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On the Economic Lot Scheduling Problem with Fuzzy Demands

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Abstract—In this paper, we investigate the economic lot scheduling problem (*ELSP*) with fuzzy demands. We assume that the demand for each product *i* can be approximated using some triangular membership functions. In this study, we solve the fuzzy ELSP using two basic solution approaches, namely, the Independent Solution (*IS*) and the Common Cycle (*CC*) approach. For both approaches, we derive the optimal fuzzy replenishment cycles and secure closed-form formula for their crisp figures in fuzzy sense, respectively. Also, we derive the conditions that assert the *CC* approach to secure the optimal solution for the fuzzy *ELSP* in many realistic situations. For the cases that deviate from those optimal-situations, we give an upper bound for the maximum error of the solution of the *CC* approach from optimality. A 10-product example demonstrates how to secure the solutions for the *IS* and the *CC* approach for the fuzzy *ELSP*, and illustrates the error bound of the *CC* approach.

Keywords—Inventory, Economic lot size scheduling, Fuzzy sets, Fuzzy replenishment cycle, Sensitivity analysis

1. INTRODUCTION

In this paper, we investigate the economic lot scheduling problem (*ELSP*) with fuzzy demands. We assume that the demand for each product *i* can be approximated using some triangular membership functions. Later, we solve the fuzzy ELSP using two basic solution approaches, namely, the Independent Solution (*IS*) and the Common Cycle (*CC*) approach. For both approaches, we derive the optimal fuzzy replenishment cycles and secure closed-form formula for their crisp figures in fuzzy sense, respectively.

In this section, we first provide some background knowledge on the *ELSP* and introduce our motivation to study the *ELSP* with fuzzy demands.

1.1 The economic lot scheduling problem

The Economic Lot Scheduling Problem (*ELSP*) is concerned with scheduling the cyclical production of $n \geq 2$ products on a single facility in lots that differ in size and, consequently, differ in production times and cycles. (A production cycle is the time from the start of production of a lot to the start of production of the next lot of the same product.) The cost items involved are two: *setup cost and inventory holding cost* (the cost of production is fixed and therefore irrelevant to optimization).

The conventional *ELSP* is characterized by the following assumptions:

1. Only one product can be produced at a time on the facility, product demands are continuous, and production rates, demand rates, setup times, setup costs, and inventory holding costs are deterministic;
2. The facility capacity is sufficient to meet all the demands of the products;
3. Setup times and costs are independent of the production sequence and lot sizes;
4. Inventory costs are directly proportional to inventory levels;
5. No shortages are allowed;
6. The planning horizon is continuous and infinite.

The *ELSP* has been studied for some forty years since Roger (1958) published the first article. Since the *ELSP* is an *NP*-hard problem (Hsu, 1983), hundreds of research articles have been addressed to the *ELSP* and its extensions. One may refer to Elmaghraby (1978), Lopez & Kingsman (1991) and Yao (1999) for the extensive literature on the solution methodologies for solving the *ELSP*.

The objective function of the conventional *ELSP* is given by

$$Minimize\ \sum_{i=1}^{n} C_i\left(T_i\right) = \sum_{i=1}^{n}\left[\frac{a_i}{T_i} + \frac{h_i}{2} d_i\left(1-\frac{d_i}{p_i}\right)T_i\right] \qquad (1)$$

where

d_i = demand per unit time for product *i*,
p_i = production rate per unit time for product *i* ($p_i > d_i$),
a_i = setup cost for product *i*,
h_i = unit-holding cost per unit time for product *i*,
T_i = cycle time for product *i*.

A production plan in the context of *ELSP* usually schedules the items within "basic periods", where a basic period, denoted by *B*, is an interval of time that is devoted to the setup and production of a subset (or all) of the products. That is, $T_i = k_iB$, and the replenishment frequency must be a positive integer. Therefore, the solution of the *ELSP* is the set of multipliers $K(B) = \{k_i \mid B\}_{i=1}^{n}$ *and* the basic period in which each product is produced.

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Depending on different ways in formulating the *ELSP* models, one may classify the conventional *ELSP* models into three categories: (1) two basic solution approaches, *i.e.*, Independent solution (*IS*) and the Common Cycle (*CC*) approach, (2) the Basic Period (*BP*) approach, and (3) the Extended Basic Period (*EBP*) approach. The *IS* ignores the interference between products (which violates the assumption that only one product can be produced at a time on the facility). In the *IS*, the optimal cycle time for product *i* is given by

$$T_i^{IS} = \max\left\{\sqrt{2a_i/\left[h_i d_i\left(1-\rho_i\right)\right]}, s_i/\left(1-\rho_i\right)\right\} \tag{2}$$

where $\rho_i = d_i/p_i$, s_i = the setup time and $\sqrt{2a_i/\left[h_i d_i\left(1-\rho_i\right)\right]}$ is the Economic Production Quantity (*EPQ*) cost expressions for product *i*. The term $s_i/\left(1-\rho_i\right)$ results from the case when the EPQ expression is not long enough, feasibility requires that it must be increased to $s_i/\left(1-\rho_i\right)$. The *IS* secures an obvious lower bound for the cost of any feasible solution. The *IS* brings infeasible solutions in general. However, whenever the *IS* has a feasible production schedule, it secures the optimal solution for the conventional *ELSP*. The *CC* approach sets k_i = 1 for all *i*, that is, all the *n* products share a common replenishment cycle. Therefore, the mathematical model for the *CC* approach is given by

$$\textit{Minimize} \quad \sum_{i=1}^{n} \frac{a_i}{T} + \frac{h_i}{2} d_i\left[1-\rho_i\right]T \tag{3}$$

$$\textit{subject to} \quad \sum_{i=1}^{n} s_i + \rho_i T \le T \tag{4}$$

where s_i is the setup time for product *i*. Ineq. (4) requires that the sum of the production time for all the products must be no longer than the common replenishment cycle. One could easily solve the optimal cycle time for the *CC* approach by a closed form, *i.e.*,

$$T_{CC}^{*} = \max\left\{\sqrt{2\sum\nolimits_{i=1}^{n} a_i \Big/ \sum\nolimits_{i=1}^{n}\left[h_i d_i\left(1-\rho_i\right)\right]}, \; \sum\nolimits_{i=1}^{n} s_i \Big/ \left(1-\sum\nolimits_{i=1}^{n}\rho_i\right)\right\}, \tag{5}$$

where s_i is the setup time for product *i*. (One may refer to Hanssmann's, 1962 book for the derivation of T_{CC}^{*}.) Note that unlike the *IS*, the *CC* is always feasible, and the *CC* approach provides an upper bound (Grznar and Riggle, 1997) for the objective value of the conventional *ELSP*. On the other hand, the *BP* approach (Bomberger, 1966) admits differing product cycles by taking the integer multiples of a fundamental cycle. The basic cycle must be long enough to accommodate the lot production for all products. Elmaghraby (1977) then extended the *BP* approach and established the extended basic period (*EBP*). One may refer to the following papers for the heuristics using the *EBP* approaches: Fujita (1978), Haessler (1979), Park and Yun (1984), Boctor (1987), and Yao (1999).

Recall that the first assumption of the conventional *ELSP* states that demand rates for all of the products are deterministic (*i.e.*, fixed and known) in the beginning of this paper. It is one of the key assumptions in the conventional *ELSP*. Though perturbation occurs in demands every day, this assumption might bother most of the managers when they employ the conventional *ELSP* models in inventory decision-making. Next, we review previous research on embedding fuzzy demands in inventory problems in the literature.

1.2 Fuzzy demands in inventory problems

Recently, fuzzy models (e.g., *EPQ*/*EOQ*) have been proposed to cope with the fluctuation problems in human subjectively originated data in the planning stage of an inventory problem. One may refer to the following articles for the review of fuzzy inventory models: Roy and Maiti (1997), Lin and Yao (2000), Chang (1999), Vujosevic et al. (1996), Lee and Yao (1998), Park (1987), Sommer (1981), Kacprzyk and Staniewski (1982), Yao and Lee (1999), and Yao and Su (2000). And, these fuzzy inventory models can be classified into two categories:

1. The total cost function is only a crisp mapping from some fuzzy variable (*e.g.*, production quantity) and thus yields a fuzzy set (e.g., Chang, 1999; Lin and Yao, 2000); *i.e.*, where the fuzziness originates only with the lot size;
2. The mapping itself is fuzzy with fuzzy parameters (e.g., Park, 1987; Vujosevic et al., 1996; Lee and Yao, 1998) and thus blurs the image of some crisp argument (e.g., production quantity).

Along the second line of research, most models seemed to concern only with the fuzzy total cost function, and researchers directly defuzzified it to obtain a compromised crisp lot size. What is neglected in this direction may be that in these circumstances, the decisions of lot sizes are also blurred. Advantages of finding such fuzzy lot sizes and thus fuzzy total costs with exact membership functions (equivalently α-cuts) are obvious. This is because fuzzy data exist and because the final decisions can be elected by the decision-makers by participating in the determination of the final crisp compromised one. In Vujosevic et al. (1996) paper, he proposed a fuzzy economic order quantity (*EOQ*). Apparently, it was a direct fuzzified version of the conventional crisp *EOQ* formula instead of deriving it from the fuzzy total cost function. Pappis and Karacapilidis (1995) also proposed to use fuzzy numbers for demands in the common cycle (*CC*) approach for solving the *ELSP* with fuzzy demands (abbreviated as fuzzy *ELSP* for the rest of the paper). But, it obviously bears a similar shortcoming. To the best of the authors' knowledge, no other research efforts in the second category of fuzzy inventory models have been addressed to the *ELSP* with fuzzy demands. Therefore, it leads our motivation to figure out the influence of the fuzzy demands on the fuzzy total cost function and to secure the fuzzy replenishment cycle times for the fuzzy *ELSP*.

