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Dynamic Pricing Model on the Internet Market

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Abstract—In many industries, sellers have the opportunity to enhance their revenues through the dynamic pricing of their perishable products such as flight seats, hotel rooms, or seasonal fashion goods that become worthless if they are not sold by a specific time. Therefore, how to dynamically adjust the prices of perishable products through differentiating the purchased time and the amount of unsold items to maximize the revenue is an important issue. Due to the immediate response and lower menu cost on the Internet, the application of the dynamic pricing to the Internet market is especially appropriate. In this paper we construct a dynamic pricing model for selling a given stock of identical perishable products over a finite time horizon on the Internet. We then propose three theorems to demonstrate the properties of the expected revenue and the time thresholds in the model. A numerical example is presented to illustrate the model and its results.

Keywords—Dynamic pricing, Perishable products, Revenue management, Internet market

1. INTRODUCTION

In many industries, sellers and service providers have the opportunity to enhance their revenues through the dynamic pricing of their perishable products that must be sold within a finite period of time. The industry which is most commonly mentioned in terms of its adopting dynamic pricing strategies is airline transportation. Similarly, items such as hotel rooms, sports tickets, and seasonal fashion goods that become worthless if they are not sold by a specific time, are all suited to dynamic pricing because these perishable items must be sold prior to the time at which they are unsalable. In all of these cases, the sellers can improve their revenue by dynamically adjusting the price of the perishable products rather than adopting a fixed price throughout the product's market life. Obviously, the price policies on perishable products are affected by the length of time remaining before the products perish as well as by the numbers of unsold inventory. Intuition suggests that when inventory is low or when there is plenty of time before the product perishes, the seller can post a higher price than when inventory is high or when the end of the product life draws near (Chatwin, 2000). In fact, how a retailer should dynamically adjust the prices of perishable products through differentiating the time of purchase and the number of yet unsold items is an important issue.

Due to the rapid growth of the Internet, the practical application of dynamic pricing to the Internet market provides the motivation for the research reported in this paper. The application of the dynamic pricing to the Internet market is especially valuable, according to several factors: (1) immediate response on the Internet (Gallego and van Ryzin, 1994; Kannan and Kopalle, 2001); (2) lower menu costs on the Internet (Brynjolfsson and Smith, 2000; Gallego and van Ryzin, 1994); and (3) purchasing convenience on the Internet (Boston Consulting Group, 2000; Brynjolfsson and Smith, 2000; Cao et al., 2003). These factors contribute to the development of dynamic pricing strategies on the Internet market, and the Internet environment is also quite well suited to dynamic pricing strategies. Therefore, dynamic pricing has increasingly become a necessary mechanism for companies on the Internet to improve profitability, maintain competitiveness, balance supply and demand, and manage risks.

Several researchers have addressed dynamic pricing models. Gallego and van Ryzin (1994) derive an optimal pricing policy when demand functions are exponential. For general demand functions, they analyze a deterministic version of the problem and find an upper bound on the expected revenue. Gallego and van Ryzin (1997) extend the single-item model to allow time-varying demand and multi-items with a network structure. There are various studies of pricing policies in the continuous-time revenue management. For example, Bitran and Mondschein (1997) present a continuous time model for the retailing industry and characterize the optimal pricing strategy as a function of the inventory and the time left in the planning horizon. In addition, Feng and Gallego (1995) obtain an optimal timing for the model with two predetermined prices and a single price switch when the demand is price sensitive and stochastic. Feng and Xiao (2000) further extend the results by incorporating multiple price levels. Furthermore, Feng and Gallego (2000) extend the previous model by assuming time-dependent or Markovian demand and fares. Feng and Xiao (2000) construct a model in which reversible price changes are allowed. Chatwin (2000) presents some properties of the dynamic pricing model in a finite set of prices, showing that the maximum expected revenue is nondecreasing and concave in both the remaining inventory and the time-to-go.

Most models in the literature (e.g., Feng and Gallego (1995)) assume that demand is a homogeneous Poisson

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process with a constant intensity. However, Zhao and Zheng (2000) study a dynamic pricing model where customers arrive according to a nonhomogeneous Poisson process and their reservation price distribution changes over time. On the other hand, by assuming that the customer's demand is represented as a negative binomial distribution, Chun (2003) presents the optimal pricing policy based on the demand rate, customers' preferences, and the length of the sales period. Lin (2004) proposes a sequential dynamic pricing model where the seller sells a given stock to a random number of customers. He also formulates the seller's problem as a stochastic dynamic programming model, and develops an algorithm to compute the optimal policy.

The remainder of this paper is organized as follows. In Section 2 we describe the problem statement along with the assumptions and notations in this study. The dynamic pricing model and formulation are given in Section 3. In Section 4 we show the properties of the optimal pricing policy. A numerical example is presented in Section 5, and Section 6 concludes the paper.

