International Journal of Operations Research

RESEARCH NOTE

International Journal of Operations Research Vol. 2, No. 1, 83−84 (2005)

Estimation of Priority Vectors

Peter Shaohua Deng[1,*], Kevin Chen[2], and Peter Chu[2]

[1] Department of Information Management, Central Police University, Taiwan ROC

[2] Department of Traffic Science, Central Police University, Taiwan ROC

Abstract—In a recent paper of Murphy (1993), he predicted that if the relative weight of two alternatives were overestimated as demonstrably more important than another then the restriction of the range for the scales would reduce the error from inflating the ratio. In this paper, we will use the sum of square residues between the normalized principal right vectors to show that the Murphy's prediction is questionable. Our method provides a suggestion to explore the properties of eigenvectors.

Keywords—Analytic hierarchy process, Comparison matrix, Inconsistency

1. INTRODUCTION

For three alternatives A_1, A_2 and A_3, Murphy (1993) studied the following 3×3 reciprocal comparison matrix as

$$A=(a_{ij})=\begin{pmatrix} 1 & x & y \\ \frac{1}{x} & 1 & 9 \\ \frac{1}{y} & \frac{1}{9} & 1 \end{pmatrix}. \tag{1}$$

When $x=7$ and $y=63$, A is a consistent matrix. Murphy (1993) found the normalized principal right eigenvector as $(0.863, 0.123, 0.014)^T$. For $x=7$ and $y=9$, Murphy (1993) solved the normalized principal right eigenvector as $(0.751, 0.205, 0.044)^T$. Then he concluded, "Inconsistency cancelled a portion of any error due to inflated ratios. (A decision maker would probably rank one alternative as demonstrably more important than another when the stronger alternative was less than seven times more important.)" Hence, Murphy (1993) predicted that if the relative weight of A_2 over A_1 is overestimated such that $x=7$ then the restriction of the range for the scales to confine $y=9$ will reduce the error from inflating the ratio of x. In Hsueh and Chu (2006), we discussed the monotonic properties for the largest and the smallest components of priority vector. In Hsueh (2006), he extended our results to consider other components. In this paper, we will use the sum of square residues between the normalized principal right vectors for $y = 9$ and $y = 9x$ to show that the Murphy's prediction is doubtable.

2. DISCUSSION

When $y = 9x$, the comparison matrix in (1) is consistent so that the maximum eigenvalue, $\lambda_{\max}$, is equal to 3 and the normalized principal right eigenvector is denoted as $P(x)=(p_1(x),p_2(x),p_3(x))^T$ with $P_1(x) = \frac{9x}{10+9x}$, $P_2(x)=\frac{9}{10+9x}$ and $P_3(x)=\frac{1}{10+9x}$.

However, from Saaty (1980), the possible ranges for the scales in a comparison matrix are $\left\{\frac{1}{9},\frac{1}{8},\ldots,\frac{1}{2},1,2,\ldots,8,9\right\}$ the value of $9x$ usually is not proper; hence, we need to consider the case for $y=9$. Under the assumption of $y=9$, we apply the results in Vargas (1982); he showed that the average of normalized columns of a reciprocal matrix to offer a good approximation for the principal right eigenvector.

We denote the row average for normalized column as $H(x)=(h_1(x),h_2(x),h_3(x))^T$ with the following three corresponding components

$$h_1(x)=\frac{1353x^2+1626x+270}{1710x^2+3439x+1710},$$

$$h_2(x)=\frac{270x^2+1626x+1353}{1710x^2+3439x+1710} \text{ and}$$

$$h_3(x)=\frac{87x^2+187x+87}{1710x^2+3439x+1710}.$$

From our previous discussion, we have $\min\{9x,9\}=9$. Therefore, we obtain a natural restriction for the domain of the variable x as $x\ge 1$. We compute the square of the distance between $P(x)$ and $H(x)$ as

$$\begin{aligned} G(x)&=\left\|P(x)-H(x)\right\|^2 \\ &=(g_1(x))^2+(g_2(x))^2+(g_3(x))^2, \end{aligned} \tag{2}$$

where $g_1(x) = p_1(x) - h_1(x)$, $g_2(x) = p_2(x) - h_2(x)$ and $g_3(x) =$

* Corresponding author email: pdeng@mail.cpu.edu.tw

$p_3(x) - h_3(x)$. Now, we begin to evaluate the first derivatives of $g_k(x)$ for k =1, 2, 3. We have

$$\frac{d}{dx}g_1(x).$$

$$=\frac{90}{81x^2+180x+100}-\frac{57\left(91x^2+180x+90\right)}{\left(90x^2+181x+90\right)^2}. \quad (3)$$