The organization of the rest of this article is as follows. In Section 2, we present two basic solution approaches for solving the fuzzy *ELSP*, namely, the independent solution (*IS*) and the Common Cycle (*CC*) approach. For both approaches, we derive the optimal fuzzy replenishment cycles and secure closed-form formula for their crisp figures in fuzzy sense, respectively. Then, we derive the conditions that assert the *CC* approach to secure the optimal for the fuzzy *ELSP* in many realistic situations in the third section. For the cases that deviate from those optimal-situations, we give an upper bound for the maximum error of the solution of the *CC* approach from optimality. A numerical example in Section 4 illustrates how to secure the *IS* and the solution of the *CC* approach for the fuzzy *ELSP*. Finally, Section 5 gives some concluding remarks.

2. TWO BASIC SOLUTION APPROACHES FOR SOLVING THE FUZZY *ELSP*

In this section, we present two basic solution approaches for solving the fuzzy *ELSP*, namely, the Independent solution (*IS*) and the Common Cycle (*CC*) approach.

2.1 Fuzzy demands in the *ELSP*

The following procedure delineates the steps for the derivation of the fuzzy replenishment cycle times for the fuzzy *ELSP*.

Step 1. Embed the fuzzy demands in the objective function (4) using fuzzy arithmetic and derive the fuzzy total cost function.

Step 2. Solve the fuzzy replenishment cycle times from the fuzzy total cost function.

Step 3. Defuzzify the fuzzy replenishment cycle times to secure their corresponding crisp ones.

In this study, the fuzzy demands (denoted as $\tilde{D}_i = (d_{i1}, d_{i0}, d_{i2})$, $i = 1, \ldots, n$) are given as "approximate d_{i0}" with triangular membership functions as follows:

$$u_{\tilde{D}_i}(d_i) = \begin{cases} (d_i - d_{i1})/(d_{i0} - d_{i1}), & d_{i1} \le d_i \le d_{i0}. \\ (d_{i2} - d_i)/(d_{i2} - d_{i0}), & d_{i0} \le d_i \le d_{i2}. \\ 0, & otherwise \end{cases} \tag{6}$$

For fuzzy arithmetic, the exact approach (Mizumoto and Tanaka, 1979) is used as follows.

Remark. For function $y = g(a, b)$, if a and b be replaced by fuzzy numbers A and B with α-cuts as A^α and B^α that are ordinary continuous bounded intervals, the fuzzy function via fuzzy arithmetic can be obtained: $Y = g(A, B)$ with $Y^\alpha = g(A^\alpha, B^\alpha)$, where $g(\cdot,\cdot)$ reduces to an interval operation.

Proof. It is quite straightforward that by extension principle, we have the membership function for $Y = g(A, B)$ as

$$\mu_{g(A,B)}(y) = \begin{cases} \sup\limits_{(a,b)\in g^{-1}(y)} [\mu_A(a) \wedge \mu_B(b)] & \text{if } g^{-1}(y) \ne \varnothing, \\ 0 & \text{if } g^{-1}(y) = \varnothing. \end{cases} \tag{7}$$

Thus consider $\mu_{g(A,B)}(y) \ge \alpha > 0$. There should exist $\{(a, b) \in g^{-1}(y)\}$ and only that with $\mu_A(a) \ge \alpha$ and $\mu_B(b) \ge \alpha$ need to be considered because of the '$\wedge$' operation. Furthermore, as the fuzzy numbers' membership functions are continuous and convex, if $\mu_{g(A,B)}(y) = \alpha$, $\{(a, b) \in g^{-1}(y)\}$ reduces to only that with $\mu_A(a) = \alpha$ and $\mu_B(b) = \alpha$ need to be concerned. Therefore, since $\mu_{g(A,B)}(y) \ge \alpha$, $\mu_A(a) \ge \alpha$, and $\mu_B(b) \ge \alpha$ define the α-cuts of fuzzy numbers, the extension principle requires that A^α and B^α are considered for Y^α, or $Y^\alpha = g(A^\alpha, B^\alpha)$ and $g(\cdot,\cdot)$ reduces to an interval operation.

2.2 The independent solution

Since the fuzzy demand for product i is given as "approximate d_{i0}" with a triangular membership function, we have the equivalent *α-cut* by

$$D_i^\alpha = \left[(d_{i0} - d_{i1})\alpha + d_{i1}, d_{i2} - \alpha(d_{i2} - d_{i0}) \right] \tag{8}$$

Next, we employ fuzzy arithmetic to derive the *IS* with fuzzy demands. Before presenting our derivation, we first define some notation that greatly simplifies the expression in the following presentation. Let $H_i^\alpha = h_i D_i^\alpha \left(1 - \frac{D_i^\alpha}{p_i}\right)$ and $H_i^\alpha = \left[H_{iL}^\alpha, H_{iU}^\alpha \right]$, where

$$H_{iL}^\alpha = h_i \left((d_{i0} - d_{i1})\alpha + d_{i1} \right) \left(1 - \tfrac{d_{i2}}{p_i} + \tfrac{(d_{i2} - d_{i0})}{p_i}\alpha \right) \tag{9}$$

and

$$H_{iU}^\alpha = h_i \left((d_{i0} - d_{i2})\alpha + d_{i2} \right) \left(1 - \tfrac{d_{i1}}{p_i} + \tfrac{(d_{i1} - d_{i0})}{p_i}\alpha \right). \tag{10}$$

We discuss the details of the three-step procedure (given at the beginning of this section) as follows.

(3) *Derive the fuzzy average total cost function – Step 1*

The fuzzy average total cost function can be derived from (1) with fuzzy demands as follows. For product i, the fuzzy average cost function, denoted as $C_i(\tilde{D}_i, T_i)$, can be expressed as follows.

$$C_i(\tilde{D}_i, T_i) = \frac{a_i}{T_i} + \frac{h_i}{2}\tilde{D}_i\left(1 - \frac{\tilde{D}_i}{p_i}\right)T_i \tag{11}$$

Therefore, the fuzzy average cost function can be given by its *α-cut*, as

$$y \in C_i^\alpha = \left[\frac{a_i}{T_i} + H_{iL}^\alpha T_i, \frac{a_i}{T_i} + H_{iU}^\alpha T \right], \tag{12}$$

iff $\mu_{C_i(\tilde{D}_i,T_i)}(y) \ge \alpha$.

For (2.7), we will denote C_i^α as $[y_L^\alpha, y_U^\alpha]$, $0 < \alpha \le 1$, for simplification.

(2) *Solve the fuzzy replenishment cycle times – Step 2*

Next, we solve *the optimal fuzzy replenishment cycle time* for each product *i*, denoted by T_i^{IS}, from its fuzzy average cost function and its fuzzy feasibility constraint, respectively.

First, from (11) and (12), the fuzzy average cost function for product *i*, the following differential equation holds:

$$dC_i(\tilde{D}_i,T_i)/dT_i = \{dC_i^\alpha/dT_i \mid 0<\alpha\le 1\}$$
$$=\{dy_L^\alpha/dT_i, dy_U^\alpha/dT_i \mid 0<\alpha\le 1\}. \tag{13}$$

By setting the first-order derivative in (13) to zero, we derive the lower and upper bounds for the fuzzy average cost function for each product *i* by $\sqrt{2a_i/H_{iU}^\alpha}$ and $\sqrt{2a_i/H_{iL}^\alpha}$, respectively. Define as $T_i^\alpha = [t_{iL}^\alpha, t_{iU}^\alpha]$ the fuzzy replenishment cycle time of product *i*. Then, one can easily secure T_i^α by

$$T_i^\alpha = [t_{iL}^\alpha, t_{iU}^\alpha] = [\sqrt{2a_i/H_{iU}^\alpha}, \sqrt{2a_i/H_{iL}^\alpha}] \tag{14}$$

The fuzzy feasibility constraint for product *i* in the *IS* leads to the following expression:

$$T_i^{\min} = [s_i\Big/\Big(1-\Big(\tfrac{d_{i1}}{p_i}+\tfrac{(d_{i0}-d_{i1})\alpha}{p_i}\Big)\Big), s_i\Big/\Big(1-\Big(\tfrac{d_{i2}}{p_i}+\tfrac{(d_{i0}-d_{i2})\alpha}{p_i}\Big)\Big)] \tag{15}$$

By (14) and (15), we have the optimal fuzzy replenishment cycle time of product *i* for the *IS* by

$$T_i^{IS} = \max\{T_i^\alpha, T_i^{\min}\}. \tag{16}$$

(3) *Defuzzify the fuzzy replenishment cycle times – Step 3*

Since the manager is unable to directly use T_i^α in (14) for his/her decision making, we need to defuzzify the fuzzy replenishment cycle time of product *i*. To avoid adding too much complexity, we utilize the method of *centroid* to secure the crisp values of the replenishment cycle times for the *IS*.

