta - r(1+\beta)} + \frac{(1-r)\beta t}{\beta - r(1+\beta)} \tag{90}$$

This shows that $T_i^{-1}r_i, v, t$ are collinear in X. This is a contradiction to the assumption that $\{T_i^{-1}r_i, \cdots T_n^{-1}r_n\}$ are non-collinear.
Furthermore, since

$$(rT_iv - rr_i) - \beta(1-r)(T_it - r_i) \in U_{(v_1,v_2)} \bigcap \ker q_j = \{0\} \ \forall q_j, \tag{91}$$

then

$$q_j[(rT_iv - rr_i) - \beta(1-r)(T_it - r_i)] = \{0\} \ \forall q_j \tag{92}$$

Let

$$\beta = \frac{r}{1-r}. \tag{93}$$

$$q_j((rT_iv - rr_i) - (rT_it - rr_i)) = 0 \ \forall q_j \tag{94}$$

This gives

$$q_j(rT_iv - rT_it) = 0. \ \forall q_j \tag{95}$$

Therefore,

$$rq_j(T_iv - T_it) = 0. \ \forall q_j \tag{96}$$

This means

$$q_j(T_iv - T_it) = 0. \ \forall q_j \tag{97}$$

Following the fact that Y is strictly convex, there exists $U_{(T_iv,T_it)}$, a linear span of T_iv and T_it such that

$$T_i v - T_i t \in U_{(T_i v, T_i t)} \bigcap \ker q_j = \{0\}. \tag{98}$$

Hence, $T_i v - T_i t = 0$. This implies that $T_i(v-t) = 0$ by linearity. Since

$$T_i^{-1} T_i = I. \tag{99}$$

Hence,

$$T_i^{-1} T_i (v-t) = 0. \tag{100}$$

That is $v - t = 0$. Which means $p_j(v-t) = 0 \quad \forall\, p_j$.

We then have

$$| p_j(v) - p_j(t) | \le p_j(v-t) = 0 \ \forall\, p_j \tag{101}$$

that is

$$p_j(v) = p_j(t) \ \forall\, p_j \tag{102}$$

which contradicts the earlier assumption and coupled with the first contradiction, $W(v)$ is strictly convex.

Theorem 4.2.4.

Suppose X and Y are Frechet spaces and (Y, τ) is strictly convex with its topology τ determined by continous seminorm $Q = \{q_j\}_{j \in N}$ and seminorm $P = \{p_j\}_{j \in N}$ determining the topology of X. Let
(i) u_k be strictly increasing and strictly convex for $k \in \{1,2,3,4,\cdots,n\}$ and $V_{(v,t)} \bigcap \ker T_k = \{0\}$ or
(ii) $T_i : X \to Y$ invertible for all $i = 1,2,3,4,\cdots,n$ and $\{(T_i^{-1} r_i)\}_i^n$ set of non-collinear fixed points in X where $\{r_1, \cdots, r_n\} \subset Y$ are fixed and distinct points.

Then the minimum point of $W(v)$ is unique.

Proof.

For (i) and (ii). Since from previous results, the minimum points exist. Here, we shall show the uniqueness of the minimum point.
Let $v, t \in V_{(v,t)} \subset X$ be distinct points of the global minimum.

Since t is also a global minimum, this implies that $t \in R$. This also shows that $\frac{v+t}{2} \in R$. From the fact that W is strictly convex, we then have

$$W(rv + (1-r)t) < rW(v) + (1-r)W(t). \quad \text{For } r \in [0,1] \tag{103}$$

i.e.

$$W(rv + (1-r)t) < rW(v) + (1-r)W(t) = W(t) \tag{104}$$

Let $r = \frac{1}{2}$

$$W(\frac{v}{2}+\frac{t}{2})<\frac{1}{2}W(v)+\frac{1}{2}W(t)=W(t) \tag{105}$$

i.e.

$$W(\frac{v+t}{2})<\frac{W(v)+W(t)}{2}=W(t) \tag{106}$$

This is a contradiction. Hence, W has a unique minimum point.

CONCLUSION

We considered the Fermat-Weber problem for Frechet spaces and have been able to show that the set of minimizers exists for a reflexive Frechet space or a reflexive metrizable locally convex space. So also, the non-emptiness of a set of minimizers in a Frechet space whose subspace is reflexive was proved. The strict convexity of a Frechet space vis-a-vis the strict convexity of each of the Banach spaces in the projective system was proved and used to discuss the uniqueness of a minimizer in the weakly compact subset of the Frechet space.

CONFLICT OF INTEREST

The authors declare no conflict of interest regarding the publication of this paper.

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