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Pooled Versus Reserved Inventory in a Two-Echelon Supply Chain

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Abstract**—**We consider a two-echelon supply chain with two retailers and one supplier. The retailers are supplied by the supplier who makes all the decisions and bears all the inventory risk. Throughout this paper, we consider two different inventory systems: a reserved inventory system and a pooled inventory system. With the reserved inventory system, the supplier keeps separate inventories for each retailer. In contrast, the pooled inventory is shared by the two retailers and the supplier makes the inventory decision based on the joint demand. Under different scenarios such as whether wholesale price is a decision variable, we study and analyze the supplier's decisions in the reserved and the pooled inventory systems. In addition, we compare the profit of the supplier and retailers in the two different systems under normally distributed demands.

Keywords**—**Inventory, Pooled inventory, Pricing, Supply chain

1. INTRODUCTION

Supply chain management is concerned with matching supply and demand, particularly through inventory management. Too much supply leads to inefficient investment and needless handling cost, while too little supply generates lost sales. The former is the inventory risk while the latter is supply risk. In reality, most supply chains cannot match supply and demand perfectly. Hence, all of the firms in a supply chain will bear some supply chain risks. However firms can decrease inventory risk.

Consider an electronics manufacturing service provider (EMS), who holds inventory of cpu chips for two or more original equipment manufacturers (OEM). The current inventory policy dictated by the OEM is to keep each company's inventory physically separated (reserved inventory). Is this the most profitable inventory policy for the EMS? Is it the most profitable inventory policy for the OEMs? In general, in this article, we are interested in knowing whether a supplier should pool inventory or reserve separate inventories for customers. If pooling is good for the supplier, is this policy also beneficial for its customers? Additionally, suppose the customers have service level requirements? We explore these questions for a two-echelon supply chain.

We consider a supply chain for a single product with a single supplier and two retailers. Only one single period or selling season is considered. We associate a customer region with each retailer and model retail customer demands as uncertain. During the selling season, each retailer receives orders from its customers, places an order to the supplier and receives product immediately for which they pay a unit wholesale price. The supplier manufactures product and holds it in inventory at his own expense until an order comes from the retailers, i.e, the supplier bears all the inventory risks. The supplier has only one chance to produce before the season starts. When a stock-out occurs at the supplier, sales are lost. The objective of the supplier is to maximize his single-period profit. Profits of retailers are maximized when they receive their full order. However, they do not have control over the inventory decision.

A key aspect of our research is the analysis of the impact of pooling inventory in the supply chain system. The literature on inventory pooling can be classified into following three categories: component commonality; inventory transshipment in supply chains; inventory pooling in multi-echelon supply chains.

If end products share common components, safety stock can be reduced and service levels maintained by pooling the inventory of the common parts. The work-to-date on component commonality concentrates mainly on the impact on safety inventory levels and does not consider the benefit of pooling to the suppliers and the retailers in the supply chain. Baker et al. (1986) study a two-product system with service level constraints and where the objective is to minimize the total safety stock. They show that total safety stock drops after pooling while total stock of specialized parts increases. Gerchak et al. (1988) extend these results to a profit maximization setting. Finally, Gerchak and Henig (1986) analyze a model in a multi-period setting and determine the optimal policy for the infinite horizon models.

Inventory transshipment involves transferring inventory from one member to another of the same echelon of a supply chain in event of a stock-out. The most relevant papers in this stream are those of Rudi et al. (2001) and Dong and Rudi (2002) in which both the retailers' and supplier's profits are considered. Transshipment creates a virtual centralization of the inventory by utilizing the benefit of inventory pooling within the same inventory echelon. Seifert and Thonemann (1999) and Seifer et al.

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(2001) model single-directional transshipments from physical to internet retailers. Anupindi et al. (2001) consider a very general decentralized transshipment model where multiple retailers not only stock inventory internally but also jointly stock it at multiple, jointly owned warehouse locations, which is similar to Anupindi and Bassok (1999a). This work is different from our work in that it concentrates on one echelon only.

There are several papers that, like us, investigate the benefits of pooling inventory in supply chain with more than one echelon. Anupindi and Bassok (1999a) consider a two-level supply chain with a single supplier and two retailers. Unlike our model, the inventory decision is made by the retailers and the retailers bear all the inventory risk. They model a system in which a fraction of the customers are willing to wait for a delivery from another retailer (market search). They show that under this setting, the manufacturer may not always benefit from inventory pooling because total sales may drop. They also discuss the possibility of optimizing wholesale prices or introducing holding cost subsidies as methods for coordinating the supply chain. In their model, demand is exogenous and not sensitive to price.

As in our work, Netessine and Rudi (2003) consider two supply chain strategies, traditional operation and drop shipping. With traditional operation, the retailer holds the inventory purchased from the supplier, while with drop shipping, the supplier holds the inventory. Although they also consider a two-echelon system, the second echelon consists of a collection of identical retailers. The retailers are only intermediaries between the end customer and the supplier and function as a single joint retailer. Netessine and Rudi compare the traditional channel and drop shipping strategy under normally distributed demands and find that the supply chain's profit may be higher or lower with drop shipping.

Cachon (2004) considers the "push contract", in which the retailer bears all the inventory risk and the "pull contract", in which the supplier bears all the inventory risk because only the supplier holds inventory. The retailer replenishes as needed during the season. His study focuses on identifying Pareto-optimal price-only contracts and studies supply chain efficiency under such contracts. However, since there is only one retailer, the benefits of inventory pooling are not reflected and in addition he only considers the case of the exogenous demands.

Of the existing literature, the work that is closest to ours is that of Bartholdi and Kemahlioglu (2003). They consider two retailers whose inventory is provided by a common supplier who bears all the inventory risk. They find that the total system profit will increase after pooling the inventory. In addition, using the Shapely value to allocation the additional profit, they analyze various schemes by which the supplier may pool inventory. By allocating Shapely value, they could coordinate the whole supply chain. However, they only consider the scenarios in which the wholesale price is fixed and the demands are not price-sensitive. We will show the optimal inventory and pricing policies when the wholesale price is a decision variable under the reserved and the pooled inventory systems. We also will analyze the comparative results for these scenarios.

2. MODEL UNDER CONSIDERATION

We consider reserved and pooled inventory systems for the two-echelon supply chain system shown in Figure 1. For the reserved inventory system, at the beginning of the period, the supplier stocks x_1 and x_2 respectively for retailers 1 and 2 at manufacturing cost c per unit. After retailer I (i = 1, 2) observes local demand, she places an order with the supplier. Retailer i receives inventory immediately and pays a wholesale price $w_i (w_i > c > 0)$ for each unit received. Let c_m be the markup on the wholesale price that the retailers charge, i.e. the retail price $p_i = w_i + c_m$ at retailer i. If the stock x_i of the supplier cannot satisfy the order from retailer i, the unsatisfied portion results in lost sales. Units remaining at the end of the season are disposed at unit cost $h(|h|\le c)$. Note that h may be negative, in which case it represents a per-unit salvage value. The supplier takes on the task of doing inventory replenishment and bears the inventory risk.

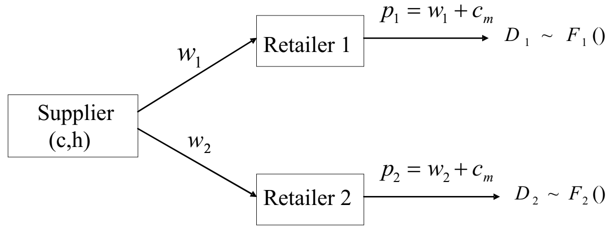

Figure 1. Two-echelon supply chain with one supplier and two retailers.

For the pooled inventory system, the supplier only has one central distribution center and the two retailers share the stock at this center. At the beginning of the period, the supplier stocks x_p at manufacturing cost c per unit. After the retailers observe their demands, they place orders with the supplier and pay a wholesale price w_p for each unit received. If the stock x_p of the supplier cannot satisfy the combined order, the unmet portion of the order is lost sales. In the pooled inventory case, when inventory cannot satisfy the total demand, the supplier needs to allocate the product to the retailers There are a number of papers discussing inventory allocation for difference scenarios (Cachon and Lariviere 1999). Our model focuses on the impact of the different policies on the profit of the supplier and the total profit of the retailers. Hence, we regard the two retailers as one joint retailer and thus need not consider the allocation policy in detail.

We denote retailer 1 and retailer 2's demands as D_1 and D_2 respectively. D_1 and D_2 are random variables with independent distributions. Let $F_1(\cdot)$, $F_2(\cdot)$ and $f_1(\cdot)$, $f_2(\cdot)$ denote the CDF and PDF of D_1 and D_2, respectively. Let D_p be the joint demand for the retailers with PDF $f_p(\cdot)$ and CDF $F_p(\cdot)$. Note that the joint demand $D_p = D_1 + D_2$ and $F_p(\cdot)$ is the convolution of $F_1(\cdot)$ and $F_2(\cdot)$. In this paper, we will analyze scenarios in which the supplier charges the retailers a fixed wholesale price and scenarios in which the wholesale prices are the supplier's decision variables. With the fixed wholesale price the demand parameters are exogenous to the system. However with the wholesale prices as variables, the demands at each retailer are treated as functions of the wholesale prices charged the retailers by the supplier.

Penalty costs associated with shortages are often hard to estimate with accuracy. It is therefore common practice for the supplier to try to maximize his profit while satisfying minimum service level requirements for retailers. Thus the service level requirement represents implicit shortage costs, e.g., loss of good will. Throughout this article, the service level requirements are measured by the probability of no stock-out. We denote ρ_i as minimum acceptable probability of no stock-out for retailer i in the reserved inventory case and ρ_p as the minimum acceptable probability meeting the retailers joint demand in the pooled inventory case.

3. FIXED WHOLESALE PRICE

Under scenario of fixed wholesale price, we will assume that the supplier charges retailers 1 and 2 a common fixed wholesale price w. In addition, the demands D_1 and D_2 at the retailers are independent random variables. We assume that the retailers 1 and 2 have minimum service level requirements.

We first present and analyze the decisions of the supplier for both the reserved and the pooled inventory policies.

3.1 Reserved inventory system

In this scenario, the retailers are powerful enough to require the supplier to use a reserved-inventory policy, i.e., the supplier maintains separate inventory x_1 and x_2 for retailers 1 and 2, respectively. In addition the retailers have minimum service level requirements ρ_1 and ρ_2. Given the inventory levels x_1 and x_2, the probability of no stock-out at retailer i(i=1,2) is

$$P(D_i \le x_i) = F_i(x_i),\ i = 1,2$$

The objective of the supplier is to maximize his profit while satisfying the service level requirements of the retailers. Thus the supplier's maximization problem is given by:

$$\begin{aligned} &\max_{x_1 \ge 0, x_2 \ge 0} \ \prod\nolimits_r (x_1, x_2) \\ &s.t.\ F_1(x_1) \ge \rho_1 \\ &\qquad F_2(x_2) \ge \rho_2 \end{aligned} \tag{1}$$

where

$$\prod\nolimits_r (x_1, x_2) = \mathrm{E}\left\{w\ \min(x_1, D_1) - h(x_1 - D_1)^+ \right.$$
$$\left. -cx_1 + w\min(x_1, D_1) - h(x_1 - D_1)^+ - cx_1 \right\}.$$

Due to the independence of the random variables D_1 and D_2. We can separate the problem (1) into following two independent problems:

$$\begin{aligned} &\max_{x_i \ge 0} \ \prod\nolimits_{ri}(x_i) \\ &s.t.\ F_i(x_i) \ge \rho_i \quad i = 1,2 \end{aligned}$$

where $\prod_{ri}(x_i) = \mathrm{E}\left\{w \min(x_i, D_i) - h(x_i - D_i)^+ - cx_1\right\}.$

Without service level requirements, the supplier's problem is a newsboy model. The optimal inventory levels correspond to a service level of $\dfrac{w-c}{w+h}$, which we called the critical ratio. Since $E\left\{w \min(x_i, D_i) - h(x_i, D_i)^+ -cx_i\right\}$ is a convex function of x_i and $F_i(x_i)$ is nondecreasing in x_i, the optimal inventory level x_i^* is

$$F_i^{-1}\left(\max(\rho_i, \frac{w-c}{w+h})\right).$$

The profits of the retailers only depend on the inventory level of supplier at the beginning of the period. Given the inventory level x_i, retailer i's (i=1,2) expected profit $\pi_{ri}^*(x_i)$ can be written as:

$$\pi_{ri}(x_i) = \mathrm{E}\left\{c_m \min(x_i, D_i)\right\} \quad i = 1,2.$$

We use π_{ri}^* to denote the optimal profit of the retailer i when the supplier holds x_i^* products for retailer *i*.