The value of the centroid is given by

$$M_i(\tilde{D}_i) = \frac{\int_{-\infty}^{\infty} u_T(z)z\,dz}{\int_{-\infty}^{\infty} u_T(z)\,dz} = \frac{M_i^L + M_i^U}{M_{il}+M_{iu}} \tag{17}$$

where

$$M_i^L = \int_{\sqrt{\frac{D_i}{C_{i2}}}}^{\sqrt{\frac{D_i}{A_i-B_{i2}+C_{i2}}}} \frac{z^2B_{i2} \pm \sqrt{z^4B_{i2}^2 - 4A_iz^2(C_{i2}z^2-D_i)}}{2z^2A_i}\cdot z\cdot dz,$$

$$M_{il} = \int_{\sqrt{\frac{D_i}{C_{i2}}}}^{\sqrt{\frac{D_i}{A_i-B_{i2}+C_{i2}}}} \frac{z^2B_{i2} \pm \sqrt{z^4B_{i2}^2 - 4A_iz^2(C_{i2}z^2-D_i)}}{2z^2A_i}\cdot dz$$

$$M_i^U = \int_{\sqrt{\frac{D_i}{A_i+B_{i1}+C_{i1}}}}^{\sqrt{\frac{D_i}{C_{i1}}}} \frac{-z^2B_{i1} \pm \sqrt{z^4B_{i1}^2 - 4A_iz^2(C_{i1}z^2-D_i)}}{2z^2A_i}\cdot z\cdot dz$$

$$M_{iu} = \int_{\sqrt{\frac{D_i}{A_i+B_{i1}+C_{i1}}}}^{\sqrt{\frac{D_i}{C_{i1}}}} \frac{-z^2B_{i1} \pm \sqrt{z^4B_{i1}^2 - 4A_iz^2(C_{i1}z^2-D_i)}}{2z^2A_i}\cdot dz$$

The closed-form expressions for M_i^L, M_{il}, M_i^U and M_{iu} are given as follows.

$$M_i^L = \frac{B_{i2}D_i}{4A_i(A_i-B_{i2}+C_{i2})} - \frac{B_{i2}D_i}{4A_iC_{i2}} - \left\{\sqrt{\frac{D_i^2}{4A_i(A_i-B_{i2}+C_{i2})}\left(1+\frac{B_{i2}^2-4A_iC_{i2}}{4A_i(A_i-B_{i2}+C_{i2})}\right)}\right.$$
$$+\frac{D_i}{\sqrt{B_{i2}^2-4A_iC_{i2}}}\ln\left|\sqrt{\frac{D_i}{A_i-B_{i2}+C_{i2}}}+\sqrt{\frac{4A_iD_i}{B_{i2}^2-4A_iC_{i2}}+\frac{D_i}{A_i-B_{i2}+C_{i2}}}\right|$$

$$-\sqrt{\frac{D_i^2}{4A_iC_{i2}}\left(1+\frac{B_{i2}^2-4A_iC_{i2}}{4A_iC_{i2}}\right)}-\frac{D_i}{\sqrt{B_{i2}^2-4A_iC_{i2}}}\ln\left|\sqrt{\frac{D_i}{C_{i2}}}+\sqrt{\frac{4A_iD_i}{B_{i2}^2-4A_iC_{i2}}+\frac{D_i}{C_{i2}}}\right|\Bigg\}$$

$$M_{il}=\frac{B_{i2}}{2A_i}\sqrt{\frac{D_i}{A_i-B_{i2}+C_{i2}}}-\frac{B_{i2}}{2A_i}\sqrt{\frac{D_i}{C_{i2}}}-\left\{\sqrt{\frac{D_i}{A_i}+\frac{D_i\left(B_{i2}^2-4A_iC_{i2}\right)}{4A_i^2\left(A_i-B_{i2}+C_{i2}\right)}}-\sqrt{\frac{D_i}{A_i}}\cdot\right.$$
$$\ln\left|\sqrt{\frac{4A_i\left(A_i-B_{i2}+C_{i2}\right)}{B_{i2}^2-4A_iC_{i2}}}+\sqrt{\frac{4A_i\left(A_i-B_{i2}+C_{i2}\right)}{B_{i2}^2-4A_iC_{i2}}+1}\right|-\sqrt{\frac{D_i}{A_i}+\frac{D_i\left(B_{i2}^2-4A_iC_{i2}\right)}{4A_i^2C_{i2}}}$$
$$\left.+\sqrt{\frac{D_i}{A_i}}\ln\left|\sqrt{\frac{4A_i\left(A_i-B_{i2}+C_{i2}\right)}{B_{i2}^2-4A_iC_{i2}}}+\sqrt{\frac{4A_iC_{i2}}{B_{i2}^2-4A_iC_{i2}}+1}\right|\right\}$$

$$M_i^U=\frac{-B_{i1}D_i}{4A_iC_{i1}}+\frac{B_{i1}D_i}{4A_i\left(A_i+B_{i1}+C_{i1}\right)}+\left\{\sqrt{\frac{D_i^2}{4A_iC_{i1}}\left(1+\frac{B_{i1}^2-4A_iC_{i1}}{4A_iC_{i1}}\right)}+\frac{D_i}{\sqrt{B_{i1}^2-4A_iC_{i1}}}\cdot\right.$$
$$\ln\left|\sqrt{\frac{D_i}{C_{i1}}}+\sqrt{\frac{4A_iD_i}{B_{i1}^2-4A_iC_{i1}}+\frac{D_i}{C_{i1}}}\right|-\sqrt{\frac{D_i^2}{4A_i\left(A_i+B_{i1}+C_{i1}\right)}\left(1+\frac{B_{i1}^2-4A_iC_{i1}}{4A_i\left(A_i+B_{i1}+C_{i1}\right)}\right)}$$
$$\left.-\frac{D_i}{\sqrt{B_{i1}^2-4A_iC_{i1}}}\ln\left|\sqrt{\frac{D_i}{A_i+B_{i1}+C_{i1}}}+\sqrt{\frac{4A_iD_i}{B_{i1}^2-4A_iC_{i1}}+\frac{D_i}{A_i+B_{i1}+C_{i1}}}\right|\right\}$$

$$M_{iu}=\frac{-B_{i1}}{2A_i}\sqrt{\frac{D_i}{C_{i1}}}+\frac{B_{i1}}{2A_i}\sqrt{\frac{D_i}{A_i+B_{i1}+C_{i1}}}+\left\{\sqrt{\frac{D_i}{A_i}+\frac{D_i}{C_{i1}}\left(\frac{B_{i1}^2-4A_iC_{i1}}{4A_i^2}\right)}\right.$$
$$-\sqrt{\frac{D_i}{A_i}}\ln\left|\sqrt{\frac{4A_iC_{i1}}{B_{i2}^2-4A_iC_{i1}}}+\sqrt{\frac{4A_iC_{i1}}{B_{i1}^2-4A_iC_{i1}}+1}\right|-\sqrt{\frac{D_i}{A_{i1}}+\frac{D_i\left(B_{i1}^2-4A_iC_{i1}\right)}{4A_i^2\left(A_i+B_{i1}+C_{i1}\right)}}$$
$$\left.+\sqrt{\frac{D_i}{A_i}}\ln\left|\sqrt{\frac{4A_i\left(A_i+B_{i1}+C_{i1}\right)}{B_{i1}^2-4A_iC_{i1}}}+\sqrt{\frac{4A_i\left(A_i+B_{i1}+C_{i1}\right)}{B_{i1}^2-4A_iC_{i1}}+1}\right|\right\}$$

where $A_i=h_i\left[\frac{1}{p_i}\left(d_{i0}-d_{i1}\right)\left(d_{i2}-d_{i0}\right)\right]$, $D_i=2a_i$,

$$B_{i1}=h_i\left[\left(d_{i0}-d_{i1}\right)\left(1-\frac{d_{i2}}{p_i}\right)+\frac{d_{i1}}{p_i}\left(d_{i2}-d_{i0}\right)\right],\quad C_{i1}=h_id_{i1}\left(1-\frac{d_{i2}}{p_i}\right),$$

$$B_{i2}=h_i\left[\left(d_{i2}-d_{i0}\right)\left(1-\frac{d_{i1}}{p_i}\right)+\frac{d_{i2}}{p_i}\left(d_{i0}-d_{i1}\right)\right],\quad C_{i2}=h_id_{i2}\left(1-\frac{d_{i1}}{p_i}\right).$$

2.3 The common cycle approach

Recall that the fuzzy demand for product i is given as "approximate d_{i0}" with a triangular membership function, and we have (8) as the equivalent α*-cut*.

Similar to the derivation in Section 2.2, we employ fuzzy arithmetic to derive the optimal solution for the *CC* with fuzzy demand using the three-step procedure as follows.

(3) *Derive the fuzzy total cost function – Step 1*

By plugging fuzzy demands $\tilde{D}_i$ into the models for the *CC* approach in (3) and (4), we may secure the lower and upper bounds for the fuzzy average total cost function and the fuzzy feasibility constraints in a format of *α-cut* as follows.

