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Reliability and Sensitivity Analysis of a System with Warm Standbys and a Repairable Service Station

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Abstract—We study the reliability and sensitivity analysis of a system with M operating machines, S warm standbys, and a repairable service station. Failure times and service times of each machine (operating or standby) are assumed to be exponentially distributed. While the service station is working, it is subject to breakdowns according to a Poisson process. When the station breaks down, it requires repair at a repair facility, where the repair times follow the negative exponential distribution. The K out of $M + S$ system is analyzed where $K = 1, 2, \ldots, M$. This paper presents derivations for the system reliability, $R_y(t)$, the mean time to system failure, $MTTF$, and numerical illustration. Several cases are analyzed to investigate the effects of various parameters on the $R_y(t)$ and the $MTTF$. Sensitivity analysis for the $R_y(t)$ and the $MTTF$ is also studied.

Keywords—reliability; sensitivity analysis, station breakdowns

1. INTRODUCTION AND LITERATURE REVIEW

In the open literature, most of the papers analyze the queueing systems where the service stations have never failed. However, in real-life situations we often encounter cases where service stations may break down and can be repaired. We study a system with $M + S$ identical machines and a single repairable service station. As many as M of these can be operating simultaneously in parallel, the rest of the S machines are warm-standby spares. A repairable service station means that the service station is typically subject to unpredictable breakdowns and can be repaired.

Several researchers have investigated some queueing systems in which a single service station subject to breakdowns is considered. Most of the papers deal with only some queueing problems of the system, rather than some reliability problems of the system. Past work may be divided into two parts according to the system is studied from the viewpoint of the queueing theory or from the viewpoint of the reliability. In the first category we review previous papers which relate to a queueing theory viewpoint only. Infinite source M/M/1 queue with breakdowns was first proposed by Wang (1989). Wang (1990) developed steady-state analytic solutions of the M/M/1 machine repair problem with a single service station subject to breakdowns. The M/E_k/1 machine repair problem with a non-reliable service station was proposed by Wang (1997). The second category of authors deal with papers which relates to a reliability viewpoint only. Cao and Cheng (1982) first introduced reliability concepts into a queueing system with a repairable service station where the life time of the service station is exponentially distributed and its repair time has a general distribution. Further, the reliability analysis of an M/G/1 queueing system in which the service station has an m-unit reliability series structure was analyzed by Cao (1994). Wang and Sivazlian (1989) studied the reliability characteristics of a multiple-server ($m + w$)-unit system with w warm standby units with exponential failure and exponential repair time distributions. Cao (1985) derived the reliability quantities of an M/G/1 machine repair model with a repairable service station which consists a single unit. Liu and Cao (1995) extended Cao's model to a repairable service station whose structure contains an m-unit reliability series. Li et al. (1997) examined the reliability analysis of an M/G/1 queueing system with server breakdowns and Bernoulli vacations. Tang (1997) investigated some reliability and queueing problems of a single-server M/G/1 queueing system subject to breakdowns. Recently, the steady-state availability and the mean time to system failure of a repairable system with warm standbys plus balking and reneging were studied by Ke and Wang (2002) and Wang and Ke (2003).

In this paper, we study the reliability characteristics of a repairable system to determine how reliability can be improved by providing sufficient spares as standbys. We also perform a sensitivity analysis for changes in the reliability characteristics along with changes in specific values of the system parameters. System failure is defined to be less than K machines in active operation, where $K = 1, 2, \ldots, M$ (K out of $M + S$ system). That is, the system failure is defined as: (i) the system fails when all $M + S$ machines fail; or (ii) the system fails when at least one of the M operating machines fails (i.e. the standby machines are emptied). This paper should be distinguished from previous works in that:

(a) the reliability problem with standbys has distinct characteristics which are different from the machine repair problem with standbys;

(b) it considers an arbitrary number of M machines operating simultaneously, and an arbitrary number of S

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machines are in preoperation (warm standby);
(c) it considers a repairable service station which is subject to breakdowns;
(d) it performs a sensitivity analysis.

We first develop the explicit expressions for the reliability, $R_Y(t)$, and the mean time to system failure, *MTTF*, by using Laplace transforms techniques. Next, we perform a parametric investigation which provides numerical results to show the effects of various system parameters to the $R_Y(t)$, and to the *MTTF*. Finally, we perform a sensitivity analyses for changes in the $R_Y(t)$ and the *MTTF* along with changes in specific values of the system parameters.

1.1 Notation

M: number of operating machines
S: number of warm standby machines
n: number of failed machines in the system
λ: failure rate of an operating machine
η: failure rate of a warm standby machine
μ: service rate of a failed machine
α: breakdown rate of a service station
β: repair rate of a service station
λ_n: mean failure rate when there are n failed machines in the system
$p_n(t)$: probability that the service station is working and there are n failed machines in the system at time t
$P(t)$: probability vector consisting of $p_n(t)$
$q_n(t)$: probability that the service station is broken down and there are n failed machines in the system at time t
$Q(t)$: probability vector consisting of $q_n(t)$
s: Laplace transform variable
$p_n^*(s)$: Laplace transform of $P_n(t)$
$P^*(s)$: Laplace transform of $P(t)$
$P(0)$: initial vector of $P(t)$ when $t=0$
$q_n^*(s)$: Laplace transform of $q_n(t)$
$Q^*(s)$: Laplace transform of $Q(t)$
$Q(0)$: initial vector of $Q(t)$ when $t=0$
Y: time to failure of the system
$R_Y(t)$: reliability function of the system
MTTF: mean time to system failure

2. DESCRIPTION OF THE SYSTEM

We consider a system with M identical machines operating simultaneously in parallel, S warm standbys, and a single service station which is subject to breakdowns.