When the wholesale price is fixed, higher service level requirements by retailers may mean that the supplier must hold more inventory. While higher inventory levels mean higher expected sales, the supplier bears higher inventory holding risk when he maintains higher inventory levels.

3.2 Pooled inventory system

Now consider the supply chain when the supplier pools the inventory but must satisfy a joint service level requirement of the retailers. In this case, the objective of the supplier is to maximize his profit subject to satisfying all demand with probability ρ_p, i.e., the supplier sets his inventory level by solving following problem:

$$\begin{aligned} &\max_{x_p \ge 0} \prod\nolimits_p (x_p) \\ &s.t.\ F_p(x_p) \ge \rho_p \end{aligned}$$

where the supplier's expected profit is

$$\prod\nolimits_p (x_p) = \mathrm{E}\left\{w\ \min(x_p, D_p) - h(x_p - D_p)^+ - cx_p\right\}$$

Recall that the optimal inventory level is $F_p^{-1}(\dfrac{w-c}{w+h})$ for the pooled inventory case in the absence of service level constraint. The convexity of $\mathrm{E}\{w \min(x_p, D_p) - h(x_p - D_p)^+ - cx_p\}$ and the fact that the CDF $F_p(\cdot)$ is nondecreasing imply that the optimal inventory level x_p^* is

$$F_p^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho_p\right)\right).$$

Given the supplier's inventory level x_p, the total expected profit of the retailers is:

$$\pi_p = \mathrm{E}\{(p-w)\min(x_p, D_p)\}.$$

We use π_p^* to denote the optimal total expected profit of the retailers when the supplier's inventory level is π_p^*.

3.3 Comparative results

In general, inventory pooling by the supplier may or may not lead to increased expected retail sales as shown by the following two examples. Example 3.1 illustrates an increase in total expected sales while Example 3.2 illustrates a decrease.

Example 3.1 *Consider a system with pooled inventory in which demands* D_1 *and* D_2 *at the retailers are independent and uniformly distributed between* [0,100], *and the CDF of the demand at each retailer is*

$$F(u) = \begin{cases} \dfrac{u}{100} & 0 \le \mathrm{u} \le 100 \\ 1 & \mathrm{u} \ge 100 \end{cases}.$$

Let $D_p = D_1 + D_2$. *The CDF of the random variable* D_p *is*

$$F_p(u) = \begin{cases} \dfrac{u^2}{10000} - \dfrac{\mathrm{u}^2}{20000} & 0 \le \mathrm{u} \le 100 \\ -\dfrac{u^2}{20000} - \dfrac{2\mathrm{u}}{100} - 1 & 100 \le \mathrm{u} \le 200 \end{cases}.$$

If the critical ratio $\frac{w-c}{w+h}$ = 0.6 *and all the service level requirements are* 0.45, *i.e.*, $\rho_1 = \rho_2 = \rho = 0.45$, *then in the reserved inventory case*, *the optimal inventory levels are*

$$x_1^* = x_2^* = F^{-1}\,(\max(0.45, 0.6)) = 60,$$

the expected sales of each retailer is 42, *and the expected total sales is* 84. *In the pooled inventory case,* $x_p^* = F_p^{-1}\,(0.6) = 111$, *and the total expected sales is 88. Therefore, the total expected profit of the retailers is increased after pooling the inventory.*

Example 3.2 *Continue to assume that the demands* D_1 *and* D_2 *at the retailers are independent and uniformly distributed between* [0,100], *that the critical ratio* $\frac{w-c}{w+h}$ = 0.8 *and all the service level requirements are* 0.7, *i.e.*, $\rho_1 = \rho_2 = \rho = 0.7$. *Then in the reserved inventory case, the optimal inventory levels are*

$$x_1^* = x_2^* = F^{-1}\,(0.8) = 80,$$

the expected sales at each retailer is 48, *and the expected total sales is* 96. *In the pooled inventory case,* $x_p^* = F_p^{-1}(0.8) = 137$, *and the expected total sales is* 95. *Therefore, the total expected profit of the retailers drops after pooling the inventory.*

Under generally distributed demands, with higher inventory levels, the expected service level provided to the retailers and their expected sales also increase. If the required service level exceeds the critical ratio, the supplier loses money by providing a higher service level. We now examine the impact of the reserved inventory and the pooled inventory policies on the profits of the supplier and retailers. We will show the results both for the case when the retailers have the same service level requirements and for the case when they have different service level requirements.

Due to its mathematical tractability, the normal distribution appears to be the distribution of choice in modeling multi-location inventory problems. In addition, a lot of random distributions can be approximated by the normal distribution. Although the range of a normally distributed variable is from − and +, if the mean value is large enough relative to its variance, the relative demand values will almost surely be nonnegative. Alfaro and Corbett (2003) perform a simulation study of the pooling effect, comparing the impact of the normal distribution with several nonnormal distributions. They conclude that the effect of pooling does not vary much between the different distributions.

Suppose D_1 and D_2 are independently distributed normal random variables with means μ_1 and μ_1 and standard deviations σ_1 and σ_2, respectively. Let $\Phi(\cdot)$ denote the CDF and $\phi(\cdot)$ the PDF of the standard normal distribution. In addition, we denote by $R(\cdot)$ the right-hand unit normal linear loss function, which is defined as follows (see Zipkin, 2000):

$$\mathrm{R}(x) = \int_x^{\infty} (u - x)\phi(u)dt$$

From Zipkin, we know that $R(x)$ is a nonnegative and nonincreasing function of x, and

$$R(x) + x \ge 0 \text{ for any } x. \tag{2}$$

For the case of normally distributed demands, we can provide a detailed comparison of the reserved and pooled inventory cases. We assume that D_1, D_2 are independent normally distributed random variables with mean μ_i, standard deviation σ_i, respectively. In addition, we suppose that all the service level requirements are same, i.e., $\rho_1 = \rho_2 = \rho_p = \rho$. We have following theorems.

Theorem 3.1: *If* D_1 *and* D_2 *are independent normally distributed random variables*, *the supplier's optimal profit is increased when the* *inventory is pooled, i.e.* $\prod_p^* \geq \prod_r^*$.

Proof: Under the reserved inventory scenario, the optimal inventory levels are given by:

$$x_1^* = \mu_2 + \sigma_1 \Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right),$$

and

$$x_2^* = \mu_2 + \sigma_2 \Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right).$$

For the pooled inventory scenario, $D_p \sim N(\mu_1 + \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2})$. Hence the optimal total inventory level for the pooled inventory scenario is:

$$x_p^* = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right).$$

The supplier's optimal profits in the reserved inventory and the pooled inventory cases are:

$$\begin{aligned}\prod_r^* &= (w+h)\mathrm{E}\left\{(\min(x_1^*, D_1) + \min(x_2^*, D_2)\right\} - (h+c)(x_1^* + x_2^*),\\ &= (w+h)\left(\mu_1 + \mu_2 - (\sigma_1 + \sigma_2)\mathrm{R}\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right)\right) - (h+c)\left(\mu_1 + \mu_2 - (\sigma_1 + \sigma_2)\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right),\end{aligned}$$

$$\begin{aligned}\prod_p^* &= (w+h)\mathrm{E}\left\{\min(x_p^*, D_p)\right\} - (h+c)x_p^*\}\\ &= (w+h)\left(\mu_1 + \mu_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\mathrm{R}\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right)\right) - (h+c)\left(\mu_1 + \mu_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right).\end{aligned}$$

Calculating the difference $\prod_r^* - \prod_p^*$, we have

$$\begin{aligned}\prod_r^* - \prod_p^* &= (w+h)\left(\sqrt{\sigma_1^2 + \sigma_2^2} - (\sigma_1 + \sigma_2)\right)\mathrm{R}\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right) + (h+c)\left(\sqrt{\sigma_1^2 + \sigma_2^2} - (\sigma_1 + \sigma_2)\right)\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\\ &\leq (h+c)\left(\sqrt{\sigma_1^2 + \sigma_2^2} - (\sigma_1 + \sigma_2)\right)\mathrm{R}\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right) + (h+c)\left(\sqrt{\sigma_1^2 + \sigma_2^2} - (\sigma_1 + \sigma_2)\right)\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\\ &= (c+h)\left(\sqrt{\sigma_1^2 + \sigma_2^2} - (\sigma_1 + \sigma_2)\right)\left\{\mathrm{R}\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right) + \Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right\}\\ &\leq 0.\end{aligned}$$

The first inequality follows from the facts that $w > c$, $R(\cdot) > 0$ and $\sqrt{\sigma_1^2 + \sigma_2^2} \leq (\sigma_1 + \sigma_2)$ for any positive σ_1 and σ_2. The second inequality is seen by applying equation (2). Hence, the supplier's profit is increased after pooling the inventory, i.e., $\prod_r^* \leq \prod_p^*$.

Theorem 3.2: *If* D_1 *and* D_2 *are independent normally distributed random variables, the retailers' total expected profit is increased when the inventory is pooled, i.e.,* $\pi_p^* \geq \pi_r^*$.

Proof: The expected retail profits of the retailers in the reserved inventory case are given by:

$$\pi_i^* = E\left\{c_m \min(x_i^*, D_i)\right\},\ i = 1,2,$$

and the total expected retail profit of the retailers in the pooled inventory case is given by:

$$\pi_p^* = E\left\{c_m \min(x_p^*, D_p)\right\}.$$

For the normally distributed demands, we have

$$\pi_{ri}^* = c_m\left(\mu_i - \sigma_i \mathrm{R}\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right)\right), \quad \text{for } i = 1, 2,$$

and

$$\pi_p^* = c_m\left(\mu_1 + \mu_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\mathrm{R}\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right)\right).$$

Therefore,

$$\begin{aligned}\pi_p^* - \pi_r^* &= \pi_p^* - \pi_{r1}^* - \pi_{r2}^*\\ &= c_m\left(\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2}\right)\mathrm{R}\left(\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right)\right).\end{aligned}$$

Since $R(\cdot)$ is nonnegative and using Cauchy's inequality $\sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2}$, we have

$$\pi_p^* - \pi_r^* \geq 0$$

Hence the total expected profit is increased after pooling the inventory.