2. PROBLEM STATEMENT

In this paper we will construct a dynamic pricing model for selling a given stock of identical perishable products over a finite time horizon on the Internet. The sale ends either when the entire stock is sold out, or when the deadline is over. Moreover, the objective of the seller is to find a dynamic pricing policy that maximizes the total expected revenue. Although the seller can use dynamic pricing policy to improve yield, it will have some negative effects on consumer's trust of the seller, especially in the nonperishable product categories (Kannan and Kopalle, 2001). Therefore, we consider only perishable products here, such as airline seats, rental cars, concert tickets, and seasonal fashion goods that will be disposed of with little salvage value left if they are unsold at the end of the horizon. In this study, demand can be modeled as a price-sensitive stochastic Poisson process with an intensity that is a known decreasing function of the price. All customers are independent, and they will either purchase an item if the current price is below their reservation price or leave empty-handed, where the reservation price of a customer is the maximum price which he or she is willing to pay for the product. Generally, the reservation price has a continuous distribution over the population of the customers. In addition, both the intensity of the customers' arrival rate and the reservation price distribution may vary with time. Furthermore, we assume that there are no other competitors on the Internet. That is, we neglect the effects of other competition. Therefore, this is a single-product, single-retail store problem. Furthermore, the list price should be posted on the Internet in advance of the sale by the seller and is not negotiable with customers.

The assumptions of this study are as follows:

(1) There is a given inventory of identical products to be sold in a given time horizon.
(2) The demand is price-sensitive and a stochastic Poisson process.
(3) Items left unsold at the end of the horizon are disposed of at a salvage value. For notational convenience, the salvage value of the items is assumed to be zero.
(4) The cost associated with price changes is ignored.
(5) All costs related to the purchase or production of items are considered sunk costs.
(6) There are no holding costs.
(7) We neglect the rate of discount.

On the other hand, there are some restrictions of this study:

(1) When the items are sold out or the deadline is over, sales must terminate immediately.
(2) Initial stock is fixed.
(3) Reordering or backlogging is not allowed during the sales horizon.
(4) Customers cannot return products after they have purchased them (i.e., no cancellations).

We have the notations below in this paper:

P_I:	the price i at which the seller offers to sell items, where i denotes predetermined available prices over the sales horizon
T:	the length of the sales horizon
$[0, T]$:	the time interval
t:	time, $0 \le t \le T$
M:	initial amount of items at time 0
$N_i(t)$:	the number of items sold up to time t at price p_i
n:	left amount of unsold items
$\lambda(t)$:	customers' arrival rate at time t, $t \in [0, T]$
$f(x, t)$:	probability density function of the reservation price at time t
$F(x, t)$:	cumulative probability distribution function of the reservation price at time t
$b(p_i, t)$:	probability that a customer arriving at time t would buy an item at price p_i
$d(p_i, t)$:	demand intensity at time t and price p_i
$r_{i(t)}$:	revenue rate at time t and price p_i
$J(n, t)$:	expected revenue over $[t, T]$ when there are n unsold items at time t
$V_i(n, t)$:	the maximum value of the expected revenue over $[t, T]$ when there are n unsold items at time t, where i means the pricing policy at time t and it may be changed afterward according to n and s ($s > t$)
z_n^i:	time threshold when there are n unsold items

3. MODEL FOMULATION

Consider a seller who has M items of perishable products to sell on the Internet market in a finite time interval $[0, T]$. At any time, the seller selects a price for the products from the set of predetermined prices $P = \{p_1, p_2, \cdots, p_k\}$, with $p_1 < p_2 < \cdots < p_k$. In addition, assume that customers arrive according to a nonhomogeneous Poisson process with rate $\lambda(t)$, $t \in [0, T]$ Each arriving customer would buy an item if the price which the seller offers at that time is below his or her reservation price. In fact, the distribution of the reservation price may vary with time. Let $f(x,t)$ denote the probability density function of the reservation price at time *t* and $F(x,t)$ denote the cumulative probability distribution function of the reservation price at time *t*. Thus, $b(p_i,t)=1-F(p_i,t)$ is the probability that a customer arriving at time *t* would buy an item at price p_i and is a decreasing function of p_i. As a result, the demand is a nonhomogeneous Poisson process with intensity $d(p_i,t)=\lambda(t)b(p_i,t)$. Moreover, if the seller adopts policy *i* to sell items at time *t*, that is, the price p_i is chosen as a pricing policy, then the corresponding demand intensity is $d(p_i,t)$, which is also decreasing in price. Furthermore, we define revenue rate to be $r_i(t)=p_i d(p_i,t)$ and assume that $r_i(t)$ is a decreasing function of p_i, i.e., if $p_j<p_k$, then $r_j(t)>r_k(t)$. This assumption must hold; otherwise, the lower price p_j would be dominated and would not be set.

Let U be the set of all pricing policies, which satisfy

$$\sum_{i=1}^{k}\int_0^T dN_i(s)\le M . \tag{1}$$

Let $S_i(t)=\begin{cases}1 & \text{if } p_i \text{ is optimal at time } t;\\ 0 & \text{otherwise.}\end{cases}$

Then given a pricing policy $u\in U$, when there are *n* unsold items at time t ($0\le t<T$), the expected revenue over [*t*, *T*] can be expressed as

$$J_u(n,t)=E\left[\sum_{i=1}^{k}\int_t^T S_i(s)\,p_i\,dN_i(s)\right] \tag{2}$$

with boundary conditions $J_u(n,T)=0,\forall n$ and $J_u(0,t)=0,\forall t$. This means that no revenue can be earned when either time or stock is running out.

The maximum of $J_u(n,t)$ over all policies $u\in U$ is denoted by $V(n,t)$; i.e.,

$$V(n,t)=\max_{u\in U} J_u(n,t). \tag{3}$$

Lemma 1. (1) $V(n,t)$ is continuous and decreasing in t for a fixed n; (2) $V(n,t)$ is increasing and concave in n for any fixed t (Chatwin, 2000, Feng and Xiao, 2000; Gallego and van Ryzin, 1994; Lee and Hersh, 1993; Zhao and Zheng, 2000).

