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An Algebraically Derived Minimal Cost Solution Technique of the Integrated Vendor-Buyer Problem

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Abstract—Recently an algebraically derived optimal cost solution policy of the integrated vendor-buyer problem with equal sized batch transfer was presented. The solution technique was illustrated with a numerical example and in comparison with two available methods in the literature a significant cost reduction was shown. This paper highlights that the optimal total cost obtained by them is about 5.8% higher than the optimal total cost derived for the same numerical example by another two methods in the literature. In this paper a model of this integrated vendor-buyer problem with equal and unequal sized batch transfer is developed. A simple minimal cost solution technique of the model, derived algebraically, is presented and a solution algorithm is provided. The model is also solved using derivatives and the same results of numerical examples are found. For simplicity the algebraic approach is presented here. For three numerical examples, a comparative study of this approach with one of the best available methods (modified) in the literature is also carried out. For the numerical examples studied, the technique developed in this paper seems to provide better solution.

Keywords—Single-vendor, Single-buyer, Minimal cost, Integrated inventory, Lot and batch sizes

1. INTRODUCTION

Integrated vendor-buyer inventory system has accelerated the success of e-business. Clark and Scarf (1960) are pioneers in developing an optimal policy for a multi-echelon inventory system. Banerjee (1986) obtained a joint economic lot size model for vendor and purchaser based on lot-for-lot policy. Goyal (1988) assumed that items are sent to the buyer in equal sized batches. Hill (1997) suggested a more general class of policy for determining successive shipment sizes. Hill (1999) determined, from first principle, the form of the globally-optimal batching and shipping policy. Hoque and Goyal (2000) proposed a model with equal and unequal sized batch transfer but imposing capacity constraint on the transport equipment. For the same two numerical examples their approach gives almost similar solutions to that obtained by Hill (1999). To determine the economic lot size Goyal and Nebebe (2000) presented an optimal policy that was a particular case of Hoque and Goyal (2000). Grubbstrom (1995) provided an EOQ without backlogging while Grubbstrom and Erden (1995) presented an EOQ with backlogging but both by algebraic method. Following the algebraic approach Cardenas-Barron (2001) derived an economic production quantity model by allowing shortage. Using the same algebraic approach Yang and Wee (2002) developed an optimal replenishment policy for an integrated vendor-buyer system by transferring the lot with equal sized batches. For a numerical problem they have shown significant cost reduction in comparison with two other techniques available in the literature. But for the same numerical problem solved by Hill (1999) and Hoque and Goyal (2000) gives 5.8% lower cost. Among all the models describedHill (1999) and Hoque and Goyal (2000) obtained the lowest cost for two numerical examples. But Hill's optimal solution procedure sometimes leads to an infeasible optimal solution. This was demonstrated with a numerical example by these authors.

In this paper a model for the same integrated vendor-buyer system is developed, based on equal and unequal sized batch transfer of a lot from the vendor to the buyer. This approach of minimizing the joint inventory cost in processing a single product in a multi-stage serial production system was used by Hoque and Kingsman (1995), originally presented by Goyal and Szendrovits (1986). The model is solved both algebraically and using derivative method. However, for simplicity of the algebraic approach, it is presented in this paper.

A comparative study of the technique developed in this paper with Hill (1999) (modified) on three numerical examples is carried out. The solution technique in this is found to provide better result. Besides, the solution technique in this paper developed algebraically, so it is simpler, straightforward and easy to follow specially for those who lack of the background of differential calculus.

2. ASSUMPTIONS AND NOTATIONS

The assumptions made in the paper are stated below:

i) The production rate is finite and greater than the

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demand.

ii) The vendor and the buyer have complete knowledge of each other's information.

iii) Shortage is not allowed.

iv) The set up or ordering times and transportation times and cost are negligible and hence ignored.

The notations used in this paper are same as in Hill (1999).

Variables

Q: Uniform lot size (Q is infinitely divisible);
C: The total cost per unit time
m: total number of batches (m is a positive integer);
e: number of unequal sized batches (e is a positive integer);
z: the smallest batch size.

Parameters

D: average demand per year;
P: annual production rate;
A_1: vendor's set up cost per set up;
A_2: purchaser's ordering cost per order;
h_1 : vendor's holding cost per unit per unit time;
h_2 : buyer's holding cost per unit per unit time;
k: ratio of the production rate and demand (P/D).