Theorem 3.3: *If* D_1 *and* D_2 *are independent normally distributed random variables, and the critical ratio* $\max(\frac{w-c}{w+h}, \rho) \geq 0.5$, *then the supplier's optimal inventory level in the pooled inventory case is less than the optimal total inventory in the reserved inventory case, i.e.* $x_p^* \leq x_1^* + x_2^*$. *Otherwise,* $x_p^* > x_1^* + x_2^*$.

Proof: We know that the optimal total inventory level in the reserved inventory case is given by,

$$x_1^* + x_2^* = \mu_1 + \mu_2 + (\sigma_1 + \sigma_2)\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right).$$

However the optimal inventory level in the pooled inventory case is given by :

$$x_p^* = \mu_1 + \mu_2 + \sqrt{\sigma_1^2 + \sigma_2^2}\Phi^{-1}\left(\max\left(\frac{w-c}{w+h}, \rho\right)\right).$$

The facts that $\Phi^{-1}(\cdot)$ is monotone nondecreasing function and $\Phi^{-1}(0.5) = 0$ imply that Φ^{-1} $(\max(\frac{w-c}{w+h}, \rho))$ ≥ 0 provided $\max(\frac{w-c}{w+h}, \rho)) \geq 0.5$. Hence $x_p^* \leq x_1^* + x_2^*$ follows from the Cauchy inequality $\sqrt{\sigma_1^2 + \sigma_2^2} \leq (\sigma_1+\sigma_2)$. Otherwise, when $\max(\frac{w-c}{w+h}, \rho)) < 0.5$, we have $x_p^* > x_1^* + x_2^*$.

Under identically and normally distributed demands, if the service level requirements in the reserved inventory case are same as the joint service level requirement under the pooled inventory case, the supplier will get benefit from the pooled inventory policy. In addition, the total expected sales is also increased by sharing the inventory between the two retailers. Because of the benefits of inventory pooling, both the retailers and the supplier will choose the pooled inventory policy. However, we find that the supplier's total inventory level may increase or decrease after pooling the inventory. As in the case without service level requirement, we find that the difference between the total optimal inventory level and the mean value of the demands is decreased after pooling the inventory.

3.4 A modified case

We have analyzed a supply chain in which the retailers have the identical service level requirements for the reserved inventory and the pooled inventory scenarios. Here, we model a supply chain in which the retailers have different service level requirements in a modified pooled inventory case.

Let x_{pi} be the stock which the supplier keeps for retailer i before the selling season, but now assume that it is sharable, i.e., if the stock kept for the retailer 1 runs out and there is stock available in the inventory for retailer 2 after the demand of retailer 2 is satisfied, then this remaining stocking can be used to satisfy the unsatisfied demand at retailer 1. Let D_1, D_2 denote the demands at retailers 1 and 2, respectively. Under the pooled inventory policy, the probability of no stock-out at retailer 1 is:

$$P(D_1 \leq x_{p1}) + P(D_1 > x_{p1}, D_2 < x_{p2}, D_1 + D_2 \leq x_{p1} + x_{p2}),$$

and at retailer 2 is:

$$P(D_2 \leq x_{p2}) + P(D_2 > x_{p2}, D_1 < x_{p1}, D_1 + D_2 \leq x_{p1} + x_{p2}).$$

Let ρ_1 and ρ_2 be the service level requirements for retailers 1 and 2, respectively. For the supplier, the problem of maximizing total profit can be formalized as:

$$\begin{aligned}
&\max_{x_{p1}, x_{p1} \geq 0} \prod\nolimits_p (x_{p1}, x_{p1}) \\
&= \max_{x_{p1}, x_{p1} \geq 0} \mathrm{E}\{w \min(x_{p1}, D_1) - h(x_{p1} - D_1)^+ - cx_{p1} \\
&\qquad + w \min(x_{p2}, D_2) - h(x_{p2} - D_{p2})^+ - cx_{p2}\} \\
&s.t.\ P(D_1 \leq x_{p1}) + P(D_1 > x_{p1}, D_2 < x_{p2}, D_1 + D_2 \leq x_{p1} + x_{p2}) \geq \rho_1 \\
&\qquad P(D_2 \leq x_{p2}) + P(D_2 > x_{p2}, D_1 < x_{p1}, D_1 + D_2 \leq x_{p1} + x_{p2}) \geq \rho_2.
\end{aligned}$$

Assume retailers require same service level ρ_1 and ρ_2 in the pooled and reserved inventory scenarios, we have the following results.

Theorem 3.4: *The supplier's optimal inventory levels in the reserved inventory case is a feasible solution for supplier's maximization problem* (3) *in the pooled inventory case.*

Proof: Let (x_1^*, x_2^*) be the optimal inventory levels in the reserved inventory case. It's sufficient to show that (x_1^*, x_2^*) satisfies the constraints in problem(3). For the first constraint in equation (3)

$$\begin{aligned}
&P(D_1 \leq x_1^*) + P(D_1 > x_1^*, D_2 < x_2^*, D_1 + D_2 \leq x_1^* + x_2^*) \\
&\geq P(D_1 \leq x_1^*) \\
&= \rho_1
\end{aligned}$$

Hence, (x_1^*, x_2^*) satisfies the first constraint. Similarly, we can prove that it also satisfies the second constraint, and thus (x_1^*, x_2^*) is a feasible solution for problem (3).

The objective function of the supplier's problem in the reserved inventory case is same as that in the pooled inventory case. However, Theorem 3.4 shows that the optimal solution of problem in the reserved inventory is a feasible solution for the problem in the pooled inventory case. Hence the optimal objective value, namely, the optimal profit of the supplier, in the pooled inventory case is at least at large as that in the reserved inventory case. We state this property in the following theorem.

Theorem 3.5: T*he optimal profit of the supplier in the pooled inventory case is at least as large as that in the reserved inventory case, i.e.,* $\prod_p^* \geq \prod_r^*$.

Theorem 3.6: *The optimal inventory level for the supplier after pooling is smaller than before pooling.*

Proof: Let (x_1^*, x_2^*) be the optimal inventory levels in the reserved inventory case. Then define (x_{p1}^*, x_{p2}^*) as the optimal inventory levels in the pooled inventory case. We have

$$x_i^* = F_i^{-1}\left(\max(\frac{w-c}{w+h}, r_i)\right),\ i = 1, 2.$$

Theorem 3.4 shows that (x_1^*, x_2^*) is a feasible solution for the supplier's maximization problem in the pooled inventory case. $\prod_p(x_{p1}, x_{p2})$ is a jointly concave function of x_{p1} and x_{p2}; and when (x_{p1}, x_{p2}) = $\left(F_1^{-1}(\frac{w-c}{w+h}), F_2^{-1}(\frac{w-c}{w+h})\right)$ it reaches its maximum value. However $\prod_p(x^*_{p1}, x^*_{p2}) \geq \prod_r(x^*_1, x^*_2)$, hence $x^*_{p1} \leq x^*_1$ and $x^*_{p2} \leq x^*_2$.

These results show that supplier gets more profit in the pooled inventory case than in the reserved inventory case. In the pooled inventory case, due to sharing the inventory, the supplier produces less stock than in the reserved inventory case. However, we cannot say whether the expected profit of the retailer will increase or drop after pooling inventory. This depends on system parameters such as the demand parameters.

4. ADDITIVE DEMAND MODEL: WITHOUT SERVICE LEVEL REQUIREMENTS

In the previous section, we considered the reserved inventory and the pooled inventory policies with service level requirements. However, wholesale prices were fixed at $w_1 = w_2 = w$. Now, we consider the scenario under which the wholesale prices are decision variables. As before, we denote retailer 1 and retailer 2's random demands as D_1 and D_2 respectively. However, here we assume that the demands are price-sensitive and they are functions of the wholesale prices.

The way the price-sensitive random demand is modeled is very important. We consider an additive demand function of the following form Mills (1959),

$$D_i(w_i) = y(w_i) + \varepsilon_i, i = 1, 2$$

where $y(w_i)$ is a deterministic and decreasing function of the product's wholesale price w_i and [*epsilon*]$_i$ is an independent random variable defined on the range [A, B] with CDF, $G_i(\cdot)$, PDF, $g_i(\cdot)$ and mean value μ_i. In addition, we assume that
$y(w) = a - bw \quad a > 0, b > 0.$

For the pooled inventory case, we use $D_p(w_p)$ to be the total joint demand for the retailers when the wholesale price is w_p. Then

$$\begin{aligned} D_p(w_p) &= D_1(w_p) + D_2(w_p) \\ &= y(w_p) + y(w_p) + \varepsilon_1 + \varepsilon_2 \\ &= 2y(w_p) + \varepsilon_p \end{aligned}$$

where
$\varepsilon_p = \varepsilon_1 + \varepsilon_2$

The random variable ε_p is defined on the range [2A, 2B] with mean value $\mu_p = \mu_1 + \mu_2$. We use $G_p(\cdot)$ and $g_p(\cdot)$ to denote the CDF and PDF of ε_p. Note that G_p is the convolution of $G_1(\cdot)$ and $G_2(\cdot)$

Specifying a feasible wholesale price range is common in the operations and economics literature (see Federguen and Heching 1996). We assume that the set of feasible wholesale prices is confined to a finite interval $[c, w_{\max}]$, where

- c: lowest possible unit wholesale price to be charged (which implies that the wholesale price should at least equal to the manufacturing cost, otherwise, the supplier cannot make a profit).
- $w_{\max}$: highest possible unit wholesale price to be charged.

In order to assure the feasible wholesale price guarantees nonnegative demands, we require that $y(w_{\max}) + A = a - bw_{\max} + A \geq 0$, which in turn implies that $y(c) + A = a - bc + A \geq 0$.

In this section, we analyze the decisions of the supplier for both the reserved inventory case and the pooled inventory case. We also compare the results of the reserved inventory scenario and the pooled inventory scenario when $\varepsilon_i (i = 1, 2)$ are normally distributed. We do this with and without service level requirements.

4.1 Reserved inventory system

To maximize the expected supplier's profit, the supplier must choose the wholesale price and inventory level for each retailer. Let $\prod_r(x_1, x_2, w_1, w_2)$ denote the supplier's expected profit when the supplier keeps inventory level x_i and charges w_i per unit for retailer *i* (*i*=1, 2). We have

$$\begin{aligned} &\prod\nolimits_r(x_1, x_2, w_1, w_2) \\ &= E\{w_1 \min(x_1, D_1(w_1)) - h(x_1 - D_1(w_1))^+ - cx_1 \\ &\quad + w_2 \min(x_2, D_2(w_2)) - h(x_2 - D_2(w_2))^+ - cx_2\}. \end{aligned}$$

Recall that we have assumed that ε_1 and ε_2 are independent random variables. Thus D_1 and D_1 are also independent and $\prod_r(x_1, x_2, w_1, w_2)$ is separable, i.e.,

$$\prod\nolimits_r(x_1, x_2, w_1, w_2) = \prod\nolimits_{r1}(x_1, w_1) + \prod\nolimits_{r2}(x_2, w_2),$$

where

$$\begin{aligned} \prod\nolimits_{r1}(x_1, w_1) &= E\{w_1 \min(x_1, D_1(w_1)) - h(x_1 - D_1(w_1))^+ - cx_1\}, \\ \prod\nolimits_{r2}(x_2, w_2) &= E\{w_2 \min(x_2, D_2(w_2)) - h(x_2 - D_2(w_2))^+ - cx_2\}. \end{aligned}$$

Hence, the supplier can maximize his profit by solving following two problems

$$\begin{aligned} &\max_{x_1 \geq 0, w_1 \in [c, w_{\max}]} \prod\nolimits_{r1}(x_1, w_1), \\ &\max_{x_2 \geq 0, w_2 \in [c, w_{\max}]} \prod\nolimits_{r2}(x_2, w_2). \end{aligned} \tag{4}$$

Due to the identical structures of $\prod_{r1}$ and $\prod_{r2}$, in the rest of the section, we focus on problem (4).