It is assumed that the switch is perfect and that the switchover time is instantaneous. Each of the operating machines fails independently of the state of the others and has an exponential time-to-failure distribution with parameter λ. If an operating machine fails, it is immediately replaced by a spare if one is available. We assume that each of the available spares fails independently of the state of all the others and has an exponential time-to-failure distribution with parameter η ($0 < \eta < \lambda$). The failed machine is sent for service, and after service is treated as a spare. It is assumed that when a spare moves into an operating state, its failure characteristics will be that of an operating machine. Whenever an operating machine or a spare fails, it is immediately sent to a service station where it is served in order of breakdowns, with a time-to-service which is exponentially distributed with parameter μ. Further, the succession of failure times and the succession of service times are independently distributed random variables. Suppose that the service station can break down at any time with breakdown rate α. Whenever the service station breaks down, it is immediately repaired at a repair rate β. Again, breakdown times and repair times of the service station are assumed to be exponentially distributed. We now assume that the service station can serve only one failed machine at a time, and that the service is independent of the failure of the machines. If the service station breaks down, then a failed machine must wait until the service station is repaired. If service of a failed machine is allowed to be interrupted by a breakdown, resumption takes place as soon as the service station is available or the repair completion terminates. If the service station breaks down, then a failed machine must wait until a service station is repaired. When the repair of a service station is completed, the service station immediately serves a failed machine. Although no service occurs during the repair period of failed service station, failed machines continue to arrive according to a Poisson process. If an operating machine fails(or spare fails) and one spare is available at an instant when the service station is available, the failed machine at once goes for service, and the spare is put into operation. Once a service station is repaired, it becomes as good as new.

System reliability is studied according to the assumptions that system failure is defined to be less than K machines in active operation, where K = 1, 2, …, M. Therefore, if n denotes the number of failed machines in the system, the system is failed if and only if $n \geq L = M + S - K + 1$

3. RELIABILITY ANALYSIS OF THE SYSTEM

At time $t=0$, the system has just started operation with no failed machines when the service station is working. The reliability function under the exponential failure time, exponential service time, exponential breakdown time, and exponential repair time distributions can then be developed through the birth and death process. Let

$p_n(t) \equiv$ probability that the service station is working and there are n failed machines in the system at time t,
$q_n(t) \equiv$ probability that the service station is broken down and there are n failed machines in the system at time t,

where

$n = 0,1,2,...,L,$ and $L = M + S - K + 1, (K = 1,2,...,M)$.

The mean failure rate λ_n is given by:

$$\lambda_n = \begin{cases} M\lambda + (S-n)\eta & \textit{if } n = 0,1,...,S-1; \\ (M+S-n)\lambda & \textit{if } n = S, S+1,...,L-1; \\ 0 & \textit{otherwise}. \end{cases}$$

The Laplace transforms of $p_n(t)$ and $q_n(t)$ are defined as:

$$p_n^*(s) = \int_0^\infty e^{-st} p_n(t)dt \ , \ n = 0,1,...,L,$$

$$q_n^*(s) = \int_0^\infty e^{-st} q_n(t)dt \ , \ n = 0,1,...,L.$$

The following Laplace transform expressions for $p_n^*(s)$ and $q_n^*(s)$ are obtained in terms of λ_n.

(i) $n = 0$

$$(\lambda_0 + \alpha + s)p_0^*(s) - \mu p_1^*(s) - \beta q_0^*(s) = p_0(0) \tag{1a}$$

(ii) $1 \le n \le L-2$

$$\begin{aligned} &-\lambda_{n-1}p_{n-1}^*(s) + (\lambda_n + \mu + \alpha + s)p_n^*(s) \\ &-\mu p_{n+1}^*(s) - \beta q_n^*(s) = p_n(0) \end{aligned} \tag{1b}$$

(iii) $n = L-1$

$$\begin{aligned} &-\lambda_{L-2}p_{L-2}^*(s) + (\lambda_{L-1} + \mu + \alpha + s)p_{L-1}^*(s) \\ &-\beta q_{L-1}^*(s) = p_{L-1}(0) \end{aligned} \tag{1c}$$

(iv) $n = L$

$$-\lambda_{L-1}p_{L-1}^*(s) + sp_L^*(s) = p_L(0) \tag{1d}$$

(v) $n = 0$

$$(\lambda_0 + \beta + s)q_0^*(s) - \alpha p_0^*(s) = q_0(0) \tag{1e}$$

(vi) $1 \le n \le L-2$

$$-\lambda_{n-1}q_{n-1}^*(s) + (\lambda_n + \beta + s)q_n^*(s) - \alpha p_n^*(s) = q_n(0) \tag{1f}$$

(vii) $n = L-1$

$$\begin{aligned} &-\lambda_{L-2}q_{L-2}^*(s) + (\lambda_{L-1} + \beta + s)q_{L-1}^*(s) \\ &-\alpha p_{L-1}^*(s) = q_{L-1}(0) \end{aligned} \tag{1g}$$

