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An Inspection Model with Discount Factor for Products having Weibull Lifetime

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Abstract—This paper investigates the case where a failed product can be detected only through inspections. During the lifetime of the product, it is inspected once and then must be replaced by either corrective or preventive replacement depending on the result of the inspection. By incorporating a discount factor, a mathematical model is established to take the time value of costs into account. Based on the model, the optimal time epoch for inspection is derived such that the present value of the expected total cost is minimized. Since there is no closed-form solution of the optimal time epoch for inspection, some properties are investigated and an efficient algorithm is provided to search for the optimal policy. Finally, numerical examples for products having Weibull lifetime distributions are given to investigate the effects of the continuous discount rate and cost parameters on the optimal policy and the corresponding present value of the expected total cost.

Keywords—Inspection policy, Discount factor, Replacement, Present value

1. INTRODUCTION

In reliability studies, there are situations where failures of a product/item can only be detected through inspections. Here, a product/item as defined in MIL-STD 721 B (Military Standard, Department of Defense, U.S.A.) could be a system, material, part, component, etc (Osaki, 1992). For some products, inspection can only be carried out once during the lifetime of a product. Practical examples for this kind of products can be found in Nelson (1982) such as a cracked component inside a turbine, fan blades of jet engines, and metal specimens in a fatigue test for endurance limit. For some cases, the product after inspection must be replaced either correctively or preventively. In this paper, we will focus on deriving an optimal time epoch for inspection for this kind of products.

Since Barlow et al. (1965) proposed a basic inspection model without any repair and replacement, there are a variety of inspection models proposed and investigated for different practical situations. Munford and Shahani (1972) focus on the computational complexity of Barlow's model and propose a nearly optimal inspection policy by introducing a single control variable. Furthermore, using a continuous inspection density function and the methods of calculus of variations, Keller (1974, 1982) makes the derivation of the optimal policy more tractable. Other extensions of using the inspection density to easily derive the optimal inspection policy can be found in Kaio and Osaki (1984) and Leung (2001).

Without considering the inspection time, Schneeweiss (1977) proposes a random check scheme and derives the probability density function of the time delay between product failure and inspection. Nakagawa and Yasui (1979) focus on the periodic inspection policies and derive an approximate checking time for a product with Weibull lifetime. Under a periodic inspection policy, Nakagawa (1984) takes preventive maintenance into account and derives the mean time to failure and the expected number of checks before failure. Furthermore, Schultz (1985) investigates a new approximation for the optimal periodic inspection policy to improve the cost performance of the approximation given in Munford (1981).

There is a common assumption in the works mentioned above, which is the time required for inspection is negligible. Luss and Kander (1974) and Luss (1977) coped with inspection models in which the duration of inspection and repair is non-negligible and the optimal policies are obtained for different objective functions. Chelbi and Ait-Kadi (1999) develop a condition-based monitoring scheme and an algorithm is provided to find the optimal inspection sequence over an infinite span.

In spite of all kinds of inspection policies developed the time value of cost (measured by the discount factor) in determining the optimal policies was seldom considered in the area of reliability engineering. In production research, Trippi and Lewin (1974) and Chung (1989) adopt the discount factor and find the optimal inventory policy in the long run. However, due to improvement in product reliability, the average lifetime of a product has increased significantly recently and hence the time value may play an important role in determining an inspection policy as pointed out in Abdel (2004).

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In this paper, we focus on deriving the optimal time epoch for inspection for a stochastically failing product, in which failures are only detected by inspection. A detailed description of the inspection scheme is given in Section 2, and a mathematical model is developed in Section 3. Furthermore, the properties of the optimal time epoch for inspection are investigated and an efficient algorithm is given in Section 3. Some numerical analysis is conducted in Section 4 to evaluate the impact of the discount factor on the optimal policy. Finally, a brief conclusion is drawn in the last section.

2. SYSTEM DESCRIPTION

Consider that the failure of a product can only be revealed by inspection. Following an inspection, the product is replaced by an identical new product. Upon the completion of a replacement, the product is renewed and the failure process starts over again. Hence, the time duration between two successive replacements is called a maintenance cycle. Suppose that each inspection incurs a cost c_1 and requires I units of time. After each inspection, the product is identified as either failed or non-failed. Once a failed product is found, a corrective replacement is carried out with cost c_2. For a non-failed product, it is also replaced by performing a preventive replacement with cost $c_3 (< c_2)$. Both corrective and preventive replacements require R units of time to complete.