Consider following optimization problem,

$$\max_{x_1 \ge 0, w_1 \in [c, w_{\max}]} \prod_{r1}(x_1, w_1). \quad (5)$$

The range of the wholesale price w_1 guarantees nonnegative demands, which implies that the optimal inventory level x_1^* is always nonnegative. Hence, problem (4) is equivalent to problem (5). We define the expected excess stock, $\Lambda_1(x)$, and the expected shortage, $\Theta_1(x)$, when inventory level is chosen as x and demand (with PDF $g_1(\cdot)$) turns out to be ε_1. Specifically,

$$\Lambda_1(x) = \int_A^x (x-u) g_1(u) du,$$

and

$$\Theta_1(x) = \int_x^B (x-u) g_1(u) du.$$

From the definition of $\Theta_1(x)$, we know that it is a nonnegative function of x. Checking the first derivative of $\Theta_1(x)$ with respect to x, we have $\Theta'_1(x) = G_1(x) - 1 \le 0$. Hence $\Theta_1(x)$ is decreasing in x. In addition, we find that $\Theta_1(x)$ and $\Lambda_1(x)$ satisfy following equation:

$\Theta_1(x) = \Lambda_1(x) - x + \mu_1$.

For the retailer 1, the supplier's profit, $\prod_{r1}(x_1, w_1)$, can be written as:

$$\begin{aligned}
&\prod_{r1}(x_1, w_1) \\
&= \int_A^{x_1 - y(w_1)} \{w_1(y(w_1) + u) - h(x_1 - y(w_1) - u)\} g_1(u) du \\
&\quad + \int_{x_1 - y(w_1)}^B \{w_1(x_1)\} g_1(u) du - c x_1 \\
&= I(w_1) - L(x_1, w_1)
\end{aligned} \quad (6)$$

where

$$I(w_1) = (w_1 - c)(y(w_1) + \mu_1),$$

and

$$L(x_1, w_1) = (c+h)\Lambda_1(x_1 - y(w_1)) + (w_1 - c)\Theta_1(x_1 - y(w_1)).$$

$I(w_1)$ represents the supplier's riskless profit function, i.e., the profit of the supplier for a given price w_1 when the demand variable ε_1 is replaced by its constant mean μ_1. Notice that without uncertainty on the demand side, the supplier can manufacture exactly the amount of inventory demanded. $L(x_1, w_1)$ is the loss function, which assesses an overage cost c+h for each unit of the expected unused inventory $\Lambda_1(x_1 - y(w_1))$ and an underage cost $(w_1 - c)$ for each unit of $\Theta_1(x_1 - y(w_1))$ expected shortages. The following lemma gives some properties of $\prod_{r1}(x_1, w_1)$ which will be used to solve the problem.

Lemma 4.1:

1. For a given w_1, $\prod_{r1}(x_1, w_1)$ is concave in x_1.

2. For a given w_1, the optimal inventory level is determined by

$$x_1^*(w_1) = y(w_1) + G_1^{-1}\left(\frac{w_1 - c}{w_1 + h}\right). \quad (7)$$

Proof: Consider the first and second partial derivatives of $\prod_{r1}(x_1, w_1)$ taken with respect to x_1:

$$\frac{\partial \prod_{r1}(x_1, w_1)}{\partial x_1} = (-c-h) + (w_1 + h)[1 - G_1(x_1 - y(w_1))],$$

$$\frac{\partial^2 \prod_{r1}(x_1, w_1)}{\partial^2 x_1} = -(w_1 + h) g_1(x_1 - y(w_1)) < 0,$$

Hence, given $w_1, \prod_{r1}(x_1, w_1)$ is concave in x_1. Part (2) follows from $\frac{\partial \prod_{r1}(x_1, w_1)}{\partial x_1} = 0.$

Lemma 4.1 shows that $\prod_{r1}(x_1, w_1)$ is concave in x_1 for a given w_1. Thus, it is possible to reduce the original problem to an optimization problem over the single variable w_1 by first solving for the optimal value of x_1 as a function of w_1 and then substituting the result back into $\prod_{r1}(x_1, w_1)$. Concavity of $\prod_{r1}(x_1, w_1)$ in x_1 for given w_1 allows us to use Zabel's method (1970) of first optimizing x_1 for a given w_1, and then searching over the resulting optimal trajectory to maximize $\prod_{r1}(x_1^*(w_1), w_1)$.

Before we give the optimal solution of the optimization problem, we introduce the concept of the failure rate. For a random variable with CDF $F(\cdot)$ and PDF $f(\cdot)$, we use $h(u)$ to denote the generalized failure rate,

$$h(u) = \frac{uf(u)}{1 - F(u)},$$

and $r(u)$ to denote the classical failure rate,

$$r(u) = \frac{uf(u)}{1 - F(u)}.$$

The classical failure rate gives roughly the percentage decrease in the probability of a stock-out from increasing the quantity stocked by one unit, while the generalized failure rate gives roughly the percentage decrease in the probability of a stock out from increasing the stocking quantity by 1%. An increasing failure rate (IFR) has appealing implications. As a supplier holds more stock for a retailer, the retailer's order quantity becomes less elastic, i.e., the probability of a stock-out becomes smaller. The IFR assumption is not restrictive because it applies for most common distributions. Distributions with IFR such as the normal or uniform distributions are clearly also IGFR, but there are IGFR distributions that are not IFR.

Returning to our optimization problem, we can now give conditions and a procedure for calculating a unique optimal solution.

Theorem 4.1: *Let $r(Z_1)$ be the failure rate of the random variable ε_1. If $r'(z_1) > 0$, then (x_1^*, w_1^*) is uniquely determined by*

$$\begin{cases} 2b(w^0 - w_1) - \Theta_1(x_1 - y(w_1)) = 0, \\ G_1(x_1 - y(w_1)) = \dfrac{w_1 - c}{w_1 + h}. \end{cases}$$

where

$$w^0 = \frac{a + bc + \mu_1}{2b}.$$

Furthermore, the optimal wholesale price w_1^* satisfies:

$$2b(w^0 - w_1) - \Theta_1(x_1^*(w_1) - y(w_1)) = 0,$$

and the optimal inventory level can be calculated by equation (7), i.e.,

$$x_1^* = y(w_1^*) + G_1^{-1}\left(\frac{w_1^* - c}{w_1^* + h}\right)$$

Proof: We know that the optimal solution (x_1^*, w_1^*) satisfies following first-order conditions,

$$\frac{\partial \prod_{r1}(x_1, w_1)}{\partial x_1} = (-c - h) + (w_1 + h)(1 - G_1(x_1 - y(w_1))) = 0,$$

$$\begin{aligned} \frac{\partial \prod_{r1}(x_1, w_1)}{\partial w_1} &= 2b(w^0 - w_1) - \Theta_1(x_1 - y(w_1)) \\ &\quad + b(-c - h) + b(w_1 + h)(1 - G_1(x_1 - y(w_1))) \\ &= b(-c - h) + b(w_1 + h)(1 - G_1(x_1 - y(w_1))) \\ &= 0. \end{aligned}$$

By Lemma 4.1, the optimal inventory level $x_1^*(w_1)$ for given w_1 is given as by:

$$x_1^*(w_1) = y(w_1) + G_1^{-1}\left(\frac{w_1 - c}{w_1 + h}\right).$$

Substituting $x_1^*(w_1)$ into equation (6), we get $\prod_{r1}(x_1^*(w_1), w_1)$ as a function of w_1, namely,

$$\begin{aligned} \prod_{r1}(x_1^*(w_1), w_1) &= (w_1 - c)(y(w_1) + \mu_1) \\ &\quad -(c + h)\Lambda_1(x_1^*(w_1) - y(w_1)) - (w_1 - c)\Theta_1(x_1^*(w_1) - y(w_1)). \end{aligned}$$

Taking the first derivative with respect to w_1, we have

$$\begin{aligned} \frac{d \prod_{r1}(x_1^*(w_1), w_1)}{dw_1} &= 2b(w^0 - w_1) - \Theta_1(x_1^*(w_1) - y(y_1)) \\ &\quad + b(-c - h) + b(w_1 + h)(1 - G_1(x_1^*(w_1) - y(w_1))) \\ &= 2b(w^0 - w_1) - \Theta_1(x_1^*(w_1) - y(y_1)). \end{aligned}$$

Defining $V(w_1) = 2b(w^0 - w_1) - \Theta_1(x_1^*(w_1) - y(w_1))$, and calculating the first derivative of V with respect to w_1, we get

$$\begin{aligned} V'(w_1) &= -2b + [1 - G_1(x_1^*(w_1) - y(w_1))]\left(\frac{dx_1^*(w_1)}{w_1} + b\right) \\ &= -2b\frac{1 - G_1(x_1^*(w_1) - y(w_1))}{(w_1 + h)r(x_1^*(w_1) - y(w_1))}. \end{aligned}$$

The second derivative is:

$$\begin{aligned} V''(w_1) &= \frac{1 - G_1(x_1^*(w_1) - y(w_1))r'(x_1^*(w_1) - y(w_1))}{(w_1 + h)r^2(x_1^*(w_1) - y(w_1))}\left(\frac{dx_1^*(w_1)}{w_1} + b\right) \\ &\quad - \frac{1 - G_1(x_1^*(w_1) - y(w_1))}{(w_1 + h)^2 r(x_1^*(w_1) - y(w_1))} - \frac{1 - g_1(x_1^*(w_1) - y(w_1))}{(w_1 + h)r(x_1(w_1) - y(w_1))} \\ &\quad \left(\frac{dx_1^*(w_1)}{dw_1} + b\right), \end{aligned}$$

where

$$\frac{dx_1^*(w_1)}{dw_1} + b = \frac{1}{(w_1 + h)r(x_1^*(w_1) - y(w_1))} \geq 0.$$

Since $r'(\cdot)$ 0 and $r(\cdot) \geq 0$, we have

$$V''(w_1) \leq 0$$

and thus $V(w_1)$ is unimodal. In addition, we assume that the wholesale price should be greater than c. When $w_1 = c$, by equation (7), the optimal inventory level is

$$x_1^*(c) = a - bc + A,$$

and

$$\begin{aligned} V(c) &= 2b(c - w^0) - \Theta_1(A) \\ &= a - bc + \mu_1 - \mu_1 + A \\ &= a - bc + A > 0. \end{aligned}$$

The inequality follows from the nonnegativity assumption. Furthermore

$$V(\infty) = -\infty.$$

Hence $V(w_1)$ has only one root. Therefore, given that ε_1 has an increasing failure rate, the problem has a unique solution given by its first order conditions.

The increasing failure rate of the random variable ε_1 guarantees the uniqueness of the optimal inventory and pricing policies. This failure rate condition is very common, and a lot of distribution functions such as the normal and exponential distributions have increasing failure rates.

Since the retailers do not hold inventory, their profits are proportional to the expected sales. Recall that the retail price $p_i = w_i + c_m$ at retailer i. Let $\pi_{ri}(x_i)$ be the profit of the retailer i when the supplier keeps inventory level x_i for her. We have

$$\pi_{ri}(x_i) = \mathrm{E}\{c_m \min(x_i, D_i)\} \quad i = 1, 2.$$

4.2 Pooled inventory system

For the pooled inventory case, the supplier sets up one common inventory x_p and charges each retailer a common unit wholesale price w_p. He sets the inventory level and wholesale price to maximize his expected profit.