Since for $x \geq 1$, it is evidently that $90(90x^2+181x+90) \geq 57(91x^2+180x+90)$ and $90x^2+181x+90 \geq 81x^2+180x+100$. Therefore, we know $\frac{d}{dx}g_1(x)>0$. Using $\lim_{x\to 1} g_1(x)=0$ and $\lim_{x\to\infty} g_1(x)=\frac{119}{570}$, we know that $g_1(x)$ is strictly increasing from o to $\frac{119}{570}$ for $x \in [1, \infty)$. We imply that

$$(g_1(x))^2 \text{ is increasing from 0 to } \left(\frac{119}{570}\right)^2. \quad (4)$$

Second, we find

$$\frac{d}{dx}g_2\left(x\right)$$

$$=\frac{-81}{81x^2+180x+100}+\frac{57\left(90x^2+180x+91\right)}{\left(90x^2+181x+90\right)^2}. \quad (5)$$

Since for $x \geq 1$, it is trivially that $81(90x^2+181x+90) \geq 57(90x^2+180x+91)$ and $90x^2+181x+90 \geq 81x^2+180x+100$. Therefore, we know $\frac{d}{dx}g_2(x)<0$. Using $\lim_{x\to 1} g_2\left(x\right)=0$ and $\lim_{x\to\infty} g_2\left(x\right)=\frac{-3}{19}$, we know that $g_2(x)$ is strictly decreasing from o to $\frac{-3}{19}$ for $x \in [1, \infty)$. We imply that

$$(g_2(x))^2 \text{ is increasing from 0 to } \left(\frac{3}{19}\right)^2. \quad (6)$$

Third, we know

$$\frac{d}{dx}g_3\left(x\right)$$

$$=\frac{-9}{81x^2+180x+100}+\frac{57\left(x^2-1\right)}{\left(90x^2+181x+90\right)^2}. \quad (7)$$

Since for $x \geq 1$, it is apparently that $9(90x^2+181x+90) \geq 57(x^2-1)$ and $90x^2+181x+90 \geq 81x^2+180x+100$. Therefore, we know $\frac{d}{dx}g_3(x)<0$. Using $\lim_{x\to 1} g_3\left(x\right)=0$ and $\lim_{x\to\infty} g_3\left(x\right)=\frac{-29}{570}$, we know that $g_3(x)$ is strictly decreasing from o to $\frac{-29}{570}$ for $x \in [1, \infty]$. We induce that

$$(g_3(x))^2 \text{ is increasing from 0 to } \left(\frac{29}{570}\right)^2. \quad (8)$$

Combining (2), (4), (6) and (8), we conclude that $G(x)=\left\|P(x)-H(x)\right\|^2$ is a strictly increasing function from $G(1) = 0$ to $G(9) = 0.2076$.

Therefore, the observation of Murphy (1993) "when $x \leq 7$, he predicted that if we increases the value of x, then the value of $G(x)$ will decrease due to the inconsistency" is invalid.

3. CONCLUSION

From the approximated principal right eigenvector we prepare a compact foundation for the distance between the priority vectors of consistent and inconsistent matrices. Our analytic work exposes a trace to detect the incompleteness of Murphy assertion. Hence, we suggest that research by a few numerical examples may overlook the true facts of the complicate problem.

REFERENCES

1. Hsueh, K.K.H. and Chu, P.. Note on Tendency of Eigenvector. submitted to *Mathematical and Computer Modeling*.
2. Hsueh, K.K.H.. Monotonic properties for eigenvector. submitted to *Journal of Information* & *Optimization Sciences*.
3. Murphy, C.K. (1993). Limits on the analytic hierarchy process from its consistency index. *European Journal of Operation Research*, 65: 138-139.
4. Saaty, T.L. (1980). *The Analytic Hierarchy Process*. McGraw-Hill, New York.
5. Vargas, L.G. (1982). Reciprocal matrices with random coefficients. *Mathematical and Computer Modeling*, 3: 69-81.