Minimize

$$\sum_{i=1}^{n} C_i\left(\tilde{D}_i, T\right) = \sum_{i=1}^{n}\left[\frac{a_i}{T} + \frac{h_i}{2}\tilde{D}_i\left(1 - \frac{d_i}{p_i}\right)T\right] \tag{19}$$

Subject to $$\sum_{i=1}^{n}\left[s_i + \frac{\tilde{D}_i}{p_i}T\right] \le T \tag{20}$$

Then, the fuzzy average total cost function for the *CC* approach can be simplified by

$$\sum_{i=1}^{n} C^{\alpha} = \left[\sum_{i=1}^{n}\left(\frac{a_i}{T} + \frac{H_{iL}^{\alpha}}{2}T\right), \sum_{i=1}^{n}\left(\frac{a_i}{T} + \frac{H_{iU}^{\alpha}}{2}T\right)\right] \tag{21}$$

and the fuzzy feasibility constraint is given by

$$\left[\sum_{i=1}^{n}\left\{s_i + \left(\frac{d_{i1}}{p_i} + \frac{\left(d_{i0} - d_{i1}\right)}{p_i}\alpha\right)T\right\}, \sum_{i=1}^{n}\left\{s_i + \left(\frac{d_{i2}}{p_i} - \frac{\left(d_{i2} - d_{i0}\right)}{p_i}\alpha\right)T\right\}\right] \le T = [T, T]. \tag{22}$$

(2) *Solve the fuzzy replenishment cycle times – Step 2*
Next, we solve *the fuzzy replenishment cycle time* for the *CC* approach, denoted by T^{CC}, from its fuzzy average cost function and fuzzy feasibility constraint, respectively.

First, from (19) and (20), the fuzzy average cost function for the *CC* approach, the following differential equation holds:

$$\left\{ d\sum_{i=1}^{n} C_i^{\alpha} \Big/ dT \mid 0 < \alpha \le 1 \right\}. \tag{23}$$

By setting the first-order derivative in (23) to zero, we derive the lower and upper bounds for the fuzzy average cost function by $\sqrt{2\sum_{i=1}^{n} a_i \Big/ \sum_{i=1}^{n} H_{iU}^{\alpha}}$ and $\sqrt{2\sum_{i=1}^{n} a_i \Big/ \sum_{i=1}^{n} H_{iL}^{\alpha}}$, respectively.

Define as $T^{\alpha} = \left[t_L^{\alpha}, t_U^{\alpha}\right]$ the fuzzy replenishment cycle time from the fuzzy average total cost function of the *CC* approach. Then, one can easily secure T^{α} by

$$T^{\alpha} = \left[t_L^{\alpha}, t_U^{\alpha}\right] = \left[\sqrt{2\sum_{i=1}^{n} a_i \Big/ \sum_{i=1}^{n} H_{iU}^{\alpha}}, \sqrt{2\sum_{i=1}^{n} a_i \Big/ \sum_{i=1}^{n} H_{iL}^{\alpha}}\right] \tag{24}$$

On the other hand, the fuzzy feasibility constraint gives the lower and upper bounds for the fuzzy replenishment cycle as follows.

$$T^{\min} = \left[\sum_{i=1}^{n} s_i \Bigg/ \left(1 - \sum_{i=1}^{n}\left(\frac{d_{i1}}{p_i} + \frac{\left(d_{i0} - d_{i1}\right)}{p_i}\alpha\right)\right), \sum_{i=1}^{n} s_i \Bigg/ \left(1 - \sum_{i=1}^{n}\left(\frac{d_{i2}}{p_i} + \frac{\left(d_{i0} - d_{i2}\right)}{p_i}\alpha\right)\right)\right] \tag{25}$$

By combining (24) and (25), we secure the optimal fuzzy replenishment cycle time for the *CC* approach, *i.e.*, T^{CC}, as follows.

$$T^{CC} = \max\left\{T^{\alpha}, T^{\min}\right\}. \tag{26}$$

(3) *Defuzzify the fuzzy replenishment cycle times – Step 3*
Next, we utilize the method of *centroid* to defuzzify the fuzzy replenishment cycle time from Step 2 to secure the crisp value of the optimal replenishment cycle time for the *CC* approach.

The value of the centroid from the objective function is given by

$$M\left(\tilde{D}_i\right) = \frac{\int_{-\infty}^{\infty} u_T(z) z dz}{\int_{-\infty}^{\infty} u_T(z) dz} = \frac{M^L + M^U}{M_l + M_u} \tag{27}$$

where

$$M^L = \int_{\sqrt{\frac{D}{C_2}}}^{\sqrt{\frac{D}{A - B_2 + C_2}}} \frac{z^2 B_2 \pm \sqrt{z^4 B_2^2 - 4Az^2\left(C_2 z^2 - D\right)}}{2z^2 A} \cdot z \cdot dz$$

$$M_l = \int_{\sqrt{\frac{D}{C_2}}}^{\sqrt{\frac{D}{A - B_2 + C_2}}} \frac{z^2 B_2 \pm \sqrt{z^4 B_2^2 - 4Az^2\left(C_2 z^2 - D\right)}}{2z^2 A} \cdot dz$$

$$M^U = \int_{\sqrt{\frac{D}{A + B_1 + C_1}}}^{\sqrt{\frac{D}{C_1}}} \frac{-z^2 B_1 \pm \sqrt{z^4 B_1^2 - 4Az^2\left(C_1 z^2 - D\right)}}{2z^2 A} \cdot z \cdot dz$$

$$M_u = \int_{\sqrt{\frac{D}{A+B_1+C_1}}}^{\sqrt{\frac{D}{C_1}}} \frac{-z^2 B_1 \pm \sqrt{z^4 B_1^2 - 4Az^2\left(C_1 z^2 - D\right)}}{2z^2 A} \cdot dz$$

The closed-form expressions for M^L, M_l, M^U and M_u are given as follows.

$$M^L = \frac{B_2 D}{4A\left(A-B_2+C_2\right)} - \frac{B_2 D}{4AC_2} - \left\{ \sqrt{\frac{D^2}{4A\left(A-B_2+C_2\right)}\left(1+\frac{B_2^2-4AC_2}{4A\left(A-B_2+C_2\right)}\right)} + \frac{D}{\sqrt{B_2^2-4AC_2}} \cdot \right.$$
$$\ln\left|\sqrt{\frac{D}{A-B_2+C_2}} + \sqrt{\frac{4AD}{B_2^2-4AC_2}+\frac{D}{A-B_2+C_2}}\right| - \sqrt{\frac{D^2}{4AC_2}\left(1+\frac{B_2^2-4AC_2}{4AC_2}\right)}$$
$$\left. - \frac{D}{\sqrt{B_2^2-4AC_2}} \ln\left|\sqrt{\frac{D}{C_2}} + \sqrt{\frac{4AD}{B_2^2-4AC_2}+\frac{D}{C_2}}\right| \right\} \frac{D}{\sqrt{B_2^2-4AC_2}} \cdot$$
$$\ln\left|\sqrt{\frac{D}{A-B_2+C_2}} + \sqrt{\frac{4AD}{B_2^2-4AC_2}+\frac{D}{A-B_2+C_2}}\right| - \sqrt{\frac{D^2}{4AC_2}\left(1+\frac{B_2^2-4AC_2}{4AC_2}\right)}$$
$$\left. - \frac{D}{\sqrt{B_2^2-4AC_2}} \ln\left|\sqrt{\frac{D}{C_2}} + \sqrt{\frac{4AD}{B_2^2-4AC_2}+\frac{D}{C_2}}\right| \right\}$$

$$M_l = \frac{B_2}{2A}\sqrt{\frac{D}{A-B_2+C_2}} - \frac{B_2}{2A}\sqrt{\frac{D}{C_2}} - \left\{ \sqrt{\frac{D}{A}+\frac{D\left(B_2^2-4AC_2\right)}{4A^2\left(A-B_2+C_2\right)}} - \sqrt{\frac{D}{A}} \cdot \right.$$
$$\ln\left|\sqrt{\frac{4A\left(A-B_2+C_2\right)}{B_2^2-4AC_2}} + \sqrt{\frac{4A\left(A-B_2+C_2\right)}{B_2^2-4AC_2}+1}\right| - \sqrt{\frac{D}{A}+\frac{D\left(B_2^2-4AC_2\right)}{4A^2 C_2}} +$$
$$\left. \sqrt{\frac{D}{A}} \ln\left|\sqrt{\frac{4AC_2}{B_2^2-4AC_2}} + \sqrt{\frac{4AC_2}{B_2^2-4AC_2}+1}\right| \right\}$$

$$M^U = \frac{-B_1 D}{4AC_1} + \frac{B_1 D}{4A\left(A+B_1+C_1\right)} + \left\{ \sqrt{\frac{D^2}{4AC_1}\left(1+\frac{B_1^2-4AC_1}{4AC_1}\right)} + \frac{D}{\sqrt{B_1^2-4AC_1}} \cdot \ln\left|\sqrt{\frac{D}{C_1}} + \sqrt{\frac{4AD}{B_1^2-4AC_1}+\frac{D}{C_1}}\right| \right.$$
$$\left. - \sqrt{\frac{D^2}{4A\left(A+B_1+C\right)_1}\left(1+\frac{B_1^2-4AC_1}{4A\left(A+B_1+C_1\right)}\right)} - \frac{D}{\sqrt{B_1^2-4AC_1}} \ln\left|\sqrt{\frac{D}{A+B_1+C_1}} + \sqrt{\frac{4AD}{B_1^2-4AC_1}+\frac{D}{A+B_1+C_1}}\right| \right\}$$

$$M_u = \frac{-B_1}{2A}\sqrt{\frac{D}{C_1}} + \frac{B_1}{2A}\sqrt{\frac{D}{A+B_1+C_1}} + \left\{ \sqrt{\frac{D}{A}+\frac{D}{C_1}\left(\frac{B_1^2-4AC_1}{4A^2}\right)} - \sqrt{\frac{D}{A}} \cdot \ln\left|\sqrt{\frac{4AC_1}{B_2^2-4AC_1}} + \sqrt{\frac{4AC_1}{B_1^2-4AC_1}+1}\right| \right.$$
$$\left. - \sqrt{\frac{D}{A_1}+\frac{D\left(B_1^2-4AC_1\right)}{4A^2\left(A+B_1+C_1\right)}} + \sqrt{\frac{D}{A}} \ln\left|\sqrt{\frac{4A\left(A+B_1+C_1\right)}{B_1^2-4AC_1}} + \sqrt{\frac{4A\left(A+B_1+C_1\right)}{B_1^2-4AC_1}+1}\right| \right\}$$

where $A = \sum_{i=1}^{n} h_i\left[\left(d_{i0}-d_{i1}\right)\left(d_{i2}-d_{i0}\right)/p_i\right]$, $D = \sum_{i=1}^{n} a_i$