Let $\prod_p(x_p, w_p)$ represent the supplier's expected profit when the wholesale price is w_p and the common inventory level is chosen as x_p. We have

$$\prod_p(x_p, w_p) = w_p \min(x_p, D_p(w_p)) - h(x_p - D_p(w_p))^+ - cx_p.$$

Recall that $D_p(w_p) = 2y(w_p) + \varepsilon_p$ and ε_p is a random variable with mean μ_p and CDF $G_p(\cdot)$ and PDF $g_p(\cdot)$.

We define the expected excess stock $\Lambda_p(x)$ and the expected shortage $\Theta_p(x)$ when inventory level is chosen as x and random demand turns out to be ε_p as

$$\Lambda_p(x) = \int_{2A}^{x} (x-u) g_p(u) du,$$

and

$$\Theta_p(x) = \int_{x}^{2B} (u-x) g_p(u) du.$$

We can rewrite the expected profit as :

$$\begin{aligned}
&\prod_p(x_p, w_p) \\
&= \int_{2A}^{x_p - 2y(w_p)} \{w_p(2y(w_p)+u) - h(x_p - 2y(w_p) - u)\} g_p(u) du \\
&\quad + \int_{x_p-2y(w_p)}^{2B} w_p(x_p) g_p(u) du - cx_p \\
&= I_p(w_p) - L_p(x_p, w_p),
\end{aligned}$$

where

$$I_p(w_p) = (w_p - c)(2y(w_p) + v_p),$$

and

$$L_p(x_p, w_p) = (c+h)\Lambda_p(x_p - 2y(w_p)) - (w_p - c)\Theta(x_p - 2y(w_p)).$$

Consequently, the expected profit again can be interpreted as the riskless profit, $I_p(w_p)$, less the expected loss due to the uncertainty, $L_p(x_p, w_p)$.

Due to the nonnegativity of demands, the supplier maximizes his profit by solving following problem,

$$\max_{x_p, w_p \in [c, w_{\max}]} \prod_p(x_p, w_p)$$

Similar to Lemma, we have following lemma concerning the properties of $\prod_p(x_p, w_p)$.

Lemma 4.2:

1. For a given w_p, $\prod_p(x_p, w_p)$ *is concave in* x_p.

2. For a given w_p, *the optimal inventory level is determined by*

$$x_p^*(w_p) = 2y(w_p) + G_p^{-1}\left(\frac{w_p - c}{w_p + h}\right)$$

The proof is similar to the proof of the Lemma 4.1. So the details are omitted.

Similarly, we can get the following theorem on the uniqueness of the optimal solution of the problem.

Theorem 4.2: *Let* $r(z_p)$ *be the failure rate of the random variable* ε_p. *If* $r'(z_p) > 0$, *then* (x_p^*, w_p^*) *is the unique solution that satisfies*

$$\begin{cases} 4b(w_p - w^0) + \Theta_p(x_p - 2y(w_p)) = 0 \\ G_p(x_p - -2y(w_p)) = \dfrac{w_p - c}{w_p + h} \end{cases}$$

where

$$w^0 = \frac{2a + 2bc + \mu_p}{4b}.$$

Furthermore, the optimal wholesale price w_p^* *satisfies:*

$$4b(w^0 - w_p) - \Theta_p(x_p^*(w_p) - 2y(w_p)) = 0,$$

and the optimal solution of inventory level can be calculated by Lemma (4.2):

$$x_p^* = 2y(w_p^*) + G_p^{-1}\left(\frac{w_p^* - c}{w_p^* + h}\right)$$

The proof follows that of Theorem . We thus omit the details.

Since the retailers do not hold inventory, their profits are proportional to the expected sales. Recall that the retail price $p_p = w_p + c_m$. Let $\pi_p(x_p)$ be the profit of the retailers when the supplier keeps inventory level x_p. We have

$$\pi_p(x_p) = \mathrm{E}\{c_m \min(x_p, D_p)\}.$$

We have derived procedures to calculate the optimal inventory and pricing policies for the reserved and the pooled inventory scenarios. In the following section, we will get the results for the two scenarios when the random components of the demands ε_1 and ε_2 are normally distributed.

4.3 Comparative results

Recall that we defined the range of the random variable ε_i $(i = 1, 2)$ as $[A, B]$. A feasibility condition (in order to ensure nonnegative demands) for the wholesale price w is $y(w) + A = a - bw + A > 0$. Although the range of normal distribution is $[-\infty, +\infty]$, if a is large enough, then the value of A can approach $-\infty$. Hence, we can use normal distribution to approximate the demand distribution. As before, we use Φ to denote the cumulative distribution function, ϕ to denote the probability density function and R to denote the right-hand unit normal linear loss function.

First we introduce a simple result that we will use in this and following sections.

Lemma 4.3: *If* ε_1 *and* ε_2 *are independent, identically, and normally distributed random variables, let* $\Theta_1(\cdot) = \Theta_2(\cdot) = \Theta(\cdot)$

and $G_1(\cdot)=G_2(\cdot)=G(\cdot)$ *, then*
$\Theta_p(2u)\le 2\Theta(u)$ for any u.

Proof: Assume that ε_1 and ε_2 have same mean μ and standard deviation σ and define

$$\Delta\Theta(u)=\Theta_p(2u)-2\Theta(u).$$

A sufficient condition for $\Theta_p(2u)\le 2\Theta(u)$ is $\Delta\Theta(u)\le 0$. Calculating the first derivative of $\Delta\Theta(u)$ with respect to u, we get

$$\Delta\Theta'(u)=-2(1-G_p(2u))+2(1-G(u))$$
$$=\Phi(\sqrt{2}(\frac{u-\mu}{\sigma}))-\Phi(\frac{u-\mu}{\sigma}).$$

$\Phi(\cdot)$ is a nondecreasing function. Hence we have

$$\Delta\Theta'(u)=\begin{cases}\ge 0 & \text{if } u\ge\mu\\ <0 & \text{if } u<\mu\end{cases}$$

We will show that (u) 0. There are two cases.

case 1 : $u\ge\mu$

$\Delta\Theta'(u)\ge 0$ indicates that $\Delta\Theta(u)$ is nondecreasing in u. Hence A sufficient condition of $\Delta\Theta(u)\le 0$ is to show that $\Delta\Theta(+\infty)$ ≤ 0. However $\Theta(+\infty)=0,\Theta_p(+\infty)=0$, hence $\Delta\Theta(u)\le 0$ for any $u\ge\mu$.

case 2 : $u<\mu$

$\Delta\Theta'(u)<0$ indicates that $\Delta\Theta(u)$ is decreasing in u. A sufficient condition for $\Delta\Theta(u)\le 0$ is $\Delta\Theta(-\infty)=0$. We will use following result from Hadley and Whitin (1963),

$$\Theta(u)=\sigma\phi(\frac{u-\mu}{\sigma})-(u-\mu)(1-\Phi(\frac{u-\mu}{\sigma})).$$

Then

$$\Delta\Theta(u)=\sqrt{2}\sigma\phi(\sqrt{2}(\frac{u-\mu}{\sigma}))-2\sigma\phi(\frac{u-\mu}{\sigma})$$
$$+2(\upsilon-\mu)(\Phi(\sqrt{2}(\frac{u-\mu}{\sigma}))-\Phi(\frac{u-\mu}{\sigma})).$$

When u approaches $-\infty$, the first two terms go to 0 and $\Phi(\sqrt{2}(\frac{u-\mu}{\sigma}))-\Phi(\frac{u-\mu}{\sigma})$ converges to 0 with exponential speed. Hence the third term also converges to 0. Therefore $\Theta_p(2u)\le 2\Theta(u)$ is obtained for any $u<\mu$.

For the case in which ε_1 and ε_2 are independent identically and normally distributed random variables, we can provide a detailed comparison of the reserved and pooled inventory cases.

Theorem 4.3: *If* ε_1 *and* ε_2 *are independent identically and normally distributed variables, then the supplier will charge the retailers an identical wholesale price in the reserved inventory case, which is smaller than the optimal wholesale price in the pooled inventory case, i.e.,* $w_1^*=w_2^*\le w_p^*$.

Proof: By Theorem and Theorem, the optimal wholesale prices w_p^*,w_1^* and w_2^* satisfy following equations,

$$V_i(w_i)=4b(w^0-w_i)-2\Theta_i(G_i^{-1}(\frac{w_i-c}{w_i+h}))=0, i=1,2,$$

$$V_p(w_p)=4b(w^0-w_p)-\Theta_p(G_p^{-1}(\frac{w_p-c}{w_p+h}))=0,$$

and V_i,V_p are unimodal functions. Here $G_1(\cdot)=G_2(\cdot)$ guarantees $w_1^*=w_2^*$. Furthermore, the normal distribution has an increasing failure rate, which satisfies the conditions of the Theorem and . Hence V_i and V_p have unique solutions. Assume that ε_i (i = 1, 2) has the mean value and standard deviation . A sufficient condition for $w_p^*\ge w_i^*$ is $V_p(w)>V_i(w)$ for any w, which is equivalent to the following:

$$V_i(w)-V_p(w)\le 0,$$
$$\Theta_p(G_p^{-1}(\frac{w-c}{w+h}))-2\Theta_i(G_i^{-1}(\frac{w-c}{w+h}))\le 0,$$
$$\Theta_p(2\mu+\sqrt{2}\sigma\Phi^{-1}(\frac{w-c}{w+h}))-2\Theta_i(\mu+\sigma\Phi^{-1}(\frac{w-c}{w+h}))\le 0.$$

Define

$$\Delta V_i(Z_\alpha)=\Theta_p(2\mu+\sqrt{2}\sigma Z_\alpha)-2\Theta_i(\mu+\sigma Z_\alpha)$$

where $Z_\alpha=\Phi^{-1}(\frac{w-c}{w+h})$.

Next we show that $\Delta V_i(Z_\alpha)$ is a monotone nondecreasing function of Z_a. Taking the first derivative of $\Delta V_i(Z_\alpha)$ with respect to Z_a, we obtain

$$\Delta V_i'(Z_\alpha)$$
$$=-\sqrt{2}\sigma(1-G_p(2\mu+\sqrt{2}\sigma Z_\alpha))+2\sigma(1-G_i(\mu+\sigma Z_\alpha))$$
$$=(2-\sqrt{2})\sigma(1-\Phi(Z_\alpha))\ge 0.$$

Hence, a sufficient condition for $\Delta V_i(Z_\alpha)\le 0$ for any Z_a is that $\Delta V_i(+\infty)\le 0$. Here

$$\Delta V_i(Z_\alpha)\Big|_{Z_\alpha\longrightarrow+\infty}=0.$$

Hence $\Delta V_i(Z_\alpha)<0$. Therefore, the wholesale price in the pooled inventory case is greater than that in the reserved inventory case.

Theorem 4.4: *If* ε_1 *and* ε_2 *are independent identically and normally distributed variables, then the optimal profit of the supplier in the pooled inventory case is at least as large as that in the reserved inventory case, i.e.,* $\prod_p^* \ge \prod_r^*$.