Since failures can only be detected by inspections, if the product failed before an inspection, it keeps operating until the time epoch of performing an inspection. During this period of time, there is a cost δ_1 per unit time incurred for operating a failed product. Furthermore, the product stops operating when inspection and replacement are performed and there is a downtime loss of δ_2 per unit time. Although frequent inspections could reveal failures early to avoid operating a failed product, it also incurs more inspection and replacement costs, which may not be cost effective. Therefore, there is a need to find an optimal time epoch for inspection such that the expected total cost in a maintenance cycle is minimized.

Let X be the failure time of the product. In this paper, we consider the case where X follows a Weibull distribution since it is one of the most frequently used lifetime distributions in the area of reliability engineering. The probability density function (p.d.f.) of a two-parameter Weibull distribution is

$$f(x) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}}, \quad x \geq 0, \tag{1}$$

where $\alpha > 0$ and $\beta > 0$ are the scale and shape parameters, respectively. Furthermore, the mean time to failure is $\mu = \alpha\,\Gamma(1 + 1/\beta)$. It is well-known that the Weibull distribution has a decreasing failure rate when $\beta < 1$, a constant failure rate when $\beta = 1$, and an increasing failure rate when $\beta > 1$. Since most products deteriorate or age as operating time increases, in this paper, we will focus on the case when $\beta > 1$.

For the product mentioned above, the Lemma below is very helpful in deriving and investigating the properties of an optimal time epoch for inspection. Let's first define the log-concavity of a function. Any function $g(x)$ is said to be log-concave if the first derivative of $\log g(x)$ is a strictly decreasing function of x (Hariga, 1996). The following Lemma shows that the p.d.f. of a Weibull distribution is log-concave.

Lemma 1: When $\beta > 1$, the p.d.f. of a Weibull distribution $f(x)$ is log-concave.

Proof: From Eq. (1), we have $\log f(x) = (\log\beta - \alpha\log\beta) + (\beta-1)\log x - (x/\alpha)^{\beta}$. It is obvious that the first and second derivatives of $\log f(x)$ are given by

$$\frac{d\log f(x)}{dx} = \frac{\beta-1}{x} - \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \tag{2}$$

and

$$\frac{d^2\log f(x)}{dx^2} = -\frac{(\beta-1)}{x^2} - \frac{\beta(\beta-1)}{\alpha^2}\left(\frac{x}{\alpha}\right)^{\beta-2}. \tag{3}$$

When $\beta > 1$, we have $d^2\log f(x)/dx^2 < 0$ for all x from Eq. (3). Hence, $d\log f(x)/dx$ is a strictly decreasing function of x and $f(x)$ is log-concave when $\beta > 1$.

Using Lemma 1, the structural property of the optimal time epoch for inspection can be easily obtained as discussed in the next section.

3. THE OPTIMAL POLICY

To derive the expected total cost in a maintenance cycle, we first introduce the discount factor. As defined in financial theory, the discount factor is the value today of \$1 received in the future. In other words, the true value of a delayed cost could be uncovered by the discount factor. Such true value generally is represented by the present value. For example, under a continuous discount rate $r > 0$, \$1 at time s has a present value e^{-rs}, which is called the discount factor.

Now, if the product is inspected at time t, then it will be replaced at time $(t + I)$ and be renewed at time $(t + I + R)$. Hence, there is an inspection cost c_1 incurred at time t. If the failure time $x \leq t$, then the product failed before inspection. In this case, there is a cost δ_1 per unit time during the interval $[x, t]$ and a corrective replacement cost c_2 occurred at time $(t + I)$. On the other hand, if $x > t$, then the product does not fail at inspection and there is a preventive replacement cost incurred at time $(t+I)$. For both cases, there is a downtime loss δ_2 per unit time during the interval $[t, t+I+R]$. Therefore, under a continuous discount rate $r>0$, the present value of the expected total cost in a maintenance cycle is

$$PV(t)=$$
$$c_1e^{-rt}+\int_0^t\left[c_2e^{-r(t+I)}+\int_x^t\delta_1e^{-ru}du+\int_t^{t+I+R}\delta_2e^{-ru}du\right]f(x)dx$$
$$+\int_t^\infty\left[c_3e^{-r(t+I)}+\int_t^{t+I+R}\delta_2e^{-ru}du\right]f(x)dx. \tag{4}$$