(viii) $n = L$

$$-\lambda_{L-1}q_{L-1}^*(s) + sq_L^*(s) = q_L(0) \tag{1h}$$

where

$$L = M + S - K + 1, \quad K = 1,2,...,M,$$

and

$$p_0(0) = 1, \quad p_n(0) = 0, \quad \textit{for } n = 1,2,...,L,$$
$$q_n(0), \quad \textit{for } n = 0,1,2,...,L$$

Equation(1) can be written in following matrix form

$$D(s)W^*(s) = W(0) \tag{2}$$

where $D(s) =$

$$\left[\begin{array}{ccccccc}
\lambda_0+\alpha+s & -\mu & 0 & \cdots & 0 & 0 & 0 \\
-\lambda_0 & \lambda_1+\mu+\alpha+s & -\mu & \cdots & 0 & 0 & 0 \\
0 & -\lambda_1 & \lambda_2+\mu+\alpha+s & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_{L-2}+\mu+\alpha+s & -\mu & 0 \\
0 & 0 & 0 & \cdots & -\lambda_{L-2} & \lambda_{L-1}+\mu+\alpha+s & 0 \\
0 & 0 & 0 & \cdots & 0 & -\lambda_{L-1} & s \\
-\alpha & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\alpha & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\alpha & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\alpha & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -\alpha & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{array}\right.$$

$$\left.\begin{matrix}
-\beta & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -\beta & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -\beta & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\beta & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & -\beta & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\lambda_0+\beta+s & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\lambda_0 & \lambda_1+\beta+s & 0 & \cdots & 0 & 0 & 0 \\
0 & -\lambda_1 & \lambda_2+\beta+s & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_{L-2}+\beta+s & 0 & 0 \\
0 & 0 & 0 & \cdots & -\lambda_{L-2} & \lambda_{L-1}+\beta+s & 0 \\
0 & 0 & 0 & \cdots & 0 & -\lambda_{L-1} & s
\end{matrix}\right]$$

is a $2(L + 1) \times 2(L + 1)$ matrix. $W^*(s)$ is a column vector containing the set of elements $\left[P^*(s), Q^*(s)\right]^T$, where

$$P^*(s)=\left[p_0^*(s), p_1^*(s), p_2^*(s), \cdots, p_{L-1}^*(s), p_L^*(s)\right]^T,$$

$$Q^*(s)=\left[q_0^*(s), q_1^*(s), q_2^*(s), \cdots, q_{L-1}^*(s), q_L^*(s)\right]^T,$$

and the symbols T denotes the transpose. $W(0)$ is a column vector containing the set of elements $[P(0), Q(0)]^T$, where

$$\begin{aligned} P(0) &= \left[p_0(0), p_1(0), p_2(0), ..., p_{L-1}(0), p_L(0)\right]^T \\ &= \left[1,0,0,...,0\right]^T, \end{aligned}$$

$$\begin{aligned} Q(0) &= \left[q_0(0), q_1(0), q_2(0), ..., q_{L-1}(0), q_L(0)\right]^T \\ &= \left[0,0,0,...,0\right]^T \end{aligned}$$

Sovling (2) in accordance with Cramer's rule, we obtain the expression for $p_L^*(s)$ and $q_L^*(s)$ given by

$$p_L^*(s)=\frac{det\left[N_{L+1}(s)\right]}{det\left[D(s)\right]}, \tag{3}$$

$$q_L^*(s)=\frac{det\left[N_{2(L+1)}(s)\right]}{det\left[D(s)\right]}, \tag{4}$$

where $det[D(s)]$ denotes the determinant of matrix $D(s)$, $det[N_{L+1}(s)]$ denotes the determinant obtained by replacing the $(L + 1)$th column in matrix $D(s)$ by the initial vector $W(0)=[1, 0, 0, 0, \ldots, 0, 0]^T$ and $det[N_{2(L+1)}(s)]$ is the determinant obtained by replacing the $2(L + 1)$th column in matrix $D(s)$ by the initial vector $W(0)=[1, 0, 0, 0, \ldots, 0, 0]$.

It is too complex to derive the explicit solutions $p_L^*(s)$ and $q_L^*(s)$ of (3) and (4), respectively. Therefore, we use the computer software *MAPLE* to obtain the solutions $p_L^*(s)$ and $q_L^*(s)$. We first consider the denominator $det[D(s)]$ in (3) and (4). It is easy to know that the equation $det[D(s)] = 0$ has double zero roots. Let $s = -r$ (r are unknown values), then we have
$D(-r)=A-rI$,
where $A = D(0)$ is an $2(L+1)\times 2(L+1)$ matrix and I is the identity matrix. Thus (2) becomes

$$(A-rI)W^*(s)=W(0). \tag{5}$$

We set the determinant of the matrix $A-rI$ equal to zero, and find the corresponding distinct eigenvalues r_l ($r_l \neq 0$ and l=1, 2, 3, …,2L) which may be real or complex. Suppose that there are i real distinct eigenvalues (excluding zero) say $r_1, r_2, \ldots, r_i$, and j pairs distinct conjugate complex eigenvalues, say $(r_{i+1}, \overline{r}_{i+1})$, $(r_{i+1}, \overline{r}_{i+2})$, …, $(r_{i+1}, \overline{r}_{i+j})$, where i and j satisfy i $i+2j=2L$ It is to be noted that i = 0 denote all eigenvalues (excluding 0) are complex, and j = 0 represents all eigenvalues are real.