3. MODEL FORMULATION.

The lot Q is transferred from the vendor to the buyer in e unequal sized batches as $z, kz, k^2z, ..., k^{e-1}z$ and $(m - e)$ batches of size $k^{e-1}z$. Following Goyal and Szendrovits (1986), the total joint inventory for the vendor and the buyer is given by

$$\frac{Q^2}{2}\left(\frac{1}{D}-\frac{1}{P}\right)+\frac{Qz}{P}$$

There are D/Q cycles per year. So the total joint inventory per year is

$$\frac{Q}{2}\left(1-\frac{D}{P}\right)+\frac{Dz}{P}$$

The inventory at the buyer per year is

$$\frac{z^2}{2Q}\left[\frac{k^{2e}-1}{k^2-1}+(m-e)k^{2(e-1)}\right]$$

The inventory at the vendor per year is

$$\left[\frac{Q}{2}\left(1-\frac{D}{P}\right)+\frac{Dz}{P}\right]-\frac{z^2}{2Q}\left[\frac{k^{2e}-1}{k^2-1}+(m-e)k^{2(e-1)}\right]$$

So the total joint inventory cost at the vendor and the buyer per year is

$$\left[\frac{Q}{2}\left(1-\frac{D}{P}\right)+\frac{Dz}{P}\right]h_1+\frac{z^2}{2Q}\left[\frac{k^{2e}-1}{k^2-1}+(m-e)k^{2(e-1)}\right](h_2-h_1)$$

The sum of all batch sizes must equal the lot size, so equating this sum to the lot size implies

$$z=\frac{Q}{f(m,e)},\text{ where } f(m,e)=(m-e)k^{e-1}+\sum_{r=0}^{e-1}k^r$$

Substituting for z in the total inventory cost, it becomes

$$\frac{Q}{2}\left[1-\frac{D}{P}+\frac{2D}{Pf(m,e)}\right]h_1+$$
$$\frac{Q}{2\{f(m,e)\}^2}\left[\frac{k^{2e}-1}{k^2-1}+(m-e)k^{2(e-1)}\right](h_2-h_1)$$

In Appendix 1 it is shown that

$$\frac{k^{2e}-1}{k^2-1}+(m-e)k^{2(e-1)}=k^{e-1}f(m,e)-\frac{(k^e-1)(k^{e-1}-1)}{k^2-1}$$

The total inventory cost then transforms to

$$\frac{Q}{2}\left[\left(1-\frac{D}{P}+\frac{2D}{Pf(m,e)}\right)h_1+\right.$$
$$\left.\left\{\frac{k^{e-1}}{f(m.e)}-\frac{(k^e-1)(k^{e-1}-1)}{(k^2-1)\{f(m,e)\}^2}\right\}(h_1-h_2)\right]$$

A_2 is the ordering cost per order for the buyer. There are m orders per lot and D/Q lots in a year. So buyer's ordering cost per year is mA_2D/Q. A1 is the set up cost per set up for a lot, so the vendor's set up cost per year is DA_1/Q.

Therefore, the total cost of ordering or set ups and inventory holding is given by

$$C=\frac{Q}{2}\left[\left(1-\frac{D}{P}+\frac{2D}{Pf(m,e)}\right)h_1\right.$$
$$\left.+\left\{\frac{k^{e-1}}{f(m.e)}-\frac{(k^e-1)(k^{e-1}-1)}{(k^2-1)\{f(m,e)\}^2}\right\}(h_2-h_1)\right]+\frac{D}{Q}\left[mA_2+A_1\right] \quad (1)$$

which is to be minimized.

4. SOLUTION TECHNIQUE

The total cost function can be written as $C=\frac{Da}{Q}+\frac{Qb}{2}$,

where

$a = mA_2 + A_1$;

$$b=\left(1-\frac{D}{P}+\frac{2D}{Pf(m,e)}\right)h_1+\left\{\frac{k^{e-1}}{f(m.e)}-\frac{(k^e-1)(k^{e-1}-1)}{(k^2-1)\{f(m,e)\}^2}\right\}(h_2-h_1).$$

That is,

$$C = \frac{b}{2Q}\left[Q^2 + \frac{2Da}{b}\right]$$
$$= \frac{b}{2Q}\left[\left(Q - \sqrt{\frac{2Da}{b}}\right)^2 + 2Q\sqrt{\frac{2Da}{b}}\right]$$
$$= \frac{b}{2Q}\left(Q - \sqrt{\frac{2Da}{b}}\right)^2 + \sqrt{2abD}$$