Using Eq. (1) and integration by parts, we have

$$\int_0^t e^{-rx}f(x)dx=e^{-rt}(1-e^{-(\frac{t}{\alpha})^\beta})+\int_0^t re^{-rx}(1-e^{-(\frac{x}{\alpha})^\beta})dx. \tag{5}$$

Combining Eqs. (4) and (5), the present value of the expected total cost becomes

$$PV(t)=c_1e^{-rt}+c_2e^{-r(t+I)}(1-e^{-(\frac{t}{\alpha})^\beta})+c_3e^{-r(t+I)}e^{-(\frac{t}{\alpha})^\beta}$$
$$+\frac{\delta_1}{r}\int_0^t\left[e^{-rx}-e^{-rt}\right]f(x)dx+\frac{\delta_2}{r}\left[e^{-rt}-e^{-r(t+I+R)}\right]$$
$$=(c_1+\frac{\delta_2}{r})e^{-rt}+\left[(c_2-c_3)(1-e^{-(\frac{t}{\alpha})^\beta})+c_3\right]e^{-r(t+I)}$$
$$-\frac{\delta_2}{r}e^{-r(t+I+R)}+\frac{\delta_1}{r}\int_0^t e^{-rx}(1-e^{-(\frac{x}{\alpha})^\beta})dx. \tag{6}$$

Hence, our objective here is to find an optimal time epoch for inspection t^* such that Eq. (6) is minimized. Obviously, there are two trivial policies, which are $t=0$ and $t=\infty$. When $t=0$, the product is constantly being inspected and replaced, which is not applicable in practice. However, we have $PV(0)=c_1+c_3e^{-rI}+\delta_2/r[1-e^{-r(I+R)}]$ that provides an upper bound for $PV(t^*)$. One the other hand, when $t=\infty$, the product is never inspected. In this case, we have $PV(\infty)=\delta_2E[e^{-rx}]/r$ from Eq. (4). Again, this value provides another upper bound for $PV(t^*)$.

Taking the first derivative of Eq. (6) with respect to t, we have

$$\frac{dPV(t)}{dt}$$
$$=e^{-r(t+I+R)}\left\{\delta_2+\left[(c_2-c_3)f(t)-r((c_2-c_3)(1-e^{-(\frac{t}{\alpha})^\beta})+c_3)\right]e^{rR}\right.$$
$$\left.+[\delta_1(1-e^{-(\frac{t}{\alpha})^\beta})-rc_1-\delta_2]e^{r(I+R)}\right\} \tag{7}$$

To simplify the notations, let $\theta_1=\delta_1e^{rI}/(c_2-c_3)-r$ and $\theta_2=\left[r(c_1e^{rI}+c_3)+\delta_2(e^{rI}-e^{-rR})\right]/(c_2-c_3)$. Since $c_2>c_3$ and $e^{rI}>e^{-rR}$, we have $\theta_2>0$ for all r. After rearranging the terms, Eq. (7) can be rewritten as

$$\frac{dPV(t)}{dt}=e^{-r(t+I)}\left(c_2-c_3\right)\left[f(t)+\theta_1(1-e^{-(\frac{t}{\alpha})^\beta})-\theta_2\right]$$
$$=e^{-r(t+I)}\left(c_2-c_3\right)\Lambda(t) \tag{8}$$

where $\Lambda(t)=f(t)+\theta_1(1-e^{-\left(\frac{t}{\alpha}\right)^\beta})-\theta_2$. It is clear that $dPV(t)/dt$ has the same sign as $\Lambda(t)$. Setting Eq. (8) equal to zero, the necessary condition for $\tilde{t}$ to be optimal is $\Lambda(\tilde{t})=0$. However, the solution for $\Lambda(t)=0$ may not be unique. Since $\lim_{t\to0}\Lambda(t)=-\theta_2<0$ and $\lim_{t\to\infty}\Lambda(t)=\theta_1-\theta_2$ may be less or larger than zero, the number of sign changes of $\Lambda(t)$ depends on the relationship between θ_1 and θ_2.