Next, we consider the numerators $det[N_{L+1}(s)]$ and $det[N_{2(L+1)}(s)]$ in (3) and (4), respectively. The computer software *MAPLE* is used to evaluate $det[N_{L+1}(s)]$ and $det[N_{2(L+1)}(s)]$. Thus, substituting $det[D(s)]$ and $det[N_{L+1}(s)]$ into (3) yields

$$\begin{aligned} p_L*(s) &= \frac{a_0}{s}+\frac{a_1}{s+r_1}+\cdots+\frac{a_i}{s+r_i} \\ &+\frac{b_1 s+c_1}{s^2+\left(r_{i+1}+\overline{r}_{i+1}\right)s+r_{i+1}\overline{r}_{i+1}} \\ &+\cdots+\frac{b_j s+c_j}{s^2+\left(r_{i+j}+\overline{r}_{i+j}\right)s+r_{i+j}\overline{r}_{i+j}} \end{aligned} \tag{6}$$

where $a_0, a_1, ..., a_i, b_1, c_1, b_2, c_2, ..., b_j, c_j$ are unknown real numbers. Likewise, substituting $det[D(s)]$ and $det[N_{2(L+1)}(s)]$ into (4), we obtain

$$q_L*(s)=\frac{d_0}{s}+\frac{d_1}{s+r_1}+\cdots+\frac{d_i}{s+r_i} +\frac{e_1s+f_1}{s^2+\left(r_{i+1}+\overline{r}_{i+1}\right)s+r_{i+1}\overline{r}_{i+1}} +\cdots+\frac{e_js+f_j}{s^2+\left(r_{i+j}+\overline{r}_{i+j}\right)s+r_{i+j}\overline{r}_{i+j}} \tag{7}$$

where $d_0, d_1, ..., d_i, e_1, f_1, e_2, f_2, ..., e_j, f_j$ are unknown real numbers .

Let u_l and v_l represent the real part and the imaginary part of complex eigenvalue r_{i+l} respectively. Inverting the Laplace transform in (6) and (7), we get the explicit expressions for

$$p_L(t)=a_0+\sum_{l=1}^{i}a_le^{-r_lt} +\sum_{l=1}^{j}\left[b_le^{-u_lt}cos(v_lt)+\frac{c_l-b_lu_l}{v_l}e^{-u_lt}sin(v_lt)\right] \tag{8}$$

$$q_L(t)=d_0+\sum_{l=1}^{i}d_le^{-r_lt} +\sum_{l=1}^{j}\left[e_le^{-u_lt}cos(v_lt)+\frac{f_l-e_lu_l}{v_l}e^{-u_lt}sin(v_lt)\right] \tag{9}$$

respectively.

Since the system has failed during the infinite period of time. Therefore we obtain

$$a_0+d_0=\lim_{t\to\infty}\left[p_L(t)+q_L(t)\right]=1 \tag{10}$$

3.1 The reliability function $R_Y(t)$

Let Y be the random variable and represent the time to failure of the system. Since $p_L(t)$ is the probability that the system has failed on or before time t when the service station is working, and $q_L(t)$ is the probability that the system has failed on or before time t when the service station is broken down, we have the reliability function given by

$$R_Y(t) \;=\; 1-p_L(t)-q_L(t),\;\; t\ge 0\,. \tag{11}$$

3.2 The mean time to system failure MTTF

If $R_Y^*(s)=\int_0^\infty e^{-st}R_Y(t)dt$ is the Laplace transform of $R_Y(t)$ and always finite, we have

$$\int_0^\infty R_Y(t)dt=\lim_{s\to0}R_Y^*(s)\,. \tag{12}$$

Thus the *MTTF* is given by

$$MTTF=\int_0^\infty R_Y(t)dt, \tag{13}$$

or equivalently

$$\begin{aligned} MTTF&=\lim_{s\to0}R_Y^*(s)\\ &=\lim_{s\to0}\begin{bmatrix}\dfrac{1-a_0-d_0}{s}-\displaystyle\sum_{l=1}^{i}\frac{a_l}{s+r_l}\\ -\displaystyle\sum_{l=1}^{j}\frac{b_ls+c_l}{s^2+\left(r_{i+1}+\overline{r}_{i+1}\right)s+r_{i+1}\overline{r}_{i+l}}-\sum_{l=1}^{i}\frac{d_l}{s+r_l}\\ -\displaystyle\sum_{l=1}^{j}\frac{e_ls+f_l}{s^2+\left(r_{i+1}+\overline{r}_{i+1}\right)s+r_{i+1}\overline{r}_{i+l}}\end{bmatrix}\\ &=-\left[\sum_{l=1}^{i}\frac{a_l}{r_l}+\sum_{l=1}^{j}\frac{c_l}{r_{i+l}\overline{r}_{i+l}}+\sum_{l=1}^{i}\frac{d_l}{r_l}+\sum_{l=1}^{j}\frac{f_l}{r_{i+l}\overline{r}_{i+l}}\right] \end{aligned} \tag{14}$$