$B_1 = \sum_{i=1}^{n} h_i\left[\left(d_{i0}-d_{i1}\right)\left(1-d_{i2}/p_i\right)+\left(d_{i2}-d_{i0}\right)d_{i1}/p_i\right]$, $C_1 = \sum_{i=1}^{n} h_i d_{i1}\left(1-d_{i2}/p_i\right)$

$B_2 = \sum_{i=1}^{n} h_i\left[\left(d_{i2}-d_{i0}\right)\left(1-d_{i1}/p_i\right)+\left(d_{i0}-d_{i1}\right)d_{i2}/p_i\right]$, $C_2 = \sum_{i=1}^{n} h_i d_{i2}\left(1-d_{i1}/p_i\right)$.

The value of the centroid from the fuzzy feasibility constraint is given by

$$M\left(\tilde{D}_i\right)=\frac{\int_{-\infty}^{\infty}u_T(z)z\,dz}{\int_{-\infty}^{\infty}u_T(z)dz}=\frac{\int_{\frac{S}{P_1}}^{\frac{S}{1-P_1-Q_1}}\frac{z-zP_1-S}{zQ_1}\cdot z\cdot dz+\int_{\frac{S}{1-P_2+Q_2}}^{\frac{S}{1-P_2}}\frac{S-z+zP_2}{zQ_2}\cdot z\cdot dz}{\int_{\frac{S}{P_1}}^{\frac{S}{1-P_1-Q_1}}\frac{z-zP_1-S}{zQ_1}\cdot dz+\int_{\frac{S}{1-P_2+Q_2}}^{\frac{S}{1-P_2}}\frac{S-z+zP_2}{zQ_2}\cdot dz} \tag{28}$$

$$=\frac{\frac{1}{Q_1}\left[\frac{S^2\left(1-P_1\right)}{2\left(1-P_1-Q_1\right)^2}+\frac{S^2}{2\left(1-P_1\right)}-\frac{S^2}{1-P_1-Q_1}\right]+\frac{1}{Q_2}\left[\frac{S^2}{2\left(1-P_2\right)}-\frac{S^2}{1-P_2+Q_2}-\frac{S^2\left(P_2-1\right)}{2\left(1-P_2+Q_2\right)^2}\right]}{\frac{1}{Q_1}\left[\frac{S\left(1-P_1\right)}{1-P_1-Q_1}-S+S\ln\left(\frac{1-P_1-Q_1}{1-P_1}\right)\right]+\frac{1}{Q_2}\left[S\ln\left(\frac{1-P_2+Q_2}{1-P_2}\right)-S-\frac{S\left(P_2-1\right)}{1-P_2+Q_2}\right]}$$

where $S=\sum_{i=1}^{n}s_i$, $P_1=\sum_{i=1}^{n}d_{i1}/p_i$, $P_2=\sum_{i=1}^{n}d_{i2}/p_i$, $Q_1=\sum_{i=1}^{n}\left(d_{i0}-d_{i1}\right)/p_i$, and $Q_2=\sum_{i=1}^{n}\left(d_{i2}-d_{i0}\right)/p_i$.

When it holds that $T^{\min}\le T^{\alpha}$, one may secure the expression for the optimal fuzzy average total cost as follows.

$$ASIC^{CC}=\sum_{i=1}^{n}\left[\frac{a_i}{T}+\frac{h_i}{2}\tilde{D}_i\left(1-\frac{\tilde{D}_i}{p_i}\right)T\right]=\left[\sqrt{2\sum_{i=1}^{n}a_i\sum_{i=1}^{n}H_{iL}^{\alpha}},\sqrt{2\sum_{i=1}^{n}a_i\sum_{i=1}^{n}H_{iU}^{\alpha}}\right] \tag{29}$$

3. THEORETICAL ANALYSIS ON THE *ELSP* WITH FUZZY DEMANDS

Since the conventional *ELSP* is *NP*-hard, it is more difficult for us to secure an optimal solution for the *ELSP* with fuzzy demands. Recall that many heuristics are proposed to solve the conventional *ELSP,* and these heuristics usually involve partial enumeration and require extensive computation efforts in general. For such a reason, one might have to use heuristics or meta-heuristics (*e.g.*, simulated annealing or genetic algorithms, *etc*) to obtain "good" and implementable solutions.

On the other hand, it is worthwhile to investigate when the *CC* approach is nearly optimal in many real world situations since the solution of the *CC* approach can be secured much more easily than other solution approaches.

In the following discussion, we first study when the fuzzy *ELSP* is easy since the *CC* approach in fact secures the optimal solution. Then, we analyze the sensitivity of these conditions; that is, to investigate how the quality of the solution from the *CC* approach is affected when we deviate from these conditions.

3.1 The optimality conditions

First, we state Lemma 1 which in fact provides the conditions for the *CC* approach to secure the optimal solution for the fuzzy *ELSP* as follows.

Lemma 1. If the following two conditions hold,

Condition 1: $\frac{a_1}{H_1^{\alpha}}=\frac{a_2}{H_2^{\alpha}}=\ldots=\frac{a_n}{H_n^{\alpha}}$

Condition 2: $T^{\min}\le T^{\alpha}$,

Then, $T_i^{IS}=T^{CC}$, *for all i=1…n* 。

Proof. Please refer to Appendix A.1 for the details of the proof.

The proof of Theorem 1 in the following discussion needs Lemma 2 as follows.

Lemma 2. The defuzzified value for $D_i^{\alpha}=\left[\left(d_{i0}-d_{i1}\right)\alpha+d_{i1},d_{i2}-\alpha\left(d_{i2}-d_{i0}\right)\right]$ is given by

$$\tfrac{1}{3}\left(d_{i1}+d_{i0}+d_{i2}\right). \tag{30}$$

Proof. Please refer to Appendix A.2 for the details of the proof.

Theorem 1. Whenever Conditions 1 and 2 in Lemma 1 hold, the *CC* approach secures the same solution as the *IS* for the fuzzy *ELSP*.

Proof. We know it from Lemma 1 that $T^{CC}=T_1^{IS}=T_2^{IS}=T_3^{IS}=\cdots=T_n^{IS}$, *i.e.*, all the *n* products share the same replenishment cycle time. The processing time and the replenishment batch quantity for product *i* for the *CC* approach are $\tau_i=T^{CC}\rho_i=\frac{1}{3}T^{CC}\left(d_{i1}+d_{i0}+d_{i2}\right)/p_i$, and $q_i=\frac{1}{3}\left(d_{i1}+d_{i0}+d_{i2}\right)T^{CC}$, respectively. And, those for the *IS* are $\tau_i=\frac{1}{3}T_i^{IS}\left(d_{i1}+d_{i0}+d_{i2}\right)/p_i$ and $q_i=\frac{1}{3}(d_{i1}+d_{i0}+d_{i2})T_i^{IS}$, respectively. Obviously, when $T^{CC}=T_i^{IS}$, one may use the production schedule of the *CC* approach to establish a feasible production schedule for the *IS*. Therefore, when Conditions 1 and 2 in Lemma 1 hold, the *CC* approach secures the same solution as the *IS*.

The following corollary concludes our assertion on the optimality conditions.

Corollary 1. Whenever Conditions 1 and 2 in Lemma 1 hold, the *CC* approach secures the optimal solution for the fuzzy *ELSP*.

Proof. First, recall that the *IS* provide a lower bound for the average total cost function for the fuzzy *ELSP*. But, the *IS* usually secures infeasible solution since the *IS* ignore the constraint that only one product can be produced at one time point. On the other hand, recall that the solution from the *CC* approach is always feasible, and the when the solution of the *IS* is feasible, it is also optimal. Therefore, when these two solutions coincide, both solutions must be feasible and optimal. By Theorem 1, when Conditions 1 and 2 in Lemma 1 hold, the *ELSP* with fuzzy demand $\tilde{D}_i$ for each product *i*, the *CC* approach secures the same solution as the *IS*. Therefore, whenever Conditions 1 and 2 in Lemma 1 hold, the *ELSP* with fuzzy demand $\tilde{D}_i$ for each product *i*, the *CC* approach secures the optimal solution.

We note that the optimality conditions discussed above commonly occur in industry. Whenever all the product's a_i/H_i^α ratios are the same and the setup time is relatively short (so that Condition 2 holds), the *CC* approach provides the optimal solution. This case often happens when the single facility produces many mirror image parts which are used to assemble a final product.

But, what if the a_i/H_i^α ratios are *not* the same? We answer this question by presenting an error bound which results from our sensitivity analysis on the optimality conditions in the next section.