So, C will be minimum when

$$Q = \sqrt{\frac{2Da}{b}} = \sqrt{\frac{2D(mA_2 + A_1)}{\left(1 - \frac{D}{P} + \frac{2D}{Pf(m,e)}\right)h_1 + \left\{\frac{k^{e-1}}{f(m.e)} - \frac{(k^e - 1)(k^{e-1} - 1)}{(k^2 - 1)\{f(m,e)\}^2}\right\}(h_2 - h_1)}} \tag{2}$$

and the minimum cost is

$$C = \sqrt{2D(mA_2 + A_1)\left[\left(1 - \frac{D}{P} + \frac{2D}{Pf(m,e)}\right)h_1 + \left\{\frac{k^{e-1}}{f(m.e)} - \frac{(k^e - 1)(k^{e-1} - 1)}{(k^2 - 1)\{f(m,e)\}^2}\right\}(h_2 - h_1)\right]} \tag{3}$$

We have $m \geq e \geq 1$. For the least value of e, the expression under the square root here reduces to $2D\left[a1m + \frac{b1}{m} + c1\right]$, where

$$a1 = (1 - D/P)h_1A_2;\; b1 = A_1\left\{\frac{2D}{P}h_1 + (h_2 - h_1)\right\};$$
$$c1 = \left\{\frac{2D}{P}h_1 + (h_2 - h_1)\right\}A_2 + A_1(1 - D/P)h_1,$$

This can be written as

$$2D\left[\frac{a1}{m}\left(m^2 + \frac{b1}{a1}\right) + c1\right]$$
$$= \;\; 2D\left[\frac{a1}{m}\left\{\left(m - \sqrt{\frac{b1}{a1}}\right)^2 + 2m\sqrt{\frac{b}{a}}\right\} + c1\right]$$
$$= \;\; 2D\left[\frac{a1}{m}\left\{\left(m - \sqrt{\frac{b1}{a1}}\right)^2 + 2\sqrt{a1b1}\right\} + c1\right]$$

The value of this expression is minimum when

$$m = \sqrt{\frac{b1}{a1}} = \sqrt{\frac{A_1\{(2D/P - 1)h_1 + h_2\}}{A_2(1 - D/P)h_1}} \tag{4}$$

The lower of the total costs calculated from (3) for the rounding up and rounding down value of m thus found gives the minimal cost for $e = 1$.

If m° and e° represent the optimal value of m and e, then from (3) we have

$$(mA_2 + A_1)\left[\left(1 - \frac{D}{P} + \frac{2D}{Pf(m,e)}\right)h_1 + \left\{\frac{k^{e-1}}{f(m.e)} - \frac{(k^e - 1)(k^{e-1} - 1)}{(k^2 - 1)\{f(m,e)\}^2}\right\}(h_2 - h_1)\right] \leq C^\circ$$

Where

$$C' = (m^\circ A_2 + A_1)\left[\left(1 - \frac{D}{P} + \frac{2D}{Pf(m^\circ, e^\circ)}\right)h_1 + \left\{\frac{k^{e^\circ-1}}{f(m^\circ.e^\circ)} - \frac{(k^{e^\circ} - 1)(k^{e^\circ-1} - 1)}{(k^2 - 1)\{f(m^\circ, e^\circ)\}^2}\right\}(h_2 - h_1)\right] \tag{4}$$

This can be written as

$$\frac{\frac{2Dh_1}{P} + k^{e-1}(h_2 - h_1)}{\{f(m,e)\}^2}\left[f(m,e) - \frac{\frac{(k^e - 1)(k^{e-1} - 1)(h_2 - h_1)}{(k^2 - 1)}}{\frac{2Dh_1}{P} + k^{e-1}(h_2 - h_1)}\right] + \left(1 - \frac{D}{P}\right)h_1 - \frac{C^\circ}{(mA_2 + A_1)} \leq 0 \tag{5}$$

Substituting the value of $f(m, e)$ inside the third brackets and then simplifying obtain

$$\frac{\left\{\frac{2Dh_1}{P}+k^{e-1}(h_2-h_1)\right\}k^{e-1}}{\{f(m,e)\}^2}$$
$$\times\left[m-e-\sum_{r=0}^{e-1}k^{-r}\left\{\frac{(k^e+1)(h_2-h_1)+\frac{2D}{P}(k+1)h_1}{(k+1)\left\{\frac{2Dh_1}{P}+k^{e-1}(h_2-h_1)\right\}}\right\}\right]$$
$$+\left(1-\frac{D}{P}\right)h_1-\frac{C^\circ}{(mA_2+A_1)}\le 0$$