Proof: If [epsilon]$_1$ and [epsilon]$_2$ are identically independently and normally distributed variables, $\prod_{r1}$ is same as $\prod_{r2}$ and hence they have same optimal solution and objective value. If we can show that $\prod_p(2x_1, w_1) \ge 2\prod_{r1}(x_1, w_1)$ for any (x_1, w_1), then $\prod_p^* \ge \prod_p(2x_1^*, w_1^*) \ge 2\prod_{r1}(x_1^*, w_1^*) = \prod_r^*$. Note that

$$\begin{aligned}&\prod_r(x_1, w_1) - \prod_p(2x_1, w_1)\\&= -2(c+h)\Lambda(c_1 - y(w_1)) - 2(w_1 - c)\Theta_1(x_1 - y(w_1))\\&\quad + (c+h)\Lambda_p(2x_1 - 2y(w_1)) + (w_1 - c)\Theta_p(2x_1 - 2y(w_1))\\&= (h + w_1)\left\{\Theta_p(2x_1 - 2y(w_1)) - 2\Theta_1(x_1 - y(w_1))\right\}.\end{aligned}$$

Applying Lemma 4.3, we have

$$\Theta_p(2x_1 - 2y(w_1)) - 2\Theta_i(x_1 - y(w_1)) \le 0.$$

Since $h+w_1> 0$, this implies that $\prod_r(x_1, w_1) - \prod_p(2x_1, w_1)$ $\ge 0$for any (x_1, w_1).

Theorem 4.5: *If* ε_1 *and* ε_2 *are independent identically and normally distributed variables and the supplier charges the retailers the identical wholesale price w in the reserved inventory case and the pooled inventory case, i.e.,* $w_1 = w_2 = w_p = w$, *then the retailers' total expected profit in the pooled inventory case is at least as large as that in the reserved inventory case, i.e.,* $\pi_{r1}(x_1) + \pi_{r2}(x_2)$ $\le \pi_p(x_p)$.

Proof: As before, given the inventory levels x_1, x_2 and x_p, we have

$\pi_{ri}(x_i) = \mathrm{E}\{c_m \min(x_i, D_i)\}$ for $i = 1, 2$,

$$\pi_p(x_p) = \mathrm{E}\{c_m \min(x_p, D_p)\}.$$

Under the reserved inventory case, given the same wholesale price w, the optimal inventory levels are same, i.e., $x_1 = x_2$. Furthermore,

$$\pi_{r1}(x_1) + \pi_{r2}(x_2) = c_m(2y(w) + 2\mu - 2\Theta_1(G_1^{-1}(\frac{w-c}{w+h}))),$$

$$\pi_p(x_p)) = c_m(2y(w) + 2\mu - \Theta_p(G_p^{-1}(\frac{w-c}{w+h}))).$$

By the proof of Theorem , we know that

$$2\Theta_1(G_1^{-1}(\frac{w-c}{w+h})) \ge \Theta_p(G_p^{-1}(\frac{w-c}{w+h})),$$

therefore, $\pi_{r1}(x_1) + \pi_{r2}(x_2) \le \pi_p(x_p)$.

If the demands are normally distributed, these results indicate that the supplier will always prefer to pool inventory. Under the pooled inventory policy, the supplier's inventory level will decrease because the variance of the total demands is decreased after pooling the inventory. If the wholesale prices are same under these two policies, the retailers will also prefer the pooled inventory policy.

5. ADDITIVE DEMAND MODEL: WITH SERVICE LEVEL REQUIREMENTS

Now we consider scenarios in which the retailers impose minimum service level requirements on the supplier. Because of the structure of the assumed demand function, the supplier may use the wholesale price to control the demands so as to meet service level requirements. We keep the same notation as in previous section. Let ρ_1 and ρ_2 be the retailers' service level requirements in the reserved inventory system and ρ_p be the joint service level requirement in the pooled inventory system.

5.1 Reserved inventory system

For the reserved inventory scenario, the supplier needs to solve following problem,

$$\begin{aligned}&\max_{x_1, x_2, w_1 \in [c, w_{\max}], w_2 \in [c, w_{\max}]} \prod_r(x_1, x_2, w_1, w_2)\\&s.t.\ \ P(D_1(w_1) \le x_1) \ge \rho_1\\&\qquad P(D_2(w_2) \le x_2) \ge \rho_2\end{aligned}$$

where

$\prod_r(x_1, x_2, w_1, w_2) = \prod_{r1}(x_1, w_1) + \prod_{r2}(x_2, w_2)$,

and

$$\begin{aligned}&\prod_{r1}(x_1, w_1)\\&= \mathrm{E}\{w_1 \min(x_1, D_1(w_1)) - h(x_1 - D_1(w_1))^+ - cx_1\},\\&\prod_{r2}(x_2, w_2)\\&= \mathrm{E}\{w_2 \min(x_2, D_2(w_2)) - h(x_2 - D_2(w_2))^+ - cx_2\}.\end{aligned} \tag{8}$$

Similar to problem without service level requirements, this problem is separable, so that the supplier needs to solve two problems with same structure, one each for retailers 1 and 2. Again, we analyze the problem for retailer 1. For retailer 1, the objective of the supplier is to solve following problem,

$$\begin{aligned}&\max_{x_1, w_1 \in [c, w_{\max}]} \prod_{r1}(x_1, w_1)\\&s.t.\ \ P(D_1(w_1) < x_1) \ge \rho_1.\end{aligned} \tag{9}$$

The method used in previous section does not work well for the problem with service level requirements. We introduce another variable, which we call the "stocking factor" defined as:

$z_1 = x_1 - y(w_1).$

Substituting for x_1 in equation (9), the problem of choosing a price w_1 and a inventory level x_1 is equivalent to choosing a price w_1 and a stocking factor z_1. The expected profit becomes:

$$\begin{aligned}
&\Pi_{r1}(z_1, w_1)\\
&= \int_A^{z_1} \{w_1(y(w_1)+u) - h(z_1 - u)\} g_1(u)du\\
&\quad + \int_{z_1}^{B} \{w_1(y(w_1)+z_1)\} g_1(u)du - c(y(w_1)+z_1)\\
&= (w_1 - c)(y(w_1)+\mu_1) - (c+h)L_1(z_1) - (w_1-c)\Theta_1(z_1),
\end{aligned}$$

and the service level constraint becomes:

$$G(z_1) \ge \rho_1.$$

Hence, the supplier's optimization problem is equivalent to the following problem,

$$\begin{aligned}
&\max_{z_1, w_1 \in [c, w_{\max}]} \Pi_{r1}(z_1, w_1)\\
&s.t.\ \ z_1 = x_1 - y(w_1)\\
&\qquad G_1(z_1) \ge \rho_1.
\end{aligned} \tag{10}$$

Considering the optimization problem without the first constraint, we have

$$\begin{aligned}
&\max_{z_1, w_1 \in [c, w_{\max}]} \Pi_{r1}(z_1, w_1)\\
&s.t.\ G_1(z_1) \ge \rho_1.
\end{aligned} \tag{11}$$

Given the optimal solution (z_1^*, w_1^*) for the problem (11), define the optimal inventory level as

$$x_1^* = z_1^* + y(w_1^*).$$

In addition, (z_1^*, w_1^*, x_1^*) is the optimal solution for the problem (10). Hence these two problems are equivalent, i.e., the supplier only needs to solve problem (11).

We will analyze the problem (11) without service level constraints first. Based on the optimal solution of the problem without constraints, then we derive the optimal solution with service level constraints.

The following lemma gives some properties of the problem without constraints.

Lemma 5.1:

1. For a given z_1, $\Pi_{r1}(z_1, w_1)$ *is concave in* w_1.

2. For a given w_1, $\Pi_{r1}(z_1, w_1)$ *is concave in* z_1.

3. For a given z_1, *the optimal price is determined by*

$$w_1^*(z_1) = w^0 - \frac{\Theta_1(z_1)}{2b},$$

where $w^0 = \dfrac{a+bc+\mu_1}{2b}$.

4. For a given w_1, *the optimal stocking factor is determined by*

$$z_1^*(w_1) = G_1^{-1}\left(\frac{w_1 - c}{w_1 + h}\right).$$

Proof: Consider the first and second partial derivatives of $\Pi_{r1}(z_1, w_1)$ taken with respected to z_1 and w_1:

$$\frac{\partial \Pi_{r1}(z_1, w_1)}{\partial z_1} = (-c-h) + (w_1+h)\int_{z_1}^{B} g_1(u)du,$$

$$\frac{\partial^2 \Pi_{r1}(z_1, w_1)}{\partial^2 z_1} = (w_1+h)g_1(z_1) < 0,$$

$$\begin{aligned}
&\frac{\partial \Pi_{r1}(z_1, w_1)}{\partial w_1}\\
&= y(w_1) + \mu_1 + (w_1 - c)\frac{dy(w_1)}{dw_1} - \Theta_1(z_1)\\
&= 2b(w^0 - w_1) - \Theta_1(z_1),
\end{aligned}$$

$$\frac{\partial^2 \Pi_{r1}(z_1, w_1)}{\partial^2 w_1} - 2b < 0,$$

where $w^0 = \dfrac{a+bc+\mu_1}{2b}$. Parts (1) and (2) follows from the negativity of the second derivatives. Parts (3) and (4) follow from $\dfrac{\partial \Pi_{r1}(z_1, w_1)}{\partial w_1} = 0$ and $\dfrac{\partial \Pi_{r1}(z_1, w_1)}{\partial z_1} = 0$.

Substituting $w_1^* = w_1(z_1)$ into $\Pi_{r1}(z_1, w_1)$, we have

$$\begin{aligned}
\Pi_{r1}(z_1, w_1^*(z_1)) &= (w_1^* - c)(y(w_1^*) + \mu_1) - (c+h)\Lambda_1(x_1)\\
&\quad - (w_1^*(x_1) - c)\Theta_1(z_1),
\end{aligned}$$

and we can translate the optimization problem to a maximization problem over a single variable z_1. The optimal inventory and pricing policies for the reserved inventory case are to hold $x_1^* = y(w_1^*(z_1^*)) + z_1^*$ units to sell at the unit price w_1^*, where w_1^* is determined by Lemma . The following theorem addresses how to determine the optimal stocking factor z_1^*.

Theorem 5.1: *Let* $r(z_1)$ *be the failure rate of the random variable* ε_1. *If* $2r(z_1)^2 + r'(z_1) > 0$, *then* z_1^* *is the unique optimal solution within the region [A,B] that satisfies*

$$-(c+h) + \left(\frac{a+bc+\mu_1}{2b} + h - \frac{\Theta_1(z_1)}{2b}\right)(1 - G_1(z_1)) = 0.$$

Proof: Applying the chain rule

$$\begin{aligned}
&\frac{d\Pi_{r1}(z_1, w_1^*(z_1))}{dz_1}\\
&= \frac{\partial \Pi_{r1}(z_1, w_1^*(z_1))}{\partial z_1} + \frac{\partial \Pi_{r1}(z_1, w_1^*(z_1))}{\partial \omega_1}\frac{dw_1^*(z_1))}{\partial z_1}\\
&= -(c+h) + \left(\frac{a+bc+\mu_1}{2b} + h - \frac{\Theta_1(z_1)}{2b}\right)(1 - G_1(z_1)).
\end{aligned}$$

Let
$V(z_1) = -(c+h) + (\frac{a+bc+\mu_1}{2b} + h - \frac{\Theta_1(z_1)}{2b})(1-G_1(z_1)).$

The optimal z_1^* should satisfy $V(z_1)=0$. Now, consider the first and second derivatives of $V(z_1)$ with respect to z_1. We have

$$\frac{dV(z_1)}{dz_1} = \frac{d}{dz_1}(\frac{d\prod_{r1}(z_1, w_1^*(z_1))}{dz_1}) = -\frac{g_1(z_1)}{2b}\left(2b(\frac{a+bc+\mu_1}{2b}+h) - \Theta_1(z_1) - \frac{1-G_1(z_1)}{r(z_1)}\right),$$

and

$$\frac{d^2V(z_1)}{dz_1^2} = \frac{dV(z_1)/dz_1}{g_1(z_1)} g_1'(z_1) - \frac{g_1(z_z)}{2b}\left\{2[1-G_1(z_1)]\right\} + \frac{[1-G_1(z_1)]r'(z_1)}{r(z_1)^2}.$$

If z_1 is a local minimum or maximum, then it always satisfies $\frac{dV(z_1)}{dz_1}=0$ and its second derivative is

$$\frac{d^2V(z_1)}{dz_1^2} = -\frac{g_1(z_1)[1-G_1(z_1)]}{2br(z_1)^2}\left\{2r(z_1)^2 + r'(z_1)\right\}.$$

If the distribution function satisfies $2r(z_1)^2 + r'(z_1) > 0$, then $\frac{d^2V(z_1)}{dz_1^2} < 0$ at $V'(z_1)=0$, which implies that $V(z_1)$ is a quasi-concave function. Hence, there are at most two roots for $V(z_1)=0$. Furthermore, $V(B)$ = $-(c+h)$ < 0. $G(A)$ > 0 plus $G(B)$ < 0 guarantees the uniqueness of z_1^* and $A < z_1^* < B$. A sufficient condition for $G(A) > 0$ is :

$$2bV(A) = -2b(c+h) + ((\frac{a+bc+\mu_1}{2b} + 2bh - (\mu_1 - A)) = a - bc + A > 0.$$

Hence, there is unique optimal solution z_1^*.

