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Component Commomality and Shortage Reduction under a Mixed Erlang Distributed Demand

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Abstract—This paper addresses an inventory control problem when common components are allowed and the goal is to minimize the expected units shortage subject to a budget constraint. A two-level assemble-to-order product structure is analyzed when the demands for two end products follow independent mixtures of Erlang distributions. Closed form expressions for the objective function under various scenarios are presented and efficient algorithms for computing the optimal inventory stock levels are developed. Relative reductions in the expected units shortage under different demand patterns and budget availability situations when introducing commonality are evaluated and compared. It is found that, for all demand patterns considered, the relative reduction can be substantial when the inventory budget is large. Thus, if a company wants to improve an already high service level for an essential item, introducing commonality may be an option without having to increase the inventory to an unbearable level. Also, given a fixed budget, our numerical results suggest that the case of independent and identically distributed demands produces the largest relative reduction in expected units shortage. Benefits from employing commonality can be insignificant, however, where the demands are very different.

Keywords—Inventory models, Component commonality, Erlang demand distributions

1. INTRODUCTION

Today's challenges of increased globalization and customization have led to increased product proliferation, which in the face of uncertain product demands aggravates the situation of high inventory levels and poor customer service. Product design incorporating component commonality, i.e., a product structure in which at least one component is common to two or more end products (Baker, 1985), plays an important role in achieving a specified service level while maintaining lower inventory levels. Thus, investigations of the effects of commonality are of interest to both academics and practitioners.

The extensive literature on inventory stocking models under component commonality can be categorized into two broad groups: (i) Analytical Studies and (ii) Computational Studies.

(i) Analytical Studies: Studies in this category may be classified further into two groups. Studies in the first group (Baker, et al., 1986; Gerchak and Henig, 1986; Jönsson and Silver, 1989; Bagchi and Gutierrez, 1992; Eynan and Rosenblatt, 1996; Mirchandani and Mishra, 2002, among others) have focused on obtaining closed form solutions. These studies have generally analyzed a prototype 2-product assemble-to-order (ATO) product structure with two unique components and one common component. In these studies independent uniform, exponential, geometric and normal distributions have been used to model the uncertainty in the product demand. The objectives of these studies include: comparing the changes in stock levels with or without component commonality, expected profits for a single-period commonality model, qualitative characterizations of the changes in stock levels, maximizing the expected total of two end products subject to a budget constraint for the components, marginal benefits from increasing component commonality, and determination of the cost-minimizing inventory of the components to meet a product-specific service level.

Studies in the second group (for example, Gerchak et al., 1988; Gerchak and Henig, 1989, among others) have considered single- and multi-period commonality models with multiple products and multiple components. These studies have sought qualitative insights into the impact of component commonality on service levels and inventory costs, and the properties of optimal solutions. An interesting finding is that in the absence of a capacity/storage constraint, a myopic, single-period

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solution is optimal even for multi-period model.

(ii) Computational studies: Studies in this group (for example, Hillier, 1999a, 1999b, 2000) have developed mathematical formulations of commonality models with multiple products and multiple components but have relied primarily on simulation techniques to explore the effect of different degrees of commonality under different parameter specifications (such as cost- and demand distribution parameters) on the total inventory cost or service level.

Our goal in this paper is to make a contribution to the literature representing the analytical studies of the first type. While we analyze a single-period prototypical 2-product model similar to the one used in earlier studies, we model the uncertain demands using mixtures of Erlang distributions. From a mathematical perspective, mixtures of Erlang distributions provide a rich class of distributions with great versatility. For example, a mixture of two Erlang distributions already has five parameters, which provides us flexibility, even when demand has a bimodal distribution. Note that Gamma distribution is often used in the literature as the demand distribution and it can be well approximated by a mixture of Erlang distributions. (The approximation is exact when the shape parameter is an integer or less than 1; see Gleser (1989)). Also, the exponential distribution is a special case of an Erlang distribution.

From a practical perspective, a mixture of Erlang distributions also has an advantage over the popular normal distribution in that it is always nonnegative. In many practical settings, the demand for a product is driven by several distinct underlying variables, heterogeneous markets, diverse customer needs, etc. In modeling the demand, one first captures the demand in each market segment by estimating a demand distribution for each market segment, and then combining them appropriately in proportion to the firm's share in each market segment. In other words, the product demand is more meaningfully modeled as a mixture of the conditional demand distributions. For example, consider that the product under consideration is a spare part whose demand is driven by failure of the products in service. Often failure can occur for more than one reason and the failure distribution for each reason can be adequately modeled by a suitable density function. The overall failure distribution, and hence the demand distribution for the product is a mixture of Erlang distributions.

In this paper we present the analysis and computational results for a prototypical 2-product model similar to that considered in Jönsson and Silver (1989), but using mixtures of Erlang distribution for the demand. We assume simple cost and demand scenarios in which the cost of the common component is the same as that of the unique components and the demand distributions are independent. We also discuss some avenues for further research to exploit the versatility of the Erlang distributions and seek further extensions to the literature.

Our basic model consists of two end products, and each end product comprises two different components that are normalized so that one component of each type is needed to make one end product. The structure of the basic model is illustrated in Figure 1a. We call it Model *N*, a model that does not have a common component.