Lemma 1 shows that more stock or more time leads to higher expected revenues. In fact, whether the demand is a homogeneous Poisson process or nonhomogeneous Poisson process, the properties in Lemma 1 always hold.

Subsequently, consider a transaction over a small time interval Δt. The seller sells one item over the next Δt with probability approximately $\lambda(t)b(p_i,t)\Delta t$ and no items with probability approximately $1-\lambda(t)b(p_i,t)\Delta t$. Thus, we have

$$\begin{aligned}V(n,t)=\max_{i=1,2,\dots,k}\big\{&\lambda(t)b(p_i,t)\Delta t\left[p_i+V(n-1,t+\Delta t)\right]\\&+\left[1-\lambda(t)b(p_i,t)\Delta t\right]V(n,t+\Delta t)\big\}\end{aligned} \tag{4}$$

Letting $r_i(t)=p_i\lambda(t)b(p_i,t)$ and $\partial V(n,t)/\partial t=\left(V(n,t+\Delta t)-V(n,t)\right)/\Delta t$, taking the limit as $\Delta t\to 0$, we can obtain

$$\begin{aligned}&\frac{\partial V(n,t)}{\partial t}+\max_{i=1,2,\dots,k}\big\{\lambda(t)b(p_i,t)\left[V(n-1,t)-V(n,t)\right]\\&+r_i(t)\big\}=0\quad \forall n\ge 1,\forall t\ge 0\end{aligned} \tag{5}$$

with boundary conditions $V(n,T)=0,\ \forall n$ and $V(0,t)=0,\ \forall t$.

As we mentioned before, the demand is a nonhomogeneous Poisson process with intensity $d(p_i,t)=\lambda(t)b(p_i,t)$. Therefore, if pi is the optimal policy at time t, from (5) we can get

$$\frac{\partial V(n,t)}{\partial t}+d(p_i,t)\left[V(n-1,t)-V(n,t)\right]+r_i(t)=0 \tag{6}$$

and

$$\frac{\partial V(n,t)}{\partial t}+d(p_j,t)\left[V(n-1,t)-V(n,t)\right]+r_j(t)\le 0,\quad \forall j\ne i. \tag{7}$$

Combining formulas (6) and (7) leads to (8):

$$V(n,t)-V(n-1,t)\le\frac{r_i(t)-r_j(t)}{d(p_i,t)-d(p_j,t)},\quad \forall j\ne i. \tag{8}$$

Lemma 2. The revenue rate $r_i(t)$ is an increasing and concave function of demand intensity $d(p_i,t)$ (Chatwin, 2000; Feng and Xiao, 2000; Feng and Xiao, 2000; Gallego and van Ryzin, 1994).

The revenue rate $r_i(t)$ in Lemma 2 is the corresponding revenue rate of the demand intensity $d(p_i,t)$ at time *t*. If the demand is a nonhomogeneous Poisson process, this means that at different time points there are different demands and with the corresponding revenue rates. However, the revenue rate $r_i(t)$ is still an increasing and concave function of demand intensity $d(p_i,t)$ at any time *t*.

If some prices fail to satisfy Lemma 2, then these prices will be automatically eliminated from the active price set. The proof of Lemma 2 can be found in (Feng and Xiao, 2000).

If Lemma 2 holds, we can see

$$\frac{r_i(t)-r_{i+1}(t)}{d(p_i,t)-d(p_{i+1},t)} \le \frac{r_i(t)-r_{i+2}(t)}{d(p_i,t)-d(p_{i+2},t)} \le \cdots \le \frac{r_i(t)-r_k(t)}{d(p_i,t)-d(p_k,t)} \tag{9}$$

and

$$\frac{r_1(t)-r_2(t)}{d(p_1,t)-d(p_2,t)} \le \frac{r_2(t)-r_3(t)}{d(p_2,t)-d(p_3,t)} \le \cdots \le \frac{r_{k-1}(t)-r_k(t)}{d(p_{k-1},t)-d(p_k,t)}. \tag{10}$$

Therefore, we have

$$V(n,t)-V(n-1,t) \le \frac{r_i(t)-r_{i+1}(t)}{d(p_i,t)-d(p_{i+1},t)}. \tag{11}$$

Since $V(0,t)=0,\ \forall t$, we proceed with n = 1. Besides, with $V(1,T)=0$, when $t\to T$, then p_1 is the optimal price. This is consistent with intuition. Indeed, with less time left, we should set the lowest price to stimulate sales. Therefore, the expected revenue when there is only one item left at time t ($t\to T$) can be solved for $V_1(1,\ t)$ in (5) as

$$V_1(1,t)=r_1(t)\int_t^T e^{-d(p_1,t)(s-t)}ds \tag{12}$$

where $\int_t^T e^{-d(p_1,t)(s-t)}ds$ means the probability that exactly one item will be sold within the time interval [t, T]. Then substituting (12) into (11), we can find

$$V_1(1,t) \le \frac{r_1(t)-r_2(t)}{d(p_1,t)-d(p_2,t)}. \tag{13}$$

According to Lemma 1, $V_1(1,t)$ is continuous and decreasing with respect to t. Hence, if $V_1(1,0) > (r_1(t)-r_2(t))/(d(p_1,t)-d(p_2,t))$, then there must exist a time point t, 0 < t < T, at that time $V_1(1,t) = (r_1(t)-r_2(t))/(d(p_1,t)-d(p_2,t))$. Therefore, we can define a time threshold as:

$$z_1^1=\inf\left\{0\le t\le T : V_1(1,t)=\frac{r_1(t)-r_2(t)}{d(p_1,t)-d(p_2,t)}\right\}. \tag{14}$$

If $0<z_1^1<T$, then it partitions the sale period into two intervals, $\left[0,z_1^1\right)$ and $\left[z_1^1,T\right]$. When there is only one unsold item left, the price p1 is offered if $z_1^1\le t\le T$. On the other hand, if $t<z_1^1$, then the seller can offer a higher price p_2. That is, z_1^1 is the switching time from p_2 markdown to p_1 if one item remains on hand.