Since $e\le m$, $h_1<h_2$ and $k>1$, m cannot be greater than $\frac{1}{A_2}\left\{\frac{C^\circ}{\left(1-D/P\right)h_1}-A_1\right\}$, which is the upper bound of m. For more tighten upper bound reduce the inequality (5) to

$$\frac{a_2}{\{f(m,e)\}^2}\left[f(m,e)-\frac{b_2}{a_2}\right]+\left(1-\frac{D}{P}\right)h_1-\frac{C^\circ}{(mA_2+A_1)}\le 0$$

where $a2=\frac{2Dh_1}{P}+k^{e-1}(h_2-h_1);$

$$b2=\frac{k^e-1)(k^{e-1}-1)(h_2\_h_1)}{(k^2-1)};$$

$$c2=\left(1-\frac{D}{P}\right)h_1.$$

which transforms to

$$a2(mA_2+A_1)f(m,e)-b2(mA_2+A_1)$$
$$+\left[c2(mA_2+A_1)-C^\circ\right]\{f(m,e)\}^2\le 0$$

Since $f(m,e)=(m-e)k^{e-1}+\sum_{r=0}^{e-1}k^r$, dividing by k^{e-1} obtain

$$a2(mA_2+A_1)(m-e1)-\frac{b2(mA_2+A_1)}{k^{e-1}}$$
$$+\left[c2(mA_2+A_1)-C^\circ\right]k^{e-1}(m-e1)^2\le 0,$$

where $e1=e-,\sum_{r}^{e-1}k^{-r}$

which on simplification reduces to

$$a3m^3+b3m^2+c3m+d3\le 0$$

where

$$a3=Ac2k^{e-1}>0;$$
$$b3=a2A_2+k^{e-1}(A_1c2-C^\circ-2A_2c2e3);$$
$$c3=A_1a2-A_2\left(a2e1+\frac{b2}{k^{e-1}}\right)+\left\{A_2c2e1-2(A_1c2-C^\circ\right\}k^{e-1}e1;$$

$$d3=(e1)^2k^{e-1}(A_1c2-C^\circ)-A_1\left(a2e1+\frac{b2}{k^{e-1}}\right).$$

Dividing by m^2 yields $a3m+b3+\frac{c3}{m}+\frac{d3}{m^2}\le 0$ (6)

For given value of e, the left hand side of this inequality is almost a form of convex function in m, so value of m converges to the optimal cost. Let

$$a3m+b3+\frac{c3}{m}+\frac{d3}{m^2}<a3(m-1)+b3+\frac{c3}{m-1}+\frac{d3}{(m-1)^2},$$

which implies

$$\frac{1}{m(m-1)}\left[c3+d3\left\{\frac{1}{m}+\frac{1}{m-1}\right\}\right]>a3>0$$

$$Or\ m(m-1)-\frac{d3}{a3}\left(\frac{1}{m}+\frac{1}{m-1}\right)\le\frac{c3}{a3}$$

Similarly

$$a3m+b3+\frac{c3}{m}+\frac{d3}{m^2}<a3(m+1)+b3+\frac{c3}{m+1}+\frac{d3}{(m+1)^2}$$

implies $m(m+1)-\frac{d3}{a3}\left(\frac{1}{m}+\frac{1}{m+1}\right)\ge\frac{c3}{a3}$

Therefore,

$$Or\ m(m-1)-\frac{d3}{a3}\left(\frac{1}{m}+\frac{1}{m-1}\right)\le\frac{c3}{a3}$$
$$\le m(m+1)-\frac{d3}{a3}\left(\frac{1}{m}+\frac{1}{m+1}\right) \quad (7)$$

For given e, the optimal value of m can be found out as the highest value of m satisfying this constraint. But we are required to determine upper bound on e. Note that $e\le m$. Thus the highest value of e is m. Substituting for $e = m$ in the expression under square root in (3) we have

$$2D\left[(mA+S)\left\{\left(1-\frac{D}{P}\right)h_1+\frac{k-1}{k^m-1}\right.\right.$$
$$\left.\left.\times\left\{\frac{2D}{P}h_1+\frac{k^m}{k}(h_2-h_1)-\frac{k^m/k-1}{k+1}(h_2-h_1)\right\}\right\}\right]$$

which on simplification yields

$$2D\left[(mA+S)\left\{\left(1-\frac{D}{P}\right)h_1+\frac{k-1}{k^m-1}\right.\right.$$
$$\left.\left.\times\left\{\frac{2D}{P}h_1+\frac{(k^m+1)}{k+1}(h_2-h_1)\right\}\right\}\right]$$

Here $\frac{k^m+1}{k^m-1}$ is a decreasing function of m;

otherwise $\frac{k^m+1}{k^m-1}<\frac{k^{m+1}+1}{k^{m+1}-1}$ implying k<1, a contradiction.