To investigate the shape of $\Lambda(t)$, let's take the first derivative of $\Lambda(t)$ with respect to t as follows.

$$\frac{d\Lambda(t)}{dt}=f(t)\left[\frac{d\log f(t)}{dt}+\theta_1\right]. \tag{9}$$

From Lemma 1, we know that $d\log f(t)/dt$ is a strictly decreasing function of t. Hence, $d\Lambda(t)/dt$ changes its sign exactly once from positive to negative. This result implies that there exists a unique solution for $d\Lambda(t)/dt=0$, say $\hat{t}$, and $\Lambda(t)$ increases for $t\le\hat{t}$ and then decreases for $t>\hat{t}$. In other words, the shape of $\Lambda(t)$ is concave downward. Let $\tilde{t}\in[0,\hat{t}]$ be the solution of $\Lambda(t)=0$ whenever it exists. Then, based on the above discussion, the possible ranges of the optimal time epoch for inspection t^* are summarized in Theorem 2.

Theorem 2: When $\beta>1$, if $\Lambda(\hat{t})<0$, then $t^*=\infty$. Otherwise,
(i) when $\theta_1-\theta_2>0$, there exists a unique $t^*\in(0,\hat{t}]$,
(ii) when $\theta_1-\theta_2\le0$, if $PV(\tilde{t})<PV(\infty)$, then $t^*=\tilde{t}$; otherwise, $t^*=\infty$.

Proof: It is clear that if $\Lambda(\hat{t})<0$, then $\Lambda(t)<0$ for all t, which means $PV(t)$ decreases in t. Therefore, $t^*=\infty$. Otherwise, we will focus on the sign of $\theta_1-\theta_2$ since $\lim_{t\to0}\Lambda(t)=-\theta_2<0$. When $\theta_1-\theta_2>0$, $\Lambda(t)$

changes its sign exactly once from negative to positive in the interval $(0,\hat{t}]$ and so does $dPV(t)/dt$. Therefore, there exists a unique $t^* \in (0,\hat{t}]$ such that $PV(t)$ is minimized.

Now, when $\Lambda(\hat{t}) \geq 0$ and $\theta_1 - \theta_2 \leq 0$, $\Lambda(t)$ changes its sign exactly twice from negative to positive in the interval $(0,\hat{t}]$ and from positive to negative in the interval $(\hat{t},\infty]$. In this case, a local minimum occurs at $\tilde{t} \in [0,\hat{t}]$. Therefore, if $PV(\tilde{t}) < PV(\infty)$, then $t^* = \tilde{t}$; otherwise, $t^* = \infty$.

Using the results in Theorem 2, the optimal time epoch for inspection t^* can be easily obtained by the following algorithm.

Step 1. Compute θ_1, θ_2, and $PV(\infty)$.

Step 2. Search for $\hat{t}$ such that $d\log f(t)/dt = -\theta_1$ and evaluate $\Lambda(\hat{t})$.

Step 3. If $\Lambda(\hat{t}) < 0$, then set $t^* = \infty$ and **STOP**.

Step 4. If $\theta_1 - \theta_2 > 0$, then search for $\tilde{t} \in [0,\hat{t}]$ such that $\Lambda(\tilde{t}) = 0$. Set $t^* = \tilde{t}$ and **STOP**.

Step 5. If $\theta_1 - \theta_2 \leq 0$, then search for $\tilde{t} \in [0,\hat{t}]$ such that $\Lambda(\tilde{t}) = 0$ and evaluate $PV(\tilde{t})$.
If $PV(\tilde{t}) < PV(\infty)$, then set $t^* = \tilde{t}$; otherwise, set $t^* = \infty$. **STOP**.

4. NUMERICAL RESULTS

In this section, some numerical examples are given to analyze the impact of the discount rate and cost parameters on the optimal time epoch for inspection t^* and the resulting present value of the expected total cost $PV(t^*)$. The lifetime distribution of the product is assumed to be Weibull distributed with $\mu = \alpha\,\Gamma(1 + 1/\beta) = 1$. The corresponding scale and shape parameters for the distributions are (α, β) = (1.1284, 2) and (α, β) = (1.1198, 3). The other cost and time parameters are listed in Table 1. Using these parameters and MATLAB 6.5 software, the numerical results are summarized in Table 2.