4. SENSITIVITY ANALYSIS FOR $R_Y(t)$ AND *MTTF*

In this section we first perform a sensitivity analysis for changes in the $R_y(t)$ along with changes in specific values of the system parameters λ, μ, α, and β. Numerical results of the sensitivity analysis for the $R_y(t)$ along with changes in λ, μ, α, and β are presented. Differentiating (2) with respect to λ, we obtain

$$\frac{\partial D(s)}{\partial\lambda}W^*(s)+D(s)\frac{\partial W*(s)}{\partial\lambda}=0,$$

or equivalently

$$\frac{\partial W^*(s)}{\partial\lambda}=-D^{-1}(s)\frac{\partial D(s)}{\partial\lambda}W^*(s)\,. \tag{15}$$

Using the computer software *MAPLE* to solve (15), we can obtain the solutions $\partial p_L{}^*(S)/\partial\lambda$ and $\partial q_L{}^*(s)/\partial\lambda$. After inverting the Laplace transform solutions, we get $\partial p_L(t)/\partial\lambda$ and $\partial q_L(t)/\partial\lambda$. Differentiating (11) with respect to λ yields

$$\frac{\partial R_Y(t)}{\partial\lambda}=-\frac{\partial p_L(t)}{\partial\lambda}-\frac{\partial q_L(t)}{\partial\lambda}\,. \tag{16}$$

Substituting $\partial p_L(t)/\partial\lambda$ and $\partial q_L(t)/\partial\lambda$ into (16), we obtain $\partial R_Y(t)/\partial\lambda$.

Using the same procedure listed above, we can get $\partial R_Y(t)/\partial\mu$, $\partial R_Y(t)/\partial\alpha$, and $\partial R_Y(t)/\partial\beta$.

Next, we perform a sensitivity analysis for changes in the *MTTF* along with changes in specific values of λ, μ, α, and β. Numerical results of the sensitivity analysis for the *MTTF* along with changes in λ, μ, α, and β

are also provided. Differentiating (13) with respect to λ, we obtain

$$\frac{\partial MTTF}{\partial \lambda} = \int_0^\infty \frac{\partial R_Y(t)}{\partial \lambda} dt . \tag{17}$$

Substituting (16) into (17) yields $\partial MTTF / \partial \lambda$.

Using the same procedure listed above, $\partial MTTF / \partial \mu$, $\partial MTTF / \partial \alpha$, and $\partial MTTF / \partial \beta$ can be obtained.

5. NUMERICAL ILLUSTRATION

The purpose of this section is fourfold. The first is to analyze graphically to study the effects of various parameters on the system reliability. We fix λ = 0.6, η = 0.05, μ = 1.0, α = 0.2, β = 3.0, choose the number of operating machines M = 3, and consider the cases when the number of warm standbys S changes from 1 to 4 and the values of K vary from 1 to 4. We can easily see from Figure 1 that moderate improvement in the system reliability is obtained by adding the number of warm standbys. Moreover, Figure 2 shows that the system reliability increases as K decreases. Obviously, the values of K affect the system reliability significantly. We shall restrict ourselves to the reliability analysis of selecting fixed values M = 3, S = 2, K = 1, and η = 0.05, for the following cases.

Case 1: We fix μ = 1.0, α = 0.2, β = 3.0, and vary the values of λ from 0.2 to 0.6.

Case 2: We fix λ = 0.6, α = 0.2, β = 3.0, and vary the values of μ from 0.5 to 2.0.

Case 3: We fix λ = 0.6, μ = 1.0, β = 3.0, and vary the values of α from 0.1 to 0.4.

Case 4: We fix λ = 0.6, μ = 1.0, α = 0.2, and vary the values of β from 3.0 to 9.0.

It can be easily observed from Figure 3 that the system reliability increases as λ decreases. Obviously, the values of λ affect the system reliability significantly. One sees from Figure 4 that the system reliability increases with increasing μ. Figures 5-6 show that the system reliability rarely changes when α or β changes. Intuitively, the system reliability may be too insensitive to changes in α or β. It appears from Figures 3-6 that the most significant parameter on the system reliability is the parameter λ.

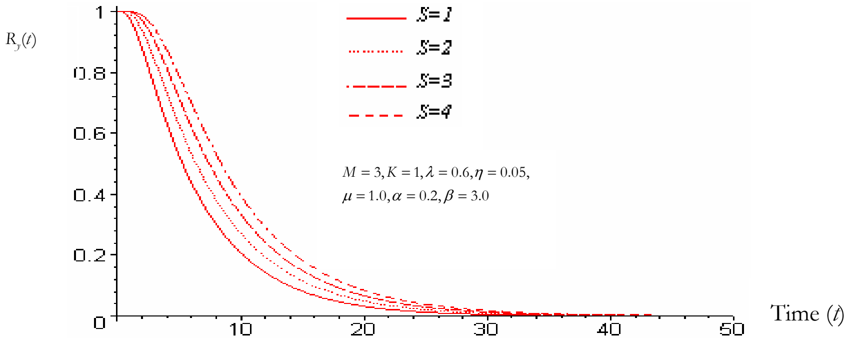

Figure 1. System reliability with warm standbys and a repairable service station. System fails when all M+S machines fail.