So the total cost converges at a certain value of m = e. Therefore, if the inequality (7) does not satisfy for e = m, then it cannot be satisfied for any higher value of m.

Hence, for optimal total cost, starting from e = 1 and increasing at each step by 1 we need to check up to $e = m$ where the inequality (7) is unsatisfied. The initial value of C° can be calculated for $e^{\circ} = 1$ and m° equal to the corresponding value of m derived using (4).

The Solution Algorithm

Step 1: Initialization.

Set $e = 1$ and $m =$ rounding up or rounding down value of $m = \sqrt{\dfrac{A_1\{(2D/P-1)h_1 + h_2\}}{A_2(1-D/P)h_1}}$, which gives the minimum cost from (3) and record details of this as the currently optimal solution with $m^{\circ}, e^{\circ}, Q^{\circ}$ and C° .

Step 2: Calculate C' with $m = m^{\circ}$ *and* $e = e^{\circ}$ from (4) and increase e by 1.

Step 3: If for m = e, $m(m-1) - \dfrac{d3}{a3}\left(\dfrac{1}{m} + \dfrac{1}{m-1}\right) \leq \dfrac{c3}{a3}$ in (7), determine the highest integral value of m (increasing m at each step by 1) satisfying (7), and for that value of m calculate the total cost from (3). If it is less than the previous value of C°, set C° equal to this cost and record details of the new current optimal solution with m°, e°, Q° and C° and go to step 2.

Step 4: The current optimal solution is the final optimal solution.

5. NUMERICAL EXAMPLE

We use the same numerical examples as used by Hill (1999). The data are as follows:

A_1 = 400, A_2 = 25, $h_1 = 4$, D=1000, P=3200

Comparative results are given below.

Example 1[1]: ($h_2 = 5$)

Table 1. Summary of comparison of the results of example 1

Policy	Shipment sizes	Batch size	Total cost
Hill (Modified[2])	23.64, 75,63, 229.27, 229.27	557.8	1792.77
This Technique	22.6, 72.32, 231.43, 231.43	557.8	1792.76

[1] For this example Yang and Wee's solution technique gives 5.8% higher cost than the cost found by Hill (1999) and Hoque and Goyal (2000).

[2] Hill's solution algorithm is modified by these authors and submitted to the international Journal of Production research

Example 2: ($h_2 = 7$)

Table 2. Summary of comparison of the results of example 2

Policy	Shipment sizes	Batch size	Total cost
Hill (Modified)	31.1, 99.53, 136.96, 136.96, 136.96	541.53	1938.97
This Technique	39.18, 125.38, 125.38, 125.38, 125.38,	540.7	1942.06

For these two numerical examples Hoque and Goyal (2000) found the same result as found by this technique. But it was developed using derivatives and imposing capacity constraint on the transport equipment.

Example 3: This is a new example set by these authors. Data are as follows:

A1=300, A2=20, $h_1 = 4.5$, $h_2 = 5.5$, D=1000, P =3000

Table 3. Summary of comparison of the results of example 3

Policy	Shipment sizes	Batch size	Total cost
Hill[3] (Modified)	42.08, 126.24, 269.31	437.63	1645.23
This Technique	21.21, 63.63, 190.89, 190.89	466.6	1629.02

[3] Hill's original method leads to an infeasible optimal solution for this example problem.

In case of example 1 both the Hill's policy and the technique developed in this paper produce almost the same result. But for example 2 the total cost obtained by Hill (1999) is less than the cost found by this technique by 3.09. Though the lot sizes are found to be almost the same, the batch sizes are different. In case of example 3 results are totally different. Hill's original solution algorithm leads to an infeasible optimal solution in this case. But it is modified by these authors and submitted to 'The International Journal of Production research'. Here all results are found based on the modified policy. For this example the total cost obtained by the technique developed in this paper is less than the total cost found following the modified policy by 16.21.