3.2 Sensitivity analysis on the optimality conditions

We are motivated to study how close to optimal the *CC* approach is when the a_i/H_i^α ratios are *almost* the same. Here, we assume that $T_i^{IS}=T_i^\alpha$ and $T^{CC}=T^\alpha$ for clarity and the setup time is relatively short. Without loss of generality, we sort the products in a descending order in their a_i/H_i^α ratios. Hence, product 1 has the largest ratio, and let λ_i be the ratio of a_i/H_i^α over a_1/H_1^α, *i.e.*, for $i = 1, \ldots, n$,

$$\frac{a_i}{H_i^\alpha}=\lambda_i\frac{a_1}{H_1^\alpha}. \tag{31}$$

Obviously, $\lambda_1=1$ and $0\le\lambda_i\le 1$, for $i=2\ldots n$. When all the values of $\lambda_i's=1$, it holds for Condition 1, the *CC* approach secures the optimal solution.

Although we do not know the optimal average total cost when Conditions 1 and 2 in Lemma 1 do not hold, the average total cost of the *IS* provides us an easy lower bound. On the other hand, the *CC* approach is always feasible, and it gives an obvious upper bound. Therefore, we may use $\left(ASIC^{CC}-ASIC^{IS}\right)/ASIC^{CC}$ as an index to evaluate the solution quality of the *CC* approach. Importantly, we shall observe how $\left(ASIC^{CC}-ASIC^{IS}\right)/ASIC^{CC}$ is bounded above by a function of $\{\lambda_i\}$.

Theorem 2. For the fuzzy *ELSP* with $T^{\min}\le T^\alpha$ and $T_i^{\min}\le T_i^\alpha$ *for all i*,

$$\frac{ASIC^{CC}-ASIC^{IS}}{ASIC^{IS}}=\left[\sqrt{\frac{\sum_{i=1}^n\lambda_iH_{iU}^\alpha\sum_{i=1}^nH_{iL}^\alpha}{\left(\sum_{i=1}^n\left(\lambda_iH_{iU}^\alpha H_{iL}^\alpha\right)^{1/2}\right)^2}}-1,\sqrt{\frac{\sum_{i=1}^n\lambda_iH_{iL}^\alpha\sum_{i=1}^nH_{iU}^\alpha}{\left(\sum_{i=1}^n\left(\lambda_iH_{iU}^\alpha H_{iL}^\alpha\right)^{1/2}\right)^2}}-1\right]. \tag{32}$$

Proof. Recall that by (18), $ASIC^{IS}=\left[\sum_{i=1}^n\sqrt{2a_iH_{iL}^\alpha},\sum_{i=1}^n\sqrt{2a_iH_{iU}^\alpha}\right]$, and

$ASIC^{CC}=\left[\sqrt{2\sum_{i=1}^na_i\sum_{i=1}^nH_{iL}^\alpha},\sqrt{2\sum_{i=1}^na_i\sum_{i=1}^nH_{iU}^\alpha}\right]$ by (29). Since $\frac{a_i}{H_i^\alpha}=\lambda_i\frac{a_1}{H_1^\alpha}$, we have

$$\frac{a_i}{H_i^\alpha}=\left[\frac{a_i}{H_{iU}^\alpha},\frac{a_i}{H_{iL}^\alpha}\right]=\left[\lambda_i\frac{a_1}{H_{1U}^\alpha},\lambda_i\frac{a_1}{H_{1L}^\alpha}\right]. \tag{33}$$

By plugging (33) into $ASIC^{IS}$ and $ASIC^{CC}$, we have

$$ASIC^{IS}=\left[\sum_{i=1}^n\sqrt{2a_iH_{iL}^\alpha},\sum_{i=1}^n\sqrt{2a_iH_{iU}^\alpha}\right]=\left[\sum_{i=1}^n\sqrt{2\lambda_iH_{iU}^\alpha\frac{a_1}{H_{1U}^\alpha}H_{iL}^\alpha},\sum_{i=1}^n\sqrt{2\lambda_iH_{iL}^\alpha\frac{a_1}{H_{1L}^\alpha}H_{iU}^\alpha}\right],$$

and

$$ASIC^{CC}=\left[\sqrt{2\sum_{i=1}^{n}a_i\sum_{i=1}^{n}H_{iL}^{\alpha}},\sqrt{2\sum_{i=1}^{n}a_i\sum_{i=1}^{n}H_{iU}^{\alpha}}\right]=\left[\sqrt{2\sum_{i=1}^{n}\lambda_i H_{iU}^{\alpha}\frac{a_1}{H_{1U}^{\alpha}}\sum_{i=1}^{n}H_{iL}^{\alpha}},\sqrt{2\sum_{i=1}^{n}\lambda_i H_{iL}^{\alpha}\frac{a_1}{H_{1L}^{\alpha}}\sum_{i=1}^{n}H_{iU}^{\alpha}}\right]$$. Therefore, we have

$$\frac{ASIC^{CC}-ASIC^{IS}}{ASIC^{IS}}=\left[\frac{\sqrt{2\sum_{i=1}^{n}\lambda_i H_{iU}^{\alpha}\frac{a_1}{H_{1U}^{\alpha}}\sum_{i=1}^{n}H_{iL}^{\alpha}}}{\sum_{i=1}^{n}\sqrt{2\lambda_i H_{iU}^{\alpha}\frac{a_1}{H_{1U}^{\alpha}}H_{iL}^{\alpha}}}-1,\frac{\sqrt{2\sum_{i=1}^{n}\lambda_i H_{iL}^{\alpha}\frac{a_1}{H_{1L}^{\alpha}}\sum_{i=1}^{n}H_{iU}^{\alpha}}}{\sum_{i=1}^{n}\sqrt{2\lambda_i H_{iL}^{\alpha}\frac{a_1}{H_{1L}^{\alpha}}H_{iU}^{\alpha}}}-1\right]$$

$$=\left[\frac{\sqrt{\sum_{i=1}^{n}\lambda_i H_{iU}^{\alpha}\sum_{i=1}^{n}H_{iL}^{\alpha}}}{\sum_{i=1}^{n}\sqrt{\lambda_i H_{iU}^{\alpha}H_{iL}^{\alpha}}}-1,\frac{\sqrt{\sum_{i=1}^{n}\lambda_i H_{iL}^{\alpha}\sum_{i=1}^{n}H_{iU}^{\alpha}}}{\sum_{i=1}^{n}\sqrt{\lambda_i H_{iL}^{\alpha}H_{iU}^{\alpha}}}-1\right]$$

$$=\left[SQRT\left(\frac{\sum_{i=1}^{n}\lambda_i H_{iU}^{\alpha}\sum_{i=1}^{n}H_{iL}^{\alpha}}{\left(\sum_{i=1}^{n}\left(\lambda_i H_{iU}^{\alpha}H_{iL}^{\alpha}\right)^{1/2}\right)^2}\right)-1,SQRT\left(\frac{\sum_{i=1}^{n}\lambda_i H_{iL}^{\alpha}\sum_{i=1}^{n}H_{iU}^{\alpha}}{\left(\sum_{i=1}^{n}\left(\lambda_i H_{iL}^{\alpha}H_{iU}^{\alpha}\right)^{1/2}\right)^2}\right)-1\right].$$

4. AN NUMERICAL EXAMPLE

In this section, we employ a 10-product example to demonstrate how to secure the solution for the *ELSP* with fuzzy demands using the *IS* and the *CC* approach. We also use this example to validate our error bound derived in Theorem 2.

In Table 1, we present the following data set (from Bomberger, 1966) for this 10-product example. We note that the third column is the crisp annual demand rate and the fourth to the sixth columns indicate the fuzzy demand rate "approximate d_{i0}" with triangular membership function.

We first summarize the computation data needed for the *IS* solution in Table 2. For the *IS*, the optimal fuzzy average total cost is secured by $ASIC^{IS}$ = [\$7,529, \$7,627]. For the conventional *ELSP*, we secure the optimal average total cost for the *IS* by \$7,589. The fuzzy replenishment cycle times, *i.e.*, (16), are listed in the 8th and the 9th columns of Table 2. The defuzzified replenishment cycle times from the method of centroid, *i.e.*, $M_i(\tilde{D}_i)$, are shown in the tenth column of Table 2.

For the *CC* approach, one can secure T^{α} = [42.599,43.021] and $T^{\min}$ = [29.040,33.200]. Therefore, we have T^{CC} = [29.040,43.021], and the defuzzified replenishment cycle time, *i.e.*, $M(\tilde{D}_i)$, is secured by 29.68. The optimal fuzzy average total cost for the *CC* approach, is secured by $ASIC^{CC}$ = [\$9,818, \$9,916]. On the other hand, for the conventional *ELSP*, we secure the crisp replenishment (common) cycle time and the optimal average total cost for the *CC* approach by 42.75 and \$9,880, respectively.