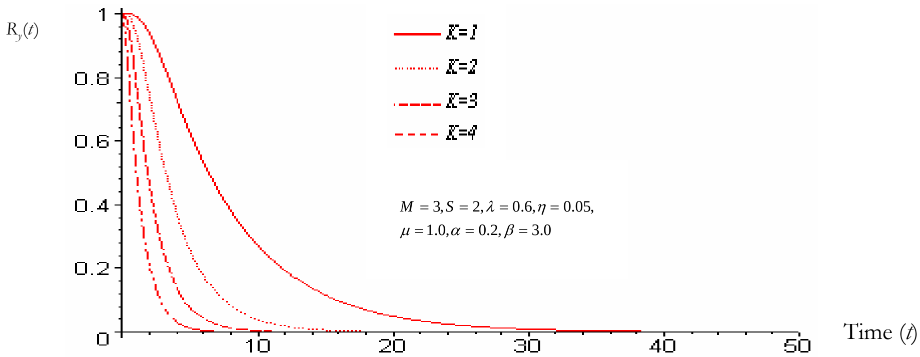

Figure 2. System reliability with warm standbys and a repairable service station for different values of K.

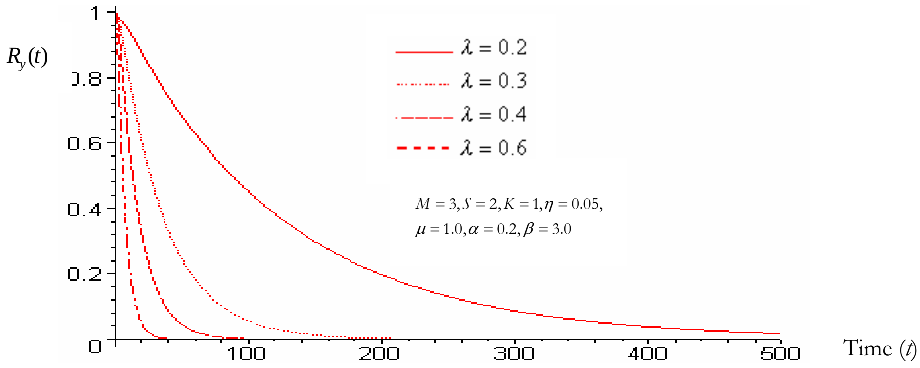

Figure 3. System reliability with warm standbys and a repairable service station. System fails when all $M+S$ machines fail.

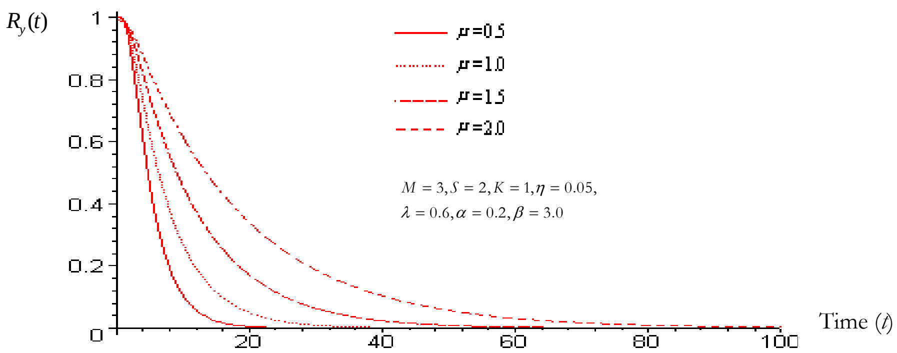

Figure 4. System reliability with warm standbys and a repairable service station. System fails when all $M+S$ machines fail.

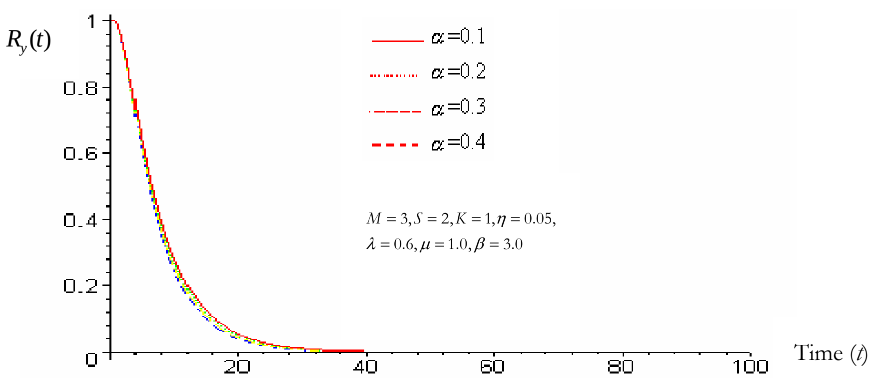

Figure 5. System reliability with warm standbys and a repairable service station. System fails when all $M+S$ machines fail.

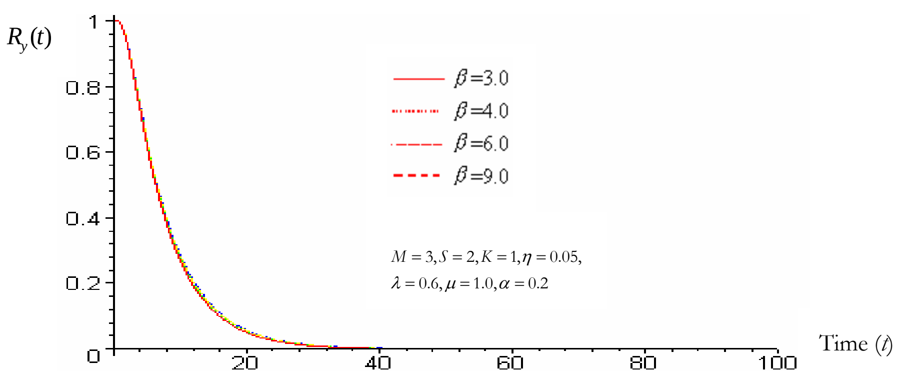

Figure 6. System reliability with warm standbys and a repairable service station. System fails when all $M+S$ machines fail.