Because p_2 is the optimal price for $t<z_1^1$ ($t\to z_1^1$), $V_2(1,t)$ can be modeled as

$$V_2(1,\ t)=r_2(t)\int_t^{z_1^1} e^{-d(p_2,t)(s-t)}ds+V_1\left(1,z_1^1\right)e^{-d(p_2,t)\left(z_1^1-t\right)} \tag{15}$$

where $\int_t^{z_1^1} e^{-d(p_2,t)(s-t)}ds$ means the probability that exactly one item will be sold within the time interval $\left[t,z_1^1\right]$ and $e^{-d(p_2,t)\left(z_1^1-t\right)}$ means the probability that no items will be sold within the time interval $\left[t,z_1^1\right]$.

Similarly, substituting (15) to (11), we can have

$$V_2(1,t)\le\frac{r_2(t)-r_3(t)}{d(p_2,t)-d(p_3,t)}. \tag{16}$$

According to Lemma 1, if $V_2(1,0)>(r_2(t)-r_3(t))/(d(p_2,t)-d(p_3,t))$, then there must exist a time point t, $0<t<z_1^1$, at the time $V_2(1,t)=\left(r_2(t)-r_3(t)\right)/\left(d(p_2,t)-d(p_3,t)\right)$. Again, we can define a time threshold as:

$$z_1^2=\inf\left\{0\le t<z_1^1 : V_2(1,t)=\frac{r_2(t)-r_3(t)}{d(p_2,t)-d(p_3,t)}\right\}. \tag{17}$$

Accordingly, once $z_1^{i-1}, i=2,\cdots,k$ have been defined, we can continue to solve $V_i(1,t)$ for $t<z_1^{i-1}$, which is given by

$$V_i(1,t)=r_i(t)\int_t^{z_1^{i-1}} e^{-d(p_i,t)(s-t)}ds+V_{i-1}\left(1,z_1^{i-1}\right)e^{-d(p_i,t)\left(z_1^{i-1}-t\right)}. \tag{18}$$

Furthermore, the time threshold can be defined as:

$$z_1^i=\inf\left\{0\le t<z_1^{i-1} : V_i(1,t)=\frac{r_i(t)-r_{i+1}(t)}{d(p_i,t)-d(p_{i+1},t)}\right\}. \tag{19}$$

The procedure is repeated until either $z_1^i=0$ for $1\le i\le k-1$ or $z_1^{k-1}>0$.

By letting $\tilde{V}(n,t)=V_i(n,t),\ for\ z_n^i\le t<z_n^{i-1}$, i = 1, …, k, where $z_n^0=T$, $z_n^k=0$, the procedure of constructing V(1, t) can be extended to V(n, t), *for n* ≥ 2. As we described above, when $t\to T$, p_1 is the optimal price. Therefore, the expected revenue when there are n unsold items at time t can be modeled as

$$V_1(n,t)=r_1(t)\int_t^T e^{-d(p_1,t)(s-t)}ds+\int_t^T d(p_1,t)\tilde{V}(n-1,s)e^{-d(p_1,t)(s-t)}ds. \tag{20}$$

Similar to the case of $n = 1$, we need to show the existence of the time threshold z_n^1, which is a switching time point from p_2 markdown to p_1 when there are n unsold items. As a result, Lemma 3 must hold.

Lemma 3. If $\tilde{V}(n-1,t)$ and $V_i(n, t)$ are both concave and decreasing in t, then $V_i(n,t)-\tilde{V}(n-1,t)$ is strictly decreasing in t (Chatwin, 2000; Feng and Xiao, 2000; Feng and Xiao, 2000; Zhao and Zheng, 2000).

Lemma 4. The marginal expected revenue is decreasing with inventory level. Namely, $V(n+1,t)-V(n,t) < V(n,t)-V(n-1,t)$, for $0 \le t < T$ (Feng and Xiao, 2000).

Lemma 3 and Lemma 4 are consistent with intuition. Indeed, when there is less time left, the expected revenue of adding one additional item is less than when there is more time left. Additionally, when more items are left, the marginal expected revenue will be reduced. The proofs of Lemma 3 and Lemma 4 can be found in Feng and Xiao (2000).