Table 1. Illustration of the parameter settings

Parameter	Assigned Values
α	1.1198, 1.1284
β	2, 3
c_1	0.1
c_2	1
c_3	0.5
δ_1	20
δ_2	1, 4, 40
I	0.01
R	0.02
r	0.001, 0.01, 0.05, 0.1, 0.3

In Table 2, the optimal time epoch for inspection t^* and the resulting present value of the expected total cost $PV(t^*)$ are obtained using the algorithm given in Section 3 for various discount rates r and ratios δ_1/δ_2. For example, when $\delta_1/\delta_2 = 5$ and r = 0.05, we have t^* = 0.1163 and $PV(t^*)$ = 0.7289 for $(\alpha,\beta) = (1.1284, 2)$. Here, the δ_1/δ_2 represents the ratio of operating a failed product to the downtime loss.

Table 2. Numerical results of t^* and $PV(t^*)$ for different parameter combinations

	α = 1.1284	β = 2		α = 1.1198	β = 3	
	r	t^*	$PV(t^*)$	r	t^*	$PV(t^*)$
$\delta_1/\delta_2 = 20$	0.001	0.0216	0.6302	0.001	0.0214	0.6300
	0.01	0.0583	0.6319	0.01	0.0577	0.6297
	0.05	0.1105	0.6380	0.05	0.1095	0.6273
	0.1	0.1440	0.6438	0.1	0.1428	0.6230
	0.3	0.2167	0.6559	0.3	0.2149	0.5996
$\delta_1/\delta_2 = 5$	0.001	0.0231	0.7203	0.001	0.0228	0.7200
	0.01	0.0616	0.7222	0.01	0.0610	0.7196
	0.05	0.1163	0.7289	0.05	0.1153	0.7167
	0.1	0.1514	0.7352	0.1	0.1501	0.7116
	0.3	0.2275	0.7470	0.3	0.2256	0.6835
$\delta_1/\delta_2 = 0.5$	0.001	0.0344	1.8006	0.001	0.0340	1.7999
	0.01	0.0889	1.8049	0.01	0.0881	1.7986
	0.05	0.1645	1.8176	0.05	0.1631	1.7883
	0.1	0.2127	1.8260	0.1	0.2109	1.7704
	0.3	0.3171	1.8167	0.3	0.3145	1.6737

From Table 2, we have the following observations.

1. When δ_1/δ_2 decreases, the optimal time epoch for inspection t^* increases. This result is reasonable since a failed product may allow operating for a longer period of time when the operating cost is relatively smaller or the downtime loss is relatively larger.
2. When the discount rate r increases, the optimal time epoch for inspection t^* also increases. In practice, the discount rate may represent the interest rate or the rate of return. If the rate of return is higher, we may prefer to pay the inspection cost and replacement cost later, even though a failed product may operate for a longer period of time. As a result, the present value of the expected total cost $PV(t^*)$ decreases.
3. When the shape parameter β increases, the optimal time epoch for inspection t^* decreases. The higher the shape parameter β is, the faster the product deteriorates. Hence, the optimal time epoch for inspection t^* becomes shorter. However, the present value of the expected total cost $PV(t^*)$ decreases in this case since there is a shorter maintenance cycle.

5. CONCLUSIONS

In this paper, an inspection model is developed for a product that is inspected once in its lifetime. After inspection, the product is replaced by either corrective or preventive replacement. Taking the time value of costs into account, the discount factor is incorporated in the model.

Based on the model, some properties of the optimal time epoch for inspection are obtained, which are then used to develop an efficient algorithm to derive the optimal time epoch for inspection such that the present value of the expected total cost is minimized.

Numerical examples for products with Weibull lifetime distributions are given to investigate the effects of the discount rate and cost parameters on the optimal policy and the corresponding present value of the expected total cost. The numerical results show that the optimal time epoch for inspection becomes longer when the discount rate increases, cost ratio δ_1/δ_2 decreases, or shape parameter β decreases.

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