The second purpose is to investigate the effects of various parameters on the *MTTF*. We fix M = 3 and choose $\eta = 0.05$. Various values of λ are considered.

Case 5: We fix K = 1, choose $\mu = 1.0$, $\alpha = 0.2$, $\beta = 3.0$, and vary the number of warm standbys S from 1 to 4.

Case 6: We fix S = 2, choose $\mu = 1.0$, $\alpha = 0.2$, $\beta = 3.0$, and vary the values of K from 1 to 4.

Case 7: We fix S = 2, K = 1, choose $\alpha = 0.2$, $\beta = 3.0$, and vary the values of μ from 0.5 to 2.0.

Case 8: We fix S = 2, K = 1, choose $\mu = 1.0$, $\beta = 3.0$, and vary the values of α from 0.1 to 0.4.

Case 9: We fix S = 2, K = 1, choose $\mu = 1.0$, $\alpha = 0.2$, and vary the values of β from 3.0 to 9.0.

The numerical results of the *MTTF* are shown in Tables 1-5. From Tables 1-5, we can easily see that the *MTTF* decreases as λ increases. Obviously, the *MTTF* can moderately decrease as λ increases for small λ. Moreover, Tables 1-5 show that (i) the addition of warm standbys S, the decrease in K, and the increase in μ can moderately increase the *MTTF* for small λ; and (ii) the increase in α or β rarely affects the *MTTF*.

The third purpose is to perform a sensitivity analysis of the system reliability for changes in the system parameters λ, μ, α, and β. We fix M = 3, S = 2, K = 1, and select λ =0.6, η =0.05, μ =1.0, α =0.2, β =3.0. In Figure 7, along the time coordinate, the system reliability will be affected even by minute change of the system parameters λ, μ, α, and β. Intuitively, increasing the values of μ and β or decreasing the values of λ and α will improve the system reliability. It seems that the order of impacts of these four parameters on the system reliability are: $\lambda > \mu > \alpha > \beta$. We observe that the effects of varying α and β on the system reliability can be neglected which matches the previous conclusions shown in Figures 5-6. Also, the effects of various parameters on the system reliability occur only in the time interval $0 < t < 50$, and the most significant effect occurs around $t = 8$.

The fourth purpose is to perform a sensitivity analysis on the change of the *MTTF* for various parameters λ, μ, α, and β. We fix M = 3, S = 2, K = 1, and select λ =0.6, η =0.05, μ =1.0, α =0.2, β =3.0. It can be easily seen from Table 6 that the order of impacts of these four parameters on the *MTTF* are: $\lambda > \mu > \alpha > \beta$. The gross effect of β is negligible when comparing with the gross effects of λ, μ, and α. It should be noted that these conclusions are only valid for the above cases. We may reach other conclusions for other cases.

Table 1. The *MTTF* for different values of λ and S
($M = 3, K = 1, \eta = 0.05, \mu = 1.0, \alpha = 0.2, \beta = 3.0$)

λ	S=1	S=2	S=3	S=4
0.20	76.99	122.76	173.85	222.45
0.25	42.58	61.62	82.40	103.56
0.30	27.27	36.71	46.08	54.98
0.35	19.24	24.70	29.79	34.42
0.40	14.51	18.04	21.24	24.10
0.45	11.48	13.97	16.20	18.19
0.50	9.42	11.29	12.96	14.47
0.60	6.83	8.04	9.13	10.15
0.70	5.31	6.19	7.00	7.76
0.80	4.32	5.01	5.65	6.27
0.90	3.63	4.20	4.73	5.25
1.00	3.13	3.61	4.07	4.52

Table 2. The *MTTF* for different values of λ and K
($M = 3, S = 2, \eta = 0.05, \mu = 1.0, \alpha = 0.2, \beta = 3.0$)

λ	K=1	K=2	K=3	K=4
0.20	122.76	31.32	12.18	5.00
0.25	61.62	18.83	8.36	3.79
0.30	36.71	12.87	6.24	3.03
0.35	24.70	9.56	4.92	2.52
0.40	18.04	7.50	4.04	2.14
0.45	13.97	6.13	3.42	1.86
0.50	11.29	5.16	2.95	1.65
0.60	8.04	3.90	2.31	1.33
0.70	6.19	3.12	1.89	1.12
0.80	5.01	2.60	1.60	0.96
0.90	4.20	2.22	1.60	0.84
1.00	3.61	1.94	1.22	0.75

Table 3. The *MTTF* for different values of λ and μ
($M = 3, S = 2, K = 1, \eta = 0.05, \alpha = 0.2, \beta = 3.0$)