According to Lemma 3, the time threshold can be defined as:

$$z_n^1 = \inf\left\{0 \le t \le T : V_1(n,t)-\tilde{V}(n-1,t) = \frac{r_1(t)-r_2(t)}{d(p_1,t)-d(p_2,t)}\right\}. \tag{21}$$

Again, when $t < z_n^1$ ($t \to z_n^1$), the seller can offer the effective price p_2. Thus, $V_2(n,t)$ can be modeled as

$$\begin{aligned} V_2(n,t) &= r_2(t)\int_t^{z_n^1} e^{-d(p_2,t)(s-t)}ds \\ &+\int_t^{z_n^1} d(p_2,t)\tilde{V}(n-1,s)e^{-d(p_2,t)(s-t)}ds \\ &+V_1(n,z_n^1)e^{-d(p_2,t)(z_n^1-t)} \end{aligned}. \tag{22}$$

In addition, the time threshold can be defined as:

$$z_n^2 = \inf\left\{0 \le t < z_n^1 : V_2(n,t)-\tilde{V}(n-1,t) = \frac{r_2(t)-r_3(t)}{d(p_2,t)-d(p_3,t)}\right\} \tag{23}$$

Similarly, once $z_n^{i-1}, i = 2,\cdots,k$ have been defined, we can continue to solve $V_i(n,t)$ for $t < z_n^{i-1}$, which is given by

$$\begin{aligned} V_i(n,t) &= r_i(t)\int_t^{z_n^{i-1}} e^{-d(p_i,t)(s-t)}ds \\ &+\int_t^{z_n^{i-1}} d(p_i,t)\tilde{V}(n-1,s)e^{-d(p_i,t)(s-t)}ds \\ &+V_{i-1}(n,z_n^{i-1})e^{-d(p_i,t)(z_n^{i-1}-t)} \end{aligned} \tag{24}$$

and

$$z_n^i = \inf\left\{ \begin{aligned} &0 \le t < z_n^{i-1} : V_i(n,t)-\tilde{V}(n-1,t) \\ &= \frac{r_i(t)-r_{i+1}(t)}{d(p_i,t)-d(p_{i+1},t)} \end{aligned}\right\}. \tag{25}$$

Accordingly, the procedure is repeated until either $z_n^i = 0$ for $1 \le i \le k-1$ or $z_n^{k-1} > 0$.

4. STRUCTURE OF THE TIME THRESHOLDS

We further discuss some properties of the time thresholds in this section. The nature of the optimal policies shows some common features, which demonstrate the fundamental relationships among price, demand, inventory, and time. As a result, we propose three theorems about the time thresholds.

Theorem 1. In the time interval where demand is a homogeneous Poisson process, for a given inventory level, the time thresholds $z_n^i, i = 1,\dots,k-1$ are decreasing in the policy i, i.e., $z_n^i \le z_n^{i-1}$.

Proof.
When we compute the time thresholds, we first solve z_1^1 from (14) and $z_1^1 \in [0,T]$. Then we solve z_1^2 from (17) and $z_1^2 \in [0,z_1^1)$. Therefore, we can see from (25) that $z_n^i \le z_n^{i-1}$.

For a given inventory, these threshold points partition the whole time period into k price zones, $T = z_n^0 \ge z_n^1 \ge \cdots \ge z_n^k = 0$. Within the time interval $[z_n^i, z_n^{i-1})$, the best pricing policy is pi. Lemma 5 shows that for a given inventory level, with more time remaining to make the sales, we can choose higher prices for the items. In contrast, if the deadline is approaching, then the lower prices should be selected.

Theorem 2. In the time interval where demand is a homogeneous Poisson process, at any given policy, the time thresholds $z_n^i, n \ge 1$ are decreasing in the number of items remaining unsold, i.e., $z_n^i \le z_{n-1}^i \quad \forall i = 1,2,\dots,k-1$.

Proof.
From (14) z_1^1 is the time point when $V_1(1,t) = (r_1(t)-r_2(t))/(d(p_1,t)-d(p_2,t))$. Similarly, from (21) we can observe that z_2^1 is the time point when $V_1(2,t)-\tilde{V}(1,t) = (r_1(t)-r_2(t))/(d(p_1,t)-d(p_2,t))$. Since the right hand sides of these two equations are equal, therefore if $z_1^1 = z_2^1$, then $V_1(1,t)$ must equal

$V_1(2,t)-\tilde{V}(1,t)$. However, Lemma 4 demonstrates that $V(2,t)-V(1,t)<V(1,t)$. Consequently, according to Lemma 3, ζ_2^1 must be less than ζ_1^1 in order for the two equations to both be satisfied. Furthermore, ζ_n^i can be extended by the same procedure.

Theorem 2 shows that at a given time point, with more inventory remaining, a lower pricing policy should be used. In addition, while inventory rises, the time thresholds are approaching to 0. That is, if the volume of the items goes to infinity, then p_1 is the only effective price.

Theorem 3. If demand is a nonhomogeneous Poisson process, then the higher the demand is, the larger ζ_n^i will be, for the same policy i and inventory n.

Proof.
Combining (12) and (14) leads to (26), as follows:

$$\zeta_1^1=\inf\left\{\ 0\le t\le T: V_1(1,t)=r_1(t)\int_t^T e^{-d(p_1,t)(s-t)}ds = \frac{r_1(t)-r_2(t)}{d(p_1,t)-d(p_2,t)}\right\}. \tag{26}$$

The length of the sales horizon T is fixed in (26). Furthermore, with higher demand, then there is larger probability that one item will be sold in a time interval, and there will be less time needed to sell exactly one item. Therefore, $t=\zeta_1^1$ is larger. Furthermore, this process can also be used to compute for ζ_n^i.

In order to investigate ζ_n^i, we must first determine the initial stock, the price set, and the length of the sales horizon. Then we collect information about customers' arrival rate and the distribution of their reservation price. Next, given these data, we can compute $V_1(1,t)$ and ζ_1^1 by (12) and (14) respectively. And then based on a known $V_1(1,t)$, we can compute $V_2(1,t)$ and ζ_1^2 by (15) and (17) respectively. This procedure can be used to compute all expected revenues according to (24) and time thresholds by (25). Finally, we can sell items on the Internet market by dynamically adjusting prices according to the calculated time thresholds.