Table 1. The data set for the 10 products in the example

Product	a_i	d_i	d_{i1}	d_{i0}	d_{i2}	p_i	s_i	h_i
Unit	(\$)	(year)	(year)	(year)	(year)	(year)	(days)	(year)
1	15	96000	92000	96000	100000	7200000	0.125	0.00065
2	20	96000	92000	96000	104000	1920000	0.125	0.01775
3	30	192000	132000	192000	200000	2280000	0.25	0.01275
4	10	384000	320000	384000	400000	1800000	0.125	0.01
5	110	19200	12000	19200	22000	480000	0.5	0.2785
6	50	19200	14000	19200	20800	1440000	0.25	0.02675
7	310	5760	4000	5760	7600	576000	1	0.15
8	130	81600	76000	81600	84000	312000	0.5	0.59
9	200	81600	78276	81600	86000	480000	0.75	0.09
10	5	96000	80000	96000	104000	3600000	0.125	0.004

Table 2. The computation for the *IS* and the error bound

Product	H_{iL}^{α}	H_{iU}^{α}	t_{iL}^{α}	t_{iU}^{α}	$T_i^{\min}$		T_i^{IS}		$M_i(\tilde{D}_i)$	a_i/H_i^{α}	λ_i
1	0.255	0.258	10.791	10.837	0.127	0.127	0.127	10.837	0.698	58.472	0.673
2	6.714	6.802	2.425	2.441	0.132	0.132	0.131	2.441	0.157	2.959	0.034
3	9.046	9.407	2.526	2.575	0.272	0.273	0.272	2.575	0.273	3.252	0.037
4	12.362	12.696	1.255	1.272	0.158	0.159	0.158	1.272	0.159	0.798	0.009
5	20.574	21.735	3.182	3.270	0.520	0.521	0.520	3.270	0.521	5.200	0.060
6	2.054	2.130	6.852	6.977	0.253	0.253	0.253	6.977	0.444	23.901	0.275
7	3.454	3.679	12.982	13.398	1.010	1.010	1.010	13.398	1.010	86.920	1
8	146.966	148.932	1.321	1.330	0.675	0.678	0.675	1.330	0.677	0.879	0.010
9	25.267	25.556	3.956	3.979	0.903	0.905	0.903	3.979	0.904	7.870	0.091
10	1.531	1.571	2.523	2.556	0.128	0.128	0.128	2.556	0.164	3.224	0.037

Next, we verify our theoretical results derived in Section 3. First, one should observe the values of λ_i in the last column of Table 2 are not close to 1 at all, and it does not meet the condition stated in Lemma 1. Clearly, in this instance, the *CC* approach is unable to secure a solution which is close to the optimal solution of the fuzzy *ELSP*, and the additional effort of using a sophisticated heuristic, *e.g.*, using a genetic algorithm (Chang, *et al*, 2002), etc., is justified. Next, we secure the error bound from Theorem 2, *i.e.*, (32), by [0.3471,0.3532]. We note that the optimal fuzzy average total costs for the *IS* and the *CC* approach are secured by $ASIC^{IS}$ = [\$7,529, \$7,627] and $ASIC^{CC}$ = [\$9,818, \$9,916], respectively. Therefore, the actual error from $ASIC^{IS}$ and $ASIC^{CC}$ is within the interval of [0.2873,0.3170] which is less than all the values in the error bound interval of [0.3471,0.3532].

5. CONCLUDING REMARKS

In this paper, we investigate the economic lot scheduling problem (*ELSP*) with fuzzy demands. We assume that the demand for each product i can be approximated using some triangular membership functions. In this study, we solve the fuzzy ELSP using two basic solution approaches, namely, using the Independent Solution (*IS*) and the Common Cycle (*CC*) approach. For both approaches, we derive the optimal fuzzy replenishment cycles and secure closed-form formula for their crisp figures in fuzzy sense, respectively. Also, we derive the conditions that assert the *CC* approach to secure the optimal solution for the fuzzy *ELSP* in many realistic situations. For the cases that deviate from those optimal-situations, we give an upper bound for the maximum error of the solution of the *CC* approach from optimality.

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APPENDIX

A.1 The Proof of Lemma 1

Proof. First recall that by (15) and (16), we have $T_i^{\min} = [s_i \Big/ \left(1-\left(\frac{d_{i1}}{p_i}+\frac{(d_{i0}-d_{i1})\alpha}{p_i}\right)\right), s_i \Big/ \left(1-\left(\frac{d_{i2}}{p_i}+\frac{(d_{i0}-d_{i2})\alpha}{p_i}\right)\right)]$ and

$$T^{\min} = \left[\sum_{i=1}^{n} s_i \Bigg/ \left(1-\sum_{i=1}^{n}\left(\frac{d_{i1}}{p_i}+\frac{(d_{i0}-d_{i1})}{p_i}\alpha\right)\right), \sum_{i=1}^{n} s_i \Bigg/ \left(1-\sum_{i=1}^{n}\left(\frac{d_{i2}}{p_i}+\frac{(d_{i0}-d_{i2})}{p_i}\alpha\right)\right)\right].$$

Since $s_i \le \sum_{i=1,\dots,n} s_i$ and $\frac{d_{i1}}{p_i}+\frac{(d_{i0}-d_{i1})\alpha}{p_i} \le \sum_{i=1,\dots n}\left(\frac{d_{i1}}{p_i}+\frac{(d_{i0}-d_{i1})\alpha}{p_i}\right)$, we have $1-\left(\frac{d_{i1}}{p_i}+\frac{(d_{i0}-d_{i1})\alpha}{p_i}\right) \ge 1-\sum_{i=1,\dots n}\left(\frac{d_{i1}}{p_i}+\frac{(d_{i0}-d_{i1})\alpha}{p_i}\right)$ and hence,

$$s_i \Big/ \left(1-\left(\frac{d_{i1}}{p_i}+\frac{(d_{i0}-d_{i1})\alpha}{p_i}\right)\right) \le \sum_{i=1,\dots n} s_i \Big/ \left(1-\sum_{i=1,\dots n}\left(\frac{d_{i1}}{p_i}+\frac{(d_{i0}-d_{i1})\alpha}{p_i}\right)\right). \tag{34}$$

Similarly, since $\frac{d_{i2}}{p_i}+\frac{(d_{i2}-d_{i0})\alpha}{p_i} \le \sum_{i=1,\dots n}\left(\frac{d_{i2}}{p_i}+\frac{(d_{i2}-d_{i0})\alpha}{p_i}\right)$, we have $1-\left(\frac{d_{i2}}{p_i}+\frac{(d_{i2}-d_{i0})\alpha}{p_i}\right) \ge 1-\sum_{i=1,\dots n}\left(\frac{d_{i2}}{p_i}+\frac{(d_{i2}-d_{i0})\alpha}{p_i}\right)$ and hence,

$$s_i \Big/ \left(1-\left(\frac{d_{i2}}{p_i}+\frac{(d_{i2}-d_{i0})\alpha}{p_i}\right)\right) \le \sum_{i=1,\dots n} s_i \Big/ \left(1-\sum_{i=1,\dots n}\left(\frac{d_{i2}}{p_i}+\frac{(d_{i2}-d_{i0})\alpha}{p_i}\right)\right). \tag{35}$$

Therefore, by (34) and (35), we assert that for all $i=1,\ldots,n$,

$$T_i^{\min} \le T^{\min}. \tag{36}$$

Recall that $T_i^\alpha = [\sqrt{2a_i / H_{iU}^\alpha}, \sqrt{2a_i / H_{iL}^\alpha}]$ and $T^\alpha = \left[\sqrt{2\sum_{i=1}^n a_i \Big/ \sum_{i=1}^n H_{iU}^\alpha}, \sqrt{2\sum_{i=1}^n a_i \Big/ \sum_{i=1}^n H_{iL}^\alpha}\right]$. By Condition 1, we have

$\dfrac{a_1}{H_1^\alpha} = \dfrac{a_i}{H_i^\alpha}$, for all $i=1\ldots n$, *i.e.,*

$$\frac{a_1}{H_1^\alpha} = \frac{a_1}{\left[H_{1L}^\alpha, H_{1U}^\alpha\right]} = \left[\frac{a_1}{H_{1U}^\alpha}, \frac{a_1}{H_{1L}^\alpha}\right], \ldots, \quad \frac{a_i}{H_i^\alpha} = \frac{a_i}{\left[H_{iL}^\alpha, H_{iU}^\alpha\right]} = \left[\frac{a_i}{H_{iU}^\alpha}, \frac{a_i}{H_{iL}^\alpha}\right]. \tag{37}$$

Observing each term in (37), we have $\dfrac{a_1}{H_{1U}^\alpha} = \dfrac{a_2}{H_{2U}^\alpha} = \cdots = \dfrac{a_i}{H_{iU}^\alpha}$ and $\dfrac{a_1}{H_{1L}^\alpha} = \dfrac{a_2}{H_{2L}^\alpha} = \cdots = \dfrac{a_i}{H_{iL}^\alpha}$ for the lower and upper bounds, respectively. Then, the following expression holds for all $i=2\ldots n$,

$$a_i = \left[\frac{H_{iU}^\alpha}{H_{1U}^\alpha} a_1, \frac{H_{iL}^\alpha}{H_{1L}} a_1\right]. \tag{38}$$

By plugging a_i in (38) into T_i^α and T^α, we have $T_1^\alpha = [\sqrt{2a_1 / H_{1U}^\alpha}, \sqrt{2a_1 / H_{1L}^\alpha}]$,

$T_2^\alpha = [\sqrt{2a_2 / H_{2U}^\alpha}, \sqrt{2a_2 / H_{2L}^\alpha}] = [\sqrt{2\left(H_{2U}^\alpha / H_{1U}^\alpha\right)a_1 / H_{2U}^\alpha}, \sqrt{2\left(H_{2L}^\alpha / H_{1L}^\alpha\right)a_1 / H_{2L}^\alpha}] = [\sqrt{2a_1 / H_{1U}^\alpha}, \sqrt{2a_1 / H_{1L}^\alpha}] = T_1^\alpha$, …,