6. CONCLUSION

In this paper a minimal cost solution technique of the integrated vendor-buyer problem is derived algebraically. The model is developed based on equal and unequal sized batch transfer of a lot from the vendor to the buyer. Though a lot of research has been carried out on this topic, a few of them derived their solution technique without using derivatives. Moreover, algebraically developed solution techniques have failed to provide better solution. The solution technique developed in this paper algebraically is compared with Hill (1999), a well-known method in the literature, on three numerical examples. In one both the methods give almost the same result. Out of the remaining two, in one Hill (1999) leads to an infeasible optimal solution but based on modification on it, the optimal cost obtained is higher than the optimal cost found by this technique by 16.21. But for the remaining one, the optimal cost obtained by Hill is less than the optimal cost by this technique by 3.09. So for the example studied it seems our solution technique gives better result. Besides, our solution technique is straightforward, easier to understand and follow specially for those who lack of knowledge of differential calculus. Therefore, we would like to recommend our solution technique.

Appendix 1

$$\frac{k^{2e}-1}{k^2-1}+(m-e)k^{2(e-1)}$$

$$=\frac{k^{2e}-1}{k^2-1}-\frac{k^e-1}{k-1}+(m-e)k^{2(e-1)}-(m-e)k^{e-1}$$

$$+\left[\frac{k^e-1}{k-1}+(m-e)k^{e-1}\right]$$

$$=\left[(k^e+1)-(k+1)\right]\frac{k^e-1}{k^2-1}+(m-e)k^{e-1}(k^{e-1}-1)+f(m,e)$$

$$=\frac{k^e-k}{k}.\frac{k(k^e-1)}{k^2-1}+(m-e)k^{e-1}(k^{e-1}-1)+f(m,e)$$

$$=(k^{e-1}-1)\left(1-\frac{1}{k+1}\right)\frac{k^e-1}{k-1}+(m-e)k^{e-1}(k^{e-1}-1)+f(m,e)$$

$$=(k^{e-1}-1)\left(\frac{k^e-1}{k-1}+(m-e)k^{e-1}-\frac{k^e-1}{k^2-1}\right)+f(m,e)$$

$$=(k^{e-1}-1)\left(f(m,e)-\frac{k^e-1}{k^2-1}\right)+f(m,e)$$

$$=(k^{e-1}f(m,e)-\frac{(k^e-1)(k^{e-1}-1)}{k^2-1}$$

REFERENCES

1. Banerjee. (1986). A joint economic lot size model for purchaser and vendor. *Decision Sciences*, 17: 292-311.
2. Clark, A.J., Scarf, H. (1960). Optimal policies for a multi-echelon inventory problem. *Management Science*, 6: 475-490.
3. cardenzas-Barron, L.E. (2001). The economic production quantity (EPQ) with shortage derived algebraically. *International Journal of Production Economics*, 70: 289-292.
4. Grubbstrom, R.W. (1995). Modeling production opportunities- an historical overview. *International Journal of Production Economics*, 41: 1-14.
5. Grubbstrom, R.W. and Erden, A. (1995). The EOQ with backlogging derived without derivatives. *International Journal of Production Economics*, 41: 1-14.
6. Goyal, S.K. (1988). A joint economic lot size model for purchaser and vendor: A comment. *Decision Sciences*, 19: 236-241.
7. Goyal, S.K. and Nebebe, F. (2000). Determination of economic production-shipment policy for a single-vendor single-buyer system. *European Journal of Operational Research*, 121: 175-178.
8. Goyal, S.K. and Szendrovits, A.Z. (1986). A constant lot size model with equal and unequal sized batch shipments between production stages. *Engineering Costs and Production Economics*, 10: 203-210.
9. Hill, R.M. (1997). The single-vendor single-buyer integrated production inventory model with a generalized policy. *European Journal of Operational Research*, 97: 493-499.
10. Hill, R.M. (1999). The optimal production and shipment policy for the single-vendor single-buyer integrated production inventory problem. *International Journal of Production Research*, 37: 2463-2475.
11. Hoque, M.A. and Kingsman, B.G. (1995). An optimal solution algorithm for the constant lot size model with equal and unequal sized batch shipments for the single product multi-stage production system. *International Journal of Production Economics*, 42: 161-174.
12. Hoque, M.A., Goyal, S.K. (2000). An Optimal policy for a single-vendor single-buyer integrated production-inventory system with capacity constraint of the transport equipment. *International Journal of Production Economics*, 65: 305-315.
13. Yang, P.C. and Wee, L.E. (2002). The economic lot size of the integrated vendor-buyer inventory system derived without derivatives. *Optimal Control Applications and Methods*, 23: 163-169.