The condition $2r(z_1)^2 + r'(z-1) > 0$ guarantees that $\prod_{r1}(z_1, w_1^*(z_1))$ is a quasi-concave in z_1. A sufficient condition is that $r'(z)>0$ which implies that distribution has increasing failure rate.

Now, we have optimal solution z_1^* for the unconstrained problem. Returning to the original problem (12) with service level constraints, since $G_1(\cdot)$ is a nondecreasing function and $\prod_{r1}(z_1, w_1^*(z_1))$ is a quasi-concave function on range $[c, w_{\max}]$, the optimal stocking factor for the problem with service level constraints is $\max(z_1^*, G_1^{-1}(r_1))$.

Since the retailers do not hold inventory, their profits are proportional to the expected sales. Recall that the retail price $p_i = w_i + c_m$ at retailer *i*. Let $\pi_{ri}(x_i)$ be the profit of the retailer *i* when the supplier keeps inventory level x_i for her. We have

$$\pi_{ri}(x_i) = \mathrm{E}\{c_m \min(x_i, D_i)\} \quad i = 1, 2.$$

5.2 Pooled inventory system

For the pooled inventory case, the supplier will set up one common inventory x_{p} and charge each retailer a common unit wholesale price w_p. We face the problem,

$$\max_{x_p, w_p \in [c, w_{\max}]} \prod_p(x_p, w_p)$$
$$s.t. \quad P(D_p \le x_p) \ge \rho_p$$

where

$$\begin{aligned}\prod_p(x_p, w_p) &= \int_{2\mathrm{A}}^{x_p - 2y(w_p)} \{w_p(2y(w_p)+u) - h(x_p - 2y(w_p) - u)\} G_p(u)du \\ &\quad + \int_{x_p - 2y(w_p)}^{2B} \left\{w_p\left(x_p\right)\right\} G_p(u)du - cx_p \\ &= I_p(w_p) - L_p(x_p, w_p),\end{aligned}$$

$$I_p(w_p) = (w_p - c)(2y(w_p) + \mu_p),$$

and

$$L_p(x_p, w_p) = (c+h)\Lambda_p(x_p - 2y(w_p)) - (w_p - c)\,\Theta_p(x_p - 2y(w_p)).$$

For this case, define the stocking factor as $z_p = x_p - 2y(w_p)$ and let

$$\Lambda_p(x_p) = \int_{2\mathrm{A}}^{x_p} (x_p - u) g_p(u) du,$$

$$\text{and } \Theta_p(x_p) = \int_{x_p}^{2\mathrm{B}} (u - x_p) g_p(u) du.$$

The supplier's expected profit can then be written as :

$$\begin{aligned}&\prod_p(x_p, w_p) \\ &= \int_{2\mathrm{A}}^{x_p} \left\{w_p\left(2y(w_p)+u\right) - h(x_p - u)\right\} g_p(u)du \\ &\quad + \int_{z_p}^{2B} \left\{w_p\left(2y(w_p)+z_p\right)\right\} g_p(u)du - c\left(2y(w_p)+z_p\right) \\ &= (w_p - c)(2y(w_p)+\mu_p) - (c+h)\Lambda_p(x_p) - (w_p - c)\Theta_p(x_p).\end{aligned}$$

The supplier's objective is to maximize his expected profit, i.e.,

$$\max_{z_p, w_p \in [c, w_{\max}]} \prod_p(z_p, w_p)$$
$$s.t. \quad P(D_p \le x_p) \ge \rho_p$$

Following arguments similar to those used in the previous section, it is straightforward to verify that $\prod_p(z_p, w_p)$ is concave in z_p when w_p is given and is concave in w_p when x_p is given. As a consequence, the first-order conditions uniquely determine w_p^* at any x_p. Specifically,

Lemma 5.2:

1. *For a given z_p, $\prod_p(z_p, w_p)$ is concave in w_p.*
2. *For a given w_p, $\prod_p(z_p, w_p)$ is concave in z_p.*
3. *For a given z_p, the optimal price is determined by*

$$w_p^*(z_p) = w^0 - \frac{\Theta_p(z_p)}{4b},$$

where $w^0 = \dfrac{2a + 2bc + \mu_p}{4b}$.

4. *For a given w_p, the optimal stocking factor is determined by*

$$z_p^*(w_p) = G_p^{-1}\left(\frac{w_p - c}{w_p + h}\right).$$

We also have the following sufficient condition for uniqueness of the supplier's optimal solution z_p^*.

Theorem 5.2: *Let $r_p(u)$ be the failure rate of the random variable ε_p. If $2r_p(u)^2 + r_p'(u) > 0$, then z_p^* is unique optimal solution within the region [2A,2B] that satisfies*

$$-(c+h) + \left(\frac{2a + 2bc + \mu_p}{4b} + h - \frac{\Theta_p(z_p)}{4b}\right)(1 - G_p(z_p)) = 0$$

Since the proof of the theorem is similar as the proof of Theorem , we will not give the details here.

If the random variable ε_p has increasing failure rate, we now have a procedure to get the optimal stocking factor z_p^* for the problem without service level requirements. Turning back to the problem with service level requirement, G_p is nondecreasing function and $\prod_p(z_p, w_p^*(z_p))$ is a quasi-concave function on range $[c, w_{\max}]$. Hence the optimal stocking factor for the pooled inventory problem with service level requirements is $\max(z_p^*, G_p^{-1}(\rho_p))$.

Since the retailers do not hold inventory, their profits are proportional to the expected sales. Recall that the retail price $p_p = w_p + c_m$ at retailer. Let $\pi_p(x_p)$ be the profit of the retailers when the supplier keeps inventory level x_p for them. We have

$$\pi_p(x_p) = \mathrm{E}\{c_m \min(x_p, D_p)\}.$$

5.3 Comparative results

For normally distributed demands, again we can provide a detailed comparison of the reserved and pooled inventory cases. We again use Φ to denote the cumulative distribution function and ϕ to denote the probability density function of the standard normal distribution.

In order to be able to compare the results, we assume that the service level requirement of each retailer under the reserved inventory case and the joint service level requirement in the pooled inventory case are identical, i.e. $\rho_1 = \rho_2 = \rho_\pi = \rho$.

Lemma 5.3: *Assume that the service level requirement of each retailer under the reserved inventory case and the joint service level requirement in the pooled inventory case are same, i.e. $\rho_1 = \rho_2 = \rho_\pi = \rho$. For both the reserved (i=1,2) and the pooled (i=p) inventory systems, the optimal stocking factor $z_i^*(w_i)$ is nondecreasing in w_i.*

Proof: We know that given the wholesale price w_i, the optimal stocking factor $z_i^*(w_i)$ can be written as:

$$z_i^*(w_i) = G_i^{-1}\left(\max\left(\frac{w_i - c}{h + w_i}, \rho\right)\right)$$

where $G_i^{-1}(\cdot)$ is nondecreasing and it is easy to show that $\dfrac{w_i - c}{h + w_i}$ is nondecreasing in w_i. Thus $G_i^{-1}\left(\max\left(\dfrac{w_i - c}{h + w_i}, \rho\right)\right)$ is nondecreasing in w_i, i.e., z_i^* is nondecreasing in w_i.

Theorem 5.3: *Assume that ε_1 and ε_2 are independent identically and normally distributed random variables and that the service level requirement of each retailer under the reserved inventory case and the joint service level requirement in the pooled inventory case are same, i.e. $\rho_1 = \rho_2 = \rho_\pi = \rho$. In addition, the supplier charges the retailers the same wholesale price w in both the reserved inventory case and the pooled inventory case,i.e., $w_1 = w_2 = w_p = w$. If $\max\left(\dfrac{w - c}{w + h}, \rho\right) \le 0.5$, then the sum of the optimal stocking factors for retailers 1 and 2 in reserved inventory case is at least as large as the optimal stocking factor in the pooled inventory case, i.e., $z_p^*(w) \le z_1^*(w) + z_2^*(w)$. Otherwise, $z_p^*(w) > z_1^*(w) + z_2^*(w)$.*

Proof: Given the wholesale price w, the optimal stocking factor z_i^* can be written as :

$$z_i^* = G_i^{-1}\left(\max\left(\frac{w - c}{w + h}, \rho\right)\right) \text{ for } i = 1, 2, p.$$

Assume ε_1 and ε_2 are independent identically and normally distributed random variables with mean and standard deviation , then,

$$z_i^*(w) = \mu + \sigma\Phi^{-1}\left(\max\left(\frac{w - c}{w + h}, \rho\right)\right) \text{ for } i = 1, 2$$

$$z_p^*(w) = 2\mu + \sqrt{2}\sigma\Phi^{-1}\left(\max\left(\frac{w - c}{w + h}, \rho\right)\right).$$

The difference between $z_1^*(w) + z_2^*(w)$ and $z_p^*(w)$ is

$$z_1^*(\omega) + z_2^*(\omega) - z_p^*(\omega) = (2 - \sqrt{2})\sigma\Phi^{-1}\left(\max\left(\frac{w - c}{w + h}, \rho\right)\right).$$

IJOR Vol. 2, No. 1, 59−76 (2005)

When $\max(\frac{w-c}{w+h}, \rho) \le 0.5$, then $\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)) \le 0$. Hence, $z_1^*(w)+z_2^*(w) \ge z_p^*(w)$. When $\max(\frac{w-c}{w+h}, \rho) > 0.5$, then $\Phi^{-1}(\max(\frac{w-c}{w+h}, \rho)) > 0$, so that $z_1^*(w) + z_2^*(w) < z_p^*(w)$.