For example, if there are currently n unsold items, then we can check time thresholds $\zeta_n^i, i=1,\cdots,k-1$ to see that t is in which time interval. If $\zeta_n^i\le t<\zeta_n^{i-1}, i=1,\cdots,k$, then p_i is the optimal pricing policy. Subsequently, if a sale occurs, namely, inventory drops to $n-1$, then we should inspect the current time thresholds $\zeta_{n-1}^i, i=1,\cdots,k-1$ to see which time interval that the current time t is within and then offer the optimal price. On the other hand, if there are no transactions prior to ζ_n^{i-1}, then the price is switched from p_i markdown to p_{i-1} at the time threshold ζ_n^{i-1}. Therefore, the optimal pricing policy can be found at any state of time and level of inventory.

5. NUMERICAL EXAMPLE

In this section we present a numerical example to illustrate how to use the proposed model to compute the time thresholds and how to apply the calculated time thresholds to adjust prices dynamically. Moreover, we discuss some properties of the time thresholds in this example.

Example 1: Ticket pricing policy
A seller has 300 baseball game tickets to be sold on the Internet beginning 30 days before the game. To maximize the total revenue, he plans to use three different prices (200, 400, 600) based on unsold tickets and elapsed time. Furthermore, customers arrive according to a nonhomogeneous Poisson process. These parameters are given in Table 1.

Table 1. Data in Example 1

t	$0\le t<10$	$10\le t<25$	$25\le t<30$
$\lambda(t)$	10	6	20
$b(p_1, t)$	0.9	0.8	0.95
$b(p_2, t)$	0.4	0.3	0.45
$b(p_3, t)$	0.2	0.15	0.25

In short, this is a case with $M=300$, $T=30$, $P_1=200$, $P_2=400$, $P_3=600$. From Table 1, we can solve demand $d(p_i,t)$ with $d(p_i,t)=\lambda(t)b(p_i,t)$ and revenue rate $r_i(t)$ with $r_i(t)=p_i d(p_i,t)$. The results are shown in Table 2. Moreover, we can observe that the demand is medium in time period 0~10, low in time period 10~25 and high in time periods 25~30.

Table 2. Demand and revenue in Example 1

t	$0\le t<10$	$10\le t<25$	$25\le t<30$
$b(p_1,t)$	9	4.8	19
$b(p_2,t)$	4	1.8	9
$b(p_3,t)$	2	0.9	5
$r_1(t)$	1800	960	3800
$r_2(t)$	1600	720	3600
$r_3(t)$	1200	540	3000

Subsequently, we can calculate all expected revenues and time thresholds based on the formulas developed in Sections 3 and 4. Some computed optimal time thresholds ζ_n^i in high, low and medium demand are listed in Tables 3, 4, and 5, respectively.

Figure 1 illustrates all optimal time thresholds for three different demand levels in the entire period. Therefore, the seller can sell tickets on the Internet market and dynamically adjust prices according to the calculated time thresholds to maximize total revenue. Because the demand is according to a nonhomogeneous Poisson process, we can observe that the entire horizon is partitioned into three time periods, 0~10, 10~25, and 25~30. At each time interval, there are two time sequences separating the pricing policies into p_1, p_2, and p_3. Therefore, at any state of time and inventory level, we can inspect the calculated time thresholds and adopt the optimal pricing policy.

Table 3. Optimal time thresholds z_n^i during high demand

n	z_n^1	z_n^2	**n**	z_n^1	z_n^2	**n**	z_n^1	z_n^2	**n**	z_n^1	z_n^2
1	29.994	29.947	15	29.380	28.264	29	28.757	26.576	60	27.377	22.837
2	29.965	29.837	16	29.336	28.144	30	28.712	26.455	65	27.155	22.234
3	29.921	29.715	17	29.291	28.023	31	28.668	26.335	70	26.932	21.631
4	29.872	29.591	18	29.247	27.902	32	28.623	26.214	75	26.710	21.028
5	29.818	29.467	19	29.202	27.782	33	28.579	26.093	80	26.487	20.425
6	29.781	29.350	20	29.158	27.661	34	28.534	25.973	85	26.265	19.822
7	29.736	29.229	21	29.113	27.541	35	28.490	25.852	90	26.042	19.219
8	29.692	29.108	22	29.069	27.420	36	28.445	25.732	95	25.820	18.616
9	29.647	28.988	23	29.024	27.299	37	28.401	25.611	100	25.597	18.013
10	29.603	28.867	24	28.980	27.179	38	28.356	25.490	105	25.375	17.410
11	29.558	28.747	25	28.935	27.058	40	28.267	25.249	110	25.152	16.807
12	29.514	28.626	26	28.891	26.938	45	28.045	24.646	111	25.108	16.687
13	29.469	28.505	27	28.846	26.817	50	27.822	24.043	112	25.063	16.566
14	29.425	28.385	28	28.801	26.696	55	27.600	23.440	113	25.019	16.445