λ	μ =0.5	μ =1.0	μ =1.5	μ =2.0
0.20	35.23	122.76	274.78	382.04
0.25	22.20	61.62	153.58	270.44
0.30	15.87	36.71	84.67	167.65
0.35	12.24	24.70	51.47	100.72
0.40	9.92	18.04	34.41	63.84
0.45	8.32	13.97	24.72	43.42
0.50	7.16	11.29	18.75	31.32
0.60	5.59	8.04	12.11	18.59
0.70	4.58	6.19	8.69	12.48
0.80	3.88	5.01	6.68	9.11
0.90	3.36	4.20	5.38	7.05
1.00	2.96	3.61	4.49	5.69

Table 4. The *MTTF* for different values of λ and α
($M = 3, S = 2, K = 1, \eta = 0.05, \mu = 1.0, \beta = 3.0$)

λ	α =0.1	α =0.2	α =0.3	α =0.4
0.20	133.72	122.76	113.24	104.92
0.25	66.57	61.62	57.38	53.72
0.30	39.16	36.71	34.59	32.75
0.35	26.08	24.70	23.49	22.43
0.40	18.90	18.04	17.29	16.63
0.45	14.54	13.97	13.47	13.02
0.50	11.68	11.29	10.93	10.61
0.60	8.26	8.04	7.84	7.66
0.70	6.32	6.19	6.06	5.95
0.80	5.10	5.01	4.92	4.85
0.90	4.26	4.20	4.14	4.08
1.00	3.66	3.61	3.56	3.52

Table 5. The *MTTF* for different values of λ and β
($M = 3, S = 2, K = 1, \eta = 0.05, \mu = 1.0, \alpha = 0.2$)

λ	β =3.0	β =4.0	β =6.0	β =9.0
0.20	122.76	128.79	134.79	138.74
0.25	61.62	64.28	67.00	68.82
0.30	36.71	38.01	39.34	40.24
0.35	24.70	25.42	26.16	26.67
0.40	18.04	18.49	18.94	19.25
0.45	13.97	14.26	14.57	14.77
0.50	11.29	11.49	11.70	11.84
0.60	8.04	8.15	8.26	8.34
0.70	6.19	6.25	6.33	6.38
0.80	5.01	5.05	5.10	5.13
0.90	4.20	4.23	4.26	4.29
1.00	3.61	3.63	3.66	3.67

Table 6. Sensitivity analysis for the *MTTF* with case
$\lambda = 0.6, \mu = 1.0, \alpha = 0.2, \beta = 3.0$

	$\theta = \lambda$	$\theta = \mu$	$\theta = \alpha$	$\theta = \beta$
$\frac{\partial MTTF}{\partial \theta}$	-23.68	6.28	-2.10	0.14

6. CONCLUSIONS

In this paper, we have developed the explicit expressions for the system reliability and the *MTTF*. It should be first noted from Figures 1-6 that α and β rarely affect the system reliability, S has moderate effect, K, λ, and μ affect the system reliability significantly. Next, we should note from Tables 1-5 that (i) α and β rarely affect the *MTTF*; and (ii) S, K, and μ affect the *MTTF* moderately for small λ. Finally, we have performed a sensitivity between the system reliability, the *MTTF* and specific values of λ, μ, α, and β. Our numerical investigations indicate that the order of impacts of these four parameters on the system reliability and the *MTTF* are: $\lambda > \mu > \alpha > \beta$.

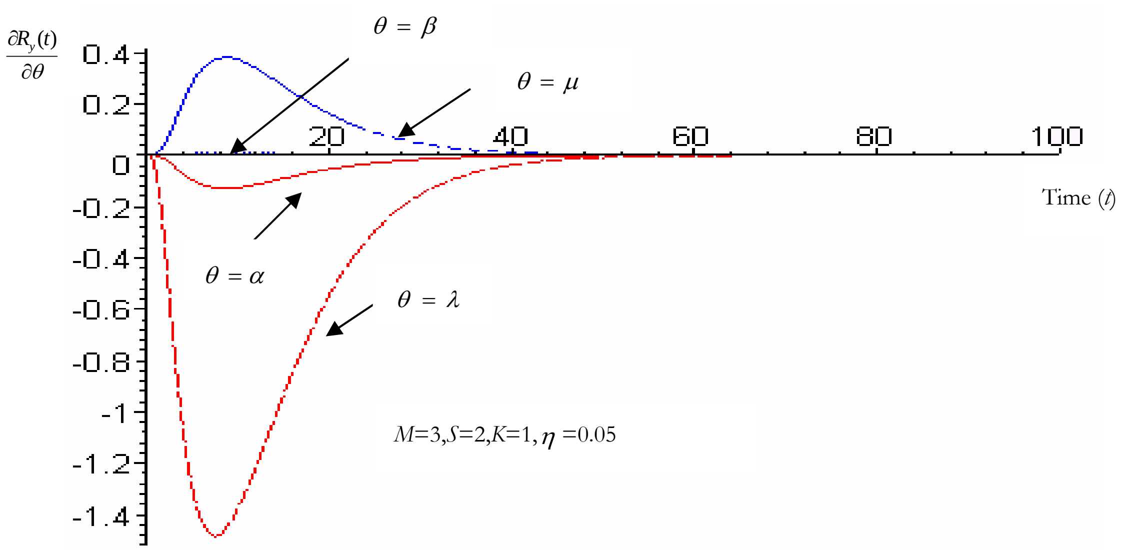

Figure 7. Sensitivity analysis for the system reliability with case $\lambda = 0.6, \mu = 1.0, \alpha = 0.2, \beta = 3.0$.

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