$T_n^\alpha = [\sqrt{2a_n / H_{nU}^\alpha}, \sqrt{2a_n / H_{nL}^\alpha}] = [\sqrt{2\left(H_{nU}^\alpha / H_{1U}^\alpha\right)a_1 / H_{nU}^\alpha}, \sqrt{2\left(H_{nL}^\alpha / H_{1L}^\alpha\right)a_1 / H_{nL}^\alpha}] = [\sqrt{2a_1 / H_{1U}^\alpha}, \sqrt{2a_1 / H_{1L}^\alpha}] = T_1^\alpha$. Therefore,

$$T_1^\alpha = T_2^\alpha = \cdots = T_n^\alpha = [\sqrt{2a_1 / H_{1U}^\alpha}, \sqrt{2a_1 / H_{1L}^\alpha}]. \tag{39}$$

On the other hand,

$$\begin{aligned}
T^\alpha &= \left[\sqrt{2\sum_{i=1}^n a_i \Big/ \sum_{i=1}^n H_{iU}^\alpha}, \sqrt{2\sum_{i=1}^n a_i \Big/ \sum_{i=1}^n H_{iL}^\alpha}\right] \\
&= \left[\sqrt{\frac{2\left(a_1 + a_2 + a_3 + \cdots + a_n\right)}{\left(H_{1U}^\alpha + H_{2U}^\alpha + H_{3U}^\alpha + \cdots + H_{nU}^\alpha\right)}}, \sqrt{\frac{2\left(a_1 + a_2 + a_3 + \cdots + a_n\right)}{\left(H_{1L}^\alpha + H_{2L}^\alpha + H_{3L}^\alpha + \cdots + H_{nL}^\alpha\right)}}\right] \\
&= \left[\sqrt{\frac{2\left(a_1 + \frac{H_{2U}^\alpha}{H_{1U}^\alpha} a_1 + \frac{H_{3U}^\alpha}{H_{1U}^\alpha} a_1 + \cdots + \frac{H_{nU}^\alpha}{H_{1U}^\alpha} a_1\right)}{\left(H_{1U}^\alpha + H_{2U}^\alpha + H_{3U}^\alpha + \cdots + H_{nU}^\alpha\right)}}, \sqrt{\frac{2\left(a_1 + \frac{H_{2L}^\alpha}{H_{1L}^\alpha} a_1 + \frac{H_{3L}^\alpha}{H_{1L}^\alpha} a_1 + \cdots + \frac{H_{nL}^\alpha}{H_{1L}^\alpha} a_1\right)}{\left(H_{1L}^\alpha + H_{2L}^\alpha + H_{3L}^\alpha + \cdots + H_{nL}^\alpha\right)}}\right] \\
&= \left[\sqrt{\frac{2a_1\left(1 + \frac{H_{2U}^\alpha}{H_{1U}^\alpha} + \frac{H_{3U}^\alpha}{H_{1U}^\alpha} + \cdots + \frac{H_{nU}^\alpha}{H_{1U}^\alpha}\right)}{\left(H_{1U}^\alpha + H_{2U}^\alpha + H_{3U}^\alpha + \cdots + H_{nU}^\alpha\right)}}, \sqrt{\frac{2a_1\left(1 + \frac{H_{2L}^\alpha}{H_{1L}^\alpha} + \frac{H_{3L}^\alpha}{H_{1L}^\alpha} + \cdots + \frac{H_{nL}^\alpha}{H_{1L}^\alpha}\right)}{\left(H_{1L}^\alpha + H_{2L}^\alpha + H_{3L}^\alpha + \cdots + H_{nL}^\alpha\right)}}\right] \\
&= \left[\sqrt{\frac{2a_1\left(H_{1U}^\alpha + H_{2U}^\alpha + H_{3U}^\alpha + \cdots + H_{nU}^\alpha\right)}{H_{1U}^\alpha\left(H_{1U}^\alpha + H_{2U}^\alpha + H_{3U}^\alpha + \cdots + H_{nU}^\alpha\right)}}, \sqrt{\frac{2a_1\left(H_{1L}^\alpha + H_{2L}^\alpha + H_{3L}^\alpha + \cdots + H_{nL}^\alpha\right)}{H_{1L}^\alpha\left(H_{1L}^\alpha + H_{2L}^\alpha + H_{3L}^\alpha + \cdots + H_{nL}^\alpha\right)}}\right] \\
&= \left[\frac{2a_1}{H_{1U}^\alpha}, \frac{2a_1}{H_{1L}^\alpha}\right].
\end{aligned} \tag{40}$$

Therefore, it holds that

$$T^{\alpha} = T_1^{\alpha} = \cdots = T_n^{\alpha}. \tag{41}$$

By combining Condition 2, (36) and (41), we assert that $T_i^{\min} \le T^{\min} \le T^{\alpha} = T_i^{\alpha}$, for all *i=1…n*. By (16) and (17), we have $T_i^{IS} = T_i^{\alpha}$ and $T^{CC} = T^{\alpha}$, for all *i=1…n*. Therefore, $T_i^{IS} = T^{CC}$, for all *i=1…n*.

A.2 The Proof of Lemma 2

Proof. Recall that $D_i^{\alpha} = \left[\left(d_{i0} - d_{i1}\right)\alpha + d_{i1}, d_{i2} - \alpha\left(d_{i2} - d_{i0}\right)\right]$. Therefore, the $\alpha - cut$ for the lower bound is given by $z = \left(d_{i0} - d_{i1}\right)\alpha + d_{i1}$, and $u_D\left(z\right) = \alpha = \dfrac{z - d_{i1}}{d_{i0} - d_{i1}}$ where $z = d_{i1}$ when $\alpha = 0$, and $z = d_{i0}$ when $\alpha = 1$. On the other hand, the $\alpha - cut$ for the upper bound is given by $z = d_{i2} - \alpha\left(d_{i2} - d_{i0}\right)$, and $u_D\left(z\right) = \alpha = \dfrac{d_{i2} - z}{d_{i2} - d_{i0}}$ where $z = d_{i2}$ when $\alpha = 0$, and $z = d_{i0}$ when $\alpha = 1$.

Hence, the defuzzified value from the method of centroid is given by

$$\frac{\int_{d_{i1}}^{d_{i0}} \frac{z - d_{i1}}{d_{i0} - d_{i1}} \cdot z \cdot dz + \int_{d_{i0}}^{d_{i2}} \frac{d_{i2} - z}{d_{i2} - d_{i0}} \cdot z \cdot dz}{\int_{d_{i1}}^{d_{i0}} \frac{z - d_{i1}}{d_{i0} - d_{i1}} \cdot dz + \int_{d_{i0}}^{d_{i2}} \frac{d_{i2} - z}{d_{i2} - d_{i0}} \cdot dz}$$

$$= \frac{\frac{1}{\left(d_{i0} - d_{i1}\right)} \int_{d_{i1}}^{d_{i0}} \left(z - d_{i1}\right) z \cdot dz + \frac{1}{\left(d_{i2} - d_{i0}\right)} \int_{d_{i0}}^{d_{i2}} \left(d_{i2} - z\right) z \cdot dz}{\frac{1}{\left(d_{i0} - d_{i1}\right)} \int_{d_{i1}}^{d_{i0}} \left(z - d_{i1}\right) \cdot dz + \frac{1}{\left(d_{i2} - d_{i0}\right)} \int_{d_{i0}}^{d_{i2}} \left(d_{i2} - z\right) \cdot dz}$$

$$= \frac{\frac{1}{\left(d_{i0} - d_{i1}\right)} \left(\frac{1}{3} z^3 - \frac{1}{2} d_{i1} z^2\right)\Bigg|_{d_{i1}}^{d_{i0}} + \frac{1}{\left(d_{i2} - d_{i0}\right)} \left(\frac{1}{2} d_{i2} z^2 - \frac{1}{3} z^3\right)\Bigg|_{d_{i0}}^{d_{i2}}}{\frac{1}{\left(d_{i0} - d_{i1}\right)} \left(\frac{1}{2} z^2 - d_{i1} z\right)\Bigg|_{d_{i1}}^{d_{i0}} + \frac{1}{\left(d_{i2} - d_{i0}\right)} \left(d_{i2} z - \frac{1}{2} z^2\right)\Bigg|_{d_{i0}}^{d_{i2}}}$$

$$= \frac{\frac{1}{\left(d_{i0} - d_{i1}\right)} \left[\frac{1}{3}\left(d_{i0}^3 - d_{i1}^3\right) - \frac{1}{2} d_{i1}\left(d_{i0}^2 - d_{i1}^2\right)\right] + \frac{1}{\left(d_{i2} - d_{i0}\right)} \left[\frac{1}{2} d_{i2}\left(d_{i2}^2 - d_{i0}^2\right) - \frac{1}{3}\left(d_{i2}^3 - d_{i0}^3\right)\right]}{\frac{1}{\left(d_{i0} - d_{i1}\right)} \left[\frac{1}{2}\left(d_{i0}^2 - d_{i1}^2\right) - d_{i1}\left(d_{i0} - d_{i1}\right)\right] + \frac{1}{\left(d_{i2} - d_{i0}\right)} \left[d_{i2}\left(d_{i2} - d_{i0}\right) - \frac{1}{2}\left(d_{i2}^2 - d_{i0}^2\right)\right]}$$

$$= \frac{\dfrac{d_{i2}^2 - d_{i1}^2 + d_{i2} d_{i0} - d_{i0} d_{i1}}{6}}{\dfrac{d_{i2} - d_{i1}}{2}} = \frac{d_{i1} + d_{i0} + d_{i2}}{3}.$$

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