Table 4. Optimal time thresholds z_n^i during low demand (Example 1)

n	z_n^1	z_n^2	**n**	z_n^1	z_n^2	**n**	z_n^1	z_n^2	**n**	z_n^1	z_n^2
1	29.894	29.633	15	26.371	20.461	29	22.849	11.277	43	19.327	2.093
2	29.650	29.002	16	26.120	19.805	30	22.597	10.621	44	19.075	1.437
3	29.382	28.338	17	25.868	19.149	31	22.346	9.965	45	18.823	0.781
4	29.132	27.674	18	25.617	18.493	32	22.094	9.309	46	18.572	0.125
5	28.895	27.017	19	25.365	17.837	33	21.843	8.653	47	18.320	0
6	28.636	26.365	20	25.113	17.181	34	21.591	7.997	48	18.069	0
7	28.384	25.709	21	24.862	16.525	35	21.339	7.341	49	17.817	0
8	28.133	25.053	22	24.610	15.869	36	21.088	6.685	50	17.565	0
9	27.881	24.397	23	24.359	15.213	37	20.836	6.029	55	16.307	0
10	27.629	23.741	24	24.107	14.557	38	20.585	5.373	60	15.049	0
11	27.378	23.085	25	23.855	13.901	39	20.333	4.717	65	13.791	0
12	27.126	22.429	26	23.604	13.245	40	20.081	4.061	70	12.533	0
13	26.875	21.773	27	23.352	12.589	41	19.830	3.405	75	11.275	0
14	26.623	21.117	28	23.101	11.933	42	19.578	2.749	80	10.017	0

Table 5. Optimal time thresholds z_n^i during medium demand (Example 1)

n	z_n^1	z_n^2	**n**	z_n^1	z_n^2	**n**	z_n^1	z_n^2	**n**	z_n^1	z_n^2
1	29.975	29.828	71	22.313	8.647	85	20.779	4.410	210	7.079	0
2	29.886	29.530	72	22.204	8.344	90	20.231	2.897	215	6.531	0
3	29.770	29.220	73	22.094	8.042	95	19.683	1.384	220	5.983	0
4	29.660	28.920	74	21.985	7.739	100	19.135	0	225	5.435	0
5	29.540	28.620	75	21.875	7.436	184	9.929	0	230	4.887	0
6	29.437	28.316	76	21.765	7.134	185	9.819	0	235	4.339	0
7	29.328	28.013	77	21.656	6.831	186	9.709	0	240	3.791	0
8	29.218	27.711	78	21.546	6.529	187	9.600	0	245	3.243	0
9	29.109	27.408	79	21.437	6.226	188	9.490	0	250	2.695	0
10	28.999	27.105	80	21.327	5.923	189	9.381	0	255	2.147	0
67	22.752	9.857	81	21.217	5.621	190	9.271	0	260	1.599	0
68	22.642	9.555	82	21.108	5.318	195	8.723	0	265	1.051	0
69	22.533	9.252	83	20.998	5.016	200	8.175	0	270	0.503	0
70	22.423	8.949	84	20.889	4.713	205	7.627	0	275	0.000	0

For example, the seller has 300 tickets to be sold in 30 days in the beginning stage. That is, he should use price p_1 at time 0 to sell the tickets because at that level of inventory $t \geq z_{300}^1$. Furthermore, if there are 200 unsold tickets at time 8, then he should offer price p_2 in that $z_{200}^2 < t < z_{200}^1$. Subsequently, if there are no transactions prior to z_{200}^1, i.e., 8.175, then the price is switched from p_2 markdown to p_1 at z_{200}^1.

We further discuss some properties of the time thresholds and demonstrate the results in Example 1 as follows.

1. Theorem 1 states that for a given inventory level, the time thresholds z_n^i, $i = 1,\ldots,k-1$ are decreasing in the policy *i*, i.e., $z_n^i \leq z_n^{i-1}$. Figure 2 shows that all $z_n^2 \leq z_n^1$, n = 1, 2, …, 5. In addition, for a given inventory, these threshold points partition the whole time period into three price zones, $T = 30 = z_n^0 \geq z_n^1 \geq z_n^2 \geq z_n^3 = 0$. We can also observe that when there is more time remaining to make the sales,

we can choose higher ticket prices. In contrast, if the deadline is approaching, lower prices should be selected. That is, in the interval $\left[z_n^1, z_n^0\right)$, the best pricing policy is p_1.

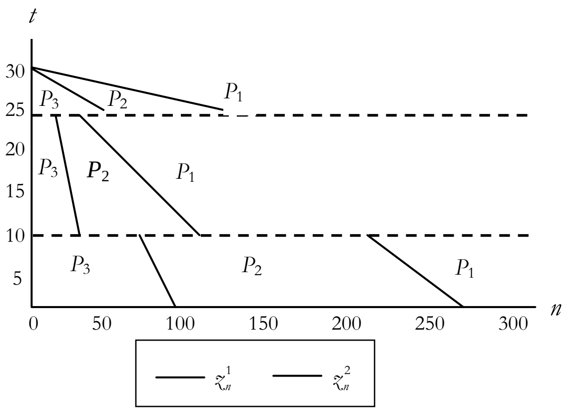

Figure 1. All time thresholds in the entire period.