Theorem 5.4: *Assume that ε_1 and ε_2 are independent identically and normally distributed random variables and that the service level requirement of each retailer under the reserved inventory case and the joint service level requirement in the pooled inventory case are same, i.e. $\rho_1 = \rho_2 = \rho_\pi = \rho$. Then the optimal profit of the supplier in the pooled inventory case is at least as large as that in the reserved inventory case, i.e., $\prod_p^* \ge \prod_r^*$.*

Proof: Assume ε_1 and ε_2 are independent identically and normally distributed random variables with mean value and standard deviation. Then $\prod_{r1}$ is same as $\prod_{r2}$ and hence they have same optimal solution and objective value. If we can show that $\prod_p(z_p^*(w_1),w_1) \ge 2\prod_{r1}(z_1^*(w_1),w_1)$ for any $(z_1^*(w_1),w_1)$, then

$$\prod_p^* \ge \prod_p(z_p^*(w_1^*),w_1^*) . \ge \ 2\prod_{r1}(z_1^*(w_1^*),w_1^*) = \prod_r^*$$

Note that

$$z_i^*(w_1) = \mu + \sigma\Phi^{-1}(\max(\frac{w_1-c}{w_1+h},\rho)) \quad \text{for } i=1,2$$

$$z_p^*(w_1) = 2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w_1-c}{w_1+h},\rho)).$$

Hence

$$\begin{aligned}
&\prod_r(z_1^*(w_1),w_1) - \prod_p(z_p^*(w_1),w_1) \\
&= -2(c+h)\Lambda_1(z_1^*(w_1)) - 2(w_1-c)\Theta_1(z_1^*(w_1)) \\
&\quad + (c+h)\Lambda_p(z_p^*(w_1)) + (w_1-c)\Theta_p(z_p^*(w_1)) \\
&= (h+w_1)\left\{\Theta_p(z_p^*(w_1)) - 2\Theta_1(z_1^*(w_1))\right\}.
\end{aligned}$$

From proof of Theorem , we know that

$$2\Theta_1(\mu+\sigma\Phi^{-1}(\max(\frac{w_1-c}{w_1+h},\rho)))$$
$$\ge \Theta_p(2\mu+\sqrt{2}\sigma\Phi^{-1}(\max(\frac{w_1-c}{w_1+h},\rho))).$$

In addition, $w_1+h>0$. Thus $\prod_r(z_1^*(w_1),w_1) - \prod_p(z_p^*(w_1),w_1) \le 0$ for any w_1.

Theorem 5.5: *Assume that ε_1 and ε_2 are independent identically and normally distributed variables, that the supplier charges the retailers the identical wholesale price w, i.e., $w_1 = w_2 = w_p = w$, and that the service level requirement of each retailer under the reserved inventory case and the joint service level requirement in the pooled inventory case are same, i.e. $\rho_1 = \rho_2 = \rho_\pi = \rho$. Then the retailers' total expected profit in the pooled inventory case is at least as large as that in the reserved inventory case, i.e., $\pi_{r1}(x_1)+\pi_{r2}(x_2) \le \pi_p(x_p)$.*

Proof: Given the inventory levels x_1, x_2 and x_p in the reserved and the pooled inventory cases, the expected profits of retailers are

$$\pi_{ri}(x_i) = \mathrm{E}\{c_m \min(x_i, D_i)\} \quad \text{for } i=1,2$$
$$\pi_p(x_p) = \mathrm{E}\{c_m \min(x_p, D_p)\}.$$

Given the identical wholesale price $w_1 = w_2 = w_p = w$, the optimal stocking factors are

$$z_1^*(w_1) = z_2^*(w_2) = \mu + \sigma\Phi^{-1}(\max(\frac{w-c}{w+h},\rho)),$$

and

$$z_p^*(w) = 2\mu + \sqrt{2}\sigma\Phi^{-1}(\max(\frac{w-c}{w+h},\rho)).$$

Furthermore, optimal inventory levels are $x_i = z_i^*(w) + y(w), i=(1,2)$ and $x_p = z_p^* + 2y(w)$. Hence

$$\pi_{r1}(x_1)+\pi_{r2}(x_2) = c_m(y(w)(2\mu - 2\Theta_1(z_1^*(w))))$$
$$\pi_p(x_p) = c_m(y(w)(2\mu - \Theta_p(z_p^*(w)))).$$

From proof of Theorem , we know that

$$2\Theta_1(\mu+\sigma\Phi^{-1}(\max(\frac{w-c}{w+h},\rho)))$$
$$\ge \Theta_p(2\mu+\sqrt{2}\sigma\Phi^{-1}(\max(\frac{w-c}{w+h},\rho))).$$

Therefore, $\pi_{r1}(x_1)+\pi_{r2}(x_2) \le \pi_p(x_p)$.

If the demands are normally distributed, the supplier will always prefer the pooled inventory policy. Under the pooled inventory policy, the supplier's profit will increase. If the wholesale prices are same under those two policies, the retailers will also prefer to the pooled inventory policy.

6. CONCLUSIONS

We have studied a two-echelon supply chain with one supplier and two retailers over one period, or selling season. The supplier bears the supply chain's inventory risk because only the supplier holds inventory while the retailer replenishes as needed during the season.

First we considered scenarios in which the wholesale price is fixed and we studied and compared the supplier's inventory decision under two policies: reserved inventory and pooled inventory. Under the first policy, the inventory is stocked in separate locations for each retailer while under the latter policy, inventory is centrally stocked by the supplier, and hence the supply chain may benefit from risk pooling. However, in general, whether the profit of the retailers and supplier increases or decreases upon inventory pooling depends on the problem parameters such as the demand parameters.

In order to obtain insights into the impact of the reserved inventory and the pooled inventory policies, we compared the profits of the supplier and retailers under those two policies assuming normal demand distributions. First we showed that the supplier's profit in the pooled inventory case is always greater than in the reserved inventory case. In addition, the total expected sales is also increased after pooling the inventory. In addition to the basic model, we also studied the case when the retailers have service level requirements, and we obtained similar conclusions.

Besides the model with fixed wholesale prices, we also developed inventory and pricing models for the supplier when the wholesale price is a decision and demand is price-sensitive. Note that these scenarios are much more complicated. We analyzed an additive demand model. For normally distributed demands, we compared the results for the reserved inventory and the pooled inventory policies with and without service level requirements. The results shows that the supplier always prefers to pool inventory while in general, this need not be so for the retailers. We also analyzed the retailers' profits under some specific conditions.

REFERENCES

1. Clarke, F. and Ekeland, I. (1982). Nonlinear oscillations and boundary-value problems for Hamiltonian systems. *Arch. Rat. Mech. Anal.*, 78: 315–333.
2. Anupindi, R. and Bassok, Y. (1999a). Centralization of stocks: retailers vs manufacturer. *Management Science*, 45: 178-191.
3. Anupindi, R., Bassok, Y., and Zemel, E. (1999b). Study of decentralized distribution sysem: part I-A General Framwork. Working paper, Kellogg Graduate School of Management, Nothwestern univeristy. *Management Science*, 45: 178-191.
4. Anupindi, R., Bassok, Y., and Zemel, E. (2001). A general framework for the study of decentralized distribution systems. *Manufacturing and service operations management*, 3(4): 349-368.
5. Bernstein, F. and Federgruen, A. (2001). *Decentrazlied Supply Chains with Competing Retailers under Demand Uncertainty*. Working paper, Duke University.
6. Bernstein, F., Chen, F., and Federgruen, A. (2002). *Vendor Managed Inventory and Supply Chain Coordination: the Case with One Supplier and Competing Retailers*. Working paper, Duke University.
7. Baker, K.R., Magazine, M.J., and Nuttle, H.L.W. (1986). The effect of the commonality of safety stock in a simple inventory model. *Management Science*, 32(8): 982-988.
8. Barthodli, J.J. and Kemahioglu, E. (2003). *Using Shapley Value to Allocate Savings in a Supply Chain*. Working paper, School of Industrial and Systems Engineering, Georgia Institute of Technology.
9. Bernstien, F. and Federgruen, A. (2001). *Pricing and Replenishment Strategies in a Distribution System with Competing Retailers*. Working paper, Fuquan School of Business, Duke University.
10. Bitran, G.R. and Mondschein, S.V. (1997). Periodic pricing of seasonal products in retailing. *Management Science*, 43: 64-79.
11. Cachon, G.P. and Lariviere, M.A. (1999). Capacity choice and allocation: strategic behavior and supply chain performance. *Management Science*, 45(8): 101-1108.
12. Cachon, G.P. (2002). *Supply Chain Coordination with Contracts*. Working paper, The Wharton Scholl of Business, University of Pennsyvania.
13. Cachon, G.P. (2004). The allocation of inventory risk in a supply chain: push, pull and advanced purchase discount contracts. *Management Science*, 50(2): 222-238.
14. Chen, F., Federgruen, A., and Zheng, Y.S. (2004). Coordination mechanism for a distribution system with one supplier and multiple retailers. *Management Science*, 47(5): 693-708.
15. Dong, L. and Rudi, N. (2001). *Supply Chain Interaction under Transshipment*. Working paper, university of Rochester.
16. Edgeworh, F. (1888). The mathematical theory of banking. *Journal of the Royal Statistical Society*, 51: 113-127.
17. Gallego, G. and Van Ryzin, G. (1994). Optimal Dynamic Pricing of inventory with stochastic over finite horizon. *Management Science*, 40: 999-1020.
18. Gerchak, Y. and Henig, M. (1986). An inventory model with component commonality. Operations Research Letters, 5(3): 157-160.
19. Gerchak, Y., Magazine, M.J., and Gamble, A.B. (1988). Component commonality with serivce requirements. *Management Science*, 34(6): 753-760.
20. Gerchak, Y. and Mossman, D. (1992). On the effect of demand randomness on inventories and costs. Operations research, 40(4): 804-807.
21. Grnot, D. and Sosic, G. (2002). A three-stage model for a decentralized dsitribution system of retailers. *Operations research*, 51(5): 771-784.
22. Hadley, G. and Whitin, T.M. (1963). *Analysis of Inventory Systems*. Prentice-Hall.
23. Mills, E.S. (1959). Uncertainty and price theory. *Quarterly Journal of Economics*, 73: 116-130.
24. Netessine, S. and Rudi, N. (2003). *Supply Chain Choice on the Internet*. Working paper, University of Rochester.
25. Pasternack, B.A. and Drezner, Z. (1991). Optimal inventory policies for substitutable commodities with stochastic demand. *Naval Research Logistics*, 38: 221-240.
26. Petruzzi, N. and Data, M. (1999). Pricing and the newsvendor problem: a review with extensions. *Operations Research*, 47(2): 183-194.
27. Rudi, N., Kapur, S. and Pyke, D. (2001). A two-location inventory model with transshipment and local decision making. *Management science*, 47: 1668-1680.
28. Serfert, R.W. and Thonemann, U.W. (2001). *Relaxing Channel Separating-Integrating a Virtual Store into the Supply Chain*. Working paper, IMD, Switzerland.
29. Serfert, R.W., Thonemann, U.W., and Rockhold, S.A. (2001). *Integrating Direct and Indirect Sales Channels under Decentralized Decision Making*. Working paper, IMD, Switzerland.
30. Portues, E.L. (1990). Stochastic Inventory Theory. Handbooks in OR &MS Eds; D.P. Heyman and M.J. Sobel, 2, 605-652.
31. Wang, Y., Jiang, L., and Shen, Z.J. (2004). Channel performance under consignment contract with revenue sharing. *Management Science*, 50(1): 34-47.
32. Whitin, T.M. (1955). Inventory control and price theory. *Management Science*, 2: 61-68.

33. Young, L. (1978). Price, inventory and the structure of uncertain demand. *New Zealand Operations Research*, 6: 157-177.
34. Zipkin, P.H. (2000). *Foundations of Inventory Management*. McGraw-Hill.