2. Theorem 2 states that at any given policy, the time thresholds $z_n^i, n \geq 1$ are decreasing in the number of items remaining unsold, i.e., $z_n^i \leq z_{n-1}^i \quad \forall i = 1, 2, \ldots, k-1$. Accordingly, Figure 2 displays that $z_1^i > z_2^i > z_3^i > z_4^i > z_5^i$, $i = 1,\ 2$. In addition, when $n \to \infty$, $z_n^i = 0$.
3. According to Theorem 3, if demand is a nonhomogeneous Poisson process, then the higher the demand is, the larger z_n^i will be, for the same policy i and inventory n. Figure 3 demonstrates that the time thresholds in high demand are larger than those in low demand. Furthermore, since the demand in time period 8~10 is larger than that in time period 10~12, then the time thresholds in time period 10~12 shift to the left (see in Figure 4). That is, in a high-to-low demand case, when there is no transaction, a lower pricing policy is usually used. In fact, we may reduce price whether a sale occurs or not because the demand in the future is lower. For example, as shown in Figure 4, if there are 81 unsold tickets at time 9.9, then we should offer price p_2. Subsequently, if no sales occur, then the price is switched from p_2 markdown to p_1 at time 10. On the other hand, if a sale occurs, i.e., 80 tickets left, then we can still offer price p_2 at time 10. On the other hand, since the demand in time period 25~26 is larger than that in time period 23~25, then the time thresholds in time period 25~26 shift to the right (see in Figure 5). In other words, in a low-to-high demand case, the price may go up when there are no transactions. This situation may happen because at that time point, the demand is rising from low to high and the time thresholds are shifting to the right. Therefore, we may use higher pricing policy after that time point. For instance, as shown in Figure 5, if there are 50 unsold tickets at time 24.9, then we should offer price p_1. However, whether or not a sale occurs, the price is switched from p_1 markup to p_2 at time 25.

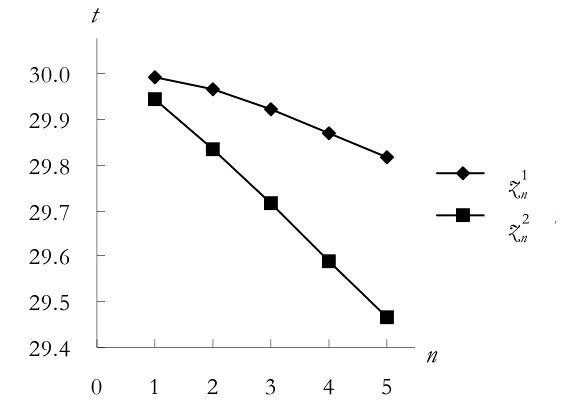

Figure 2. Time thresholds with only 5 tickets left.

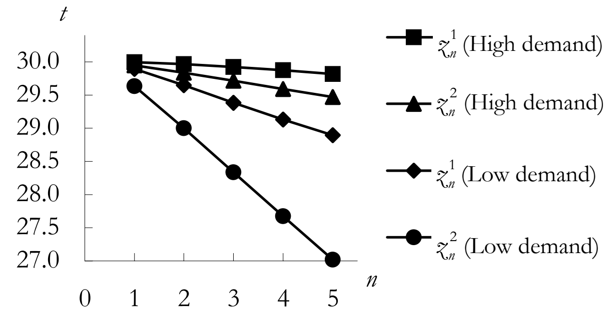

Figure 3. Comparisons of high demand and low demand.

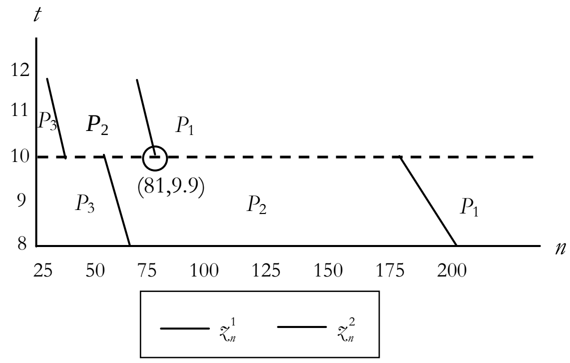

Figure 4. Time thresholds in time periods 8 to 12.

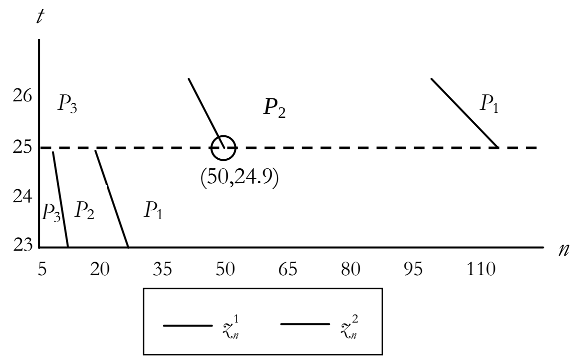

Figure 5. Time thresholds in time periods 23 to 26.

6. CONCLUSIONS

In this paper we construct a dynamic pricing model for selling a given stock of identical perishable products over a finite time horizon on the Internet. Here, the objective of the seller is to find a dynamic pricing policy that maximizes the total expected revenue. We assume that perishable items are priced at a finite set of the predetermined levels, and that the demand obeys a nonhomogeneous Poisson process. According to the proposed model, we can compute all optimal time thresholds. Moreover, we can determine the optimal price level based on the length of the remaining sales time and the unsold inventory level, and make switches among the prices by the calculated time thresholds. Furthermore, we present a numerical example to demonstrate the model and its results.

Dynamic pricing policy adjusts the price of the perishable products in response to market changes. The proposed model is particularly suitable for perishable products such as airline seats, sports tickets, and hotel rooms. Furthermore, the application of the dynamic pricing to the Internet market is especially valuable because of minimal menu cost in price update. It is possible that several customers make transactions at the same time on the Internet market. In fact, when one customer purchases multiple products once or there are several customers coming to buy items simultaneously, the computed time thresholds still perform adequately.

Directions for further research could investigate several extensions where overbooking, re-supply and cancellations are allowed. Besides, in order to demonstrate the practicability of the proposed model, it would be important to implement an empirical study for a further investigation.

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