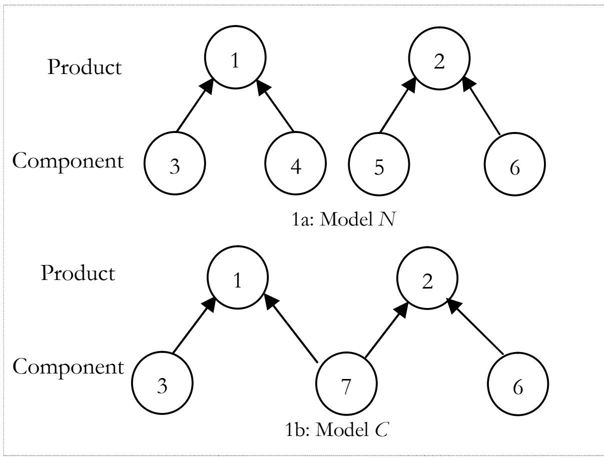

Figure 1. Structures of models *N* and *C*.

When a common component, say Component 7, is used to replace Components 4 and 5, we have product structure as illustrated in Figure 1b, which we call Model *C*, a model that has one common component.

Our objective function is to minimize the expected units storage (EUS) (or equivalently to maximize the expected units sold) subject to a budget constraint. Recognizing that the objective function is convex for any unbounded continuous demand distributions, we show that it can be evaluated in closed form for the class of mixtures of Erlang distributions and develop an efficient algorithm to compute the optimal stock levels.

The rest of the paper is organized as follows: In Section 2, we describe a mixture of two Erlang distributions and introduce some notation. In Section 3, we present the results for Model *N*. In Section 4, we develop a closed form expression for the objective function and provide an efficient algorithm to compute the optimal stock levels for Model *C*. In Section 5, we evaluate the benefits from employing commonality under different demand scenarios. Finally, in Section 6 we present our conclusions and avenues for further research. Section 7 lists references.

2. NOTATION

The density function of an Erlang distribution with parameters $\alpha > 0$ and $\beta > 0$ is

$$f(x\,|\,\alpha\,,\,\beta) = \frac{\beta^{\alpha} x^{\alpha-1}\exp(-\beta x)}{(\alpha-1)!},\ x > 0;\ \alpha > 0 \text{ and } \beta > 0$$

α is taken to be an integer. Note that the mean for this distribution is $E = \alpha/\beta$ and the variance is $V = \alpha/\beta^2$, so that $E^2/V = \alpha$ and $E/V = \beta$. When α = 1, the Erlang distribution becomes an exponential distribution.

The density function of a mixture of two Erlang distributions is

$$r f(x|\alpha_1,\beta_1)+(1-r)f(x|\alpha_2,\beta_2),$$

where $0 \le r \le 1$, and for i = 1, 2, $\beta_i > 0$ and α_i is a positive integer. The mean and the variance of the mixture are respectively given by

$$\mu_x=\frac{r\alpha_1}{\beta_1}+\frac{(1-r)\alpha_2}{\beta_2} \text{ and } \sigma_x^2=(\frac{r\alpha_1}{\beta_1^2}+\frac{(1-r)\alpha_2}{\beta_2^2}) +r(1-r)(\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2})^2.$$

Note that there are five parameters in this distribution, thus it allows great flexibility to model demand with a given mean and variance. In general, a finite mixture of Erlang distributions is given by $\sum_i r_i f(x|\alpha_i,\beta_i)$ where $\sum_i r_i$ = 1 and $r_i \ge 0$.

Let X and Y be the demand for products 1 and 2. To make a clear presentation, we only consider mixtures with two components here. When the densities of X and Y are needed for calculations in this paper, they are given by

$$f_X(x)=r_1 f(x \mid \alpha_1,\beta_1)+(1-r_1)f(x \mid \alpha_2,\beta_2),\ 0\le r_1\le 1,$$
$$f_Y(y)=r_2 f(y \mid \alpha_3,\beta_3)+(1-r_2)f(y \mid \alpha_4,\beta_4),\ 0\le r_2\le 1$$

Define

$$\phi_1(a)=\int_a^\infty f_X(x)dx,\phi_2(a)=\int_a^\infty xf_X(x)dx,$$
$$\phi_3(a)=\int_a^\infty f_Y(y)dy,\phi_4(a)=\int_a^\infty yf_Y(y)dy,$$
$$F_1(a_1,a_2,a_3)=\int_{a_3-a_2}^{\alpha_1} f_X(x)\phi_3(a_3-x)dx,$$
$$F_2(a_1,a_2,a_3)=\int_{a_3-a_2}^{\alpha_1} f_X(x)[(x-a_3)\phi_3(a_3-x)+\phi_4(a_3-x)]dx$$

These functions are all continuous and differentiable. The following notation will also be used in this paper:

S_{Ni} : stock level for component i in Model N, i = 3, 4, 5, 6

S_i : stock level for component i in Model C, i = 3, 6, 7

EUS_N : expected units shortage for Model N

The superscript * will be used to denote the optimal values of the variables. T is the budget limit (total number of units available among all components)

3. OPTIMAL INVENTORY LEVELS FOR MODEL N

Referring to Figure 1a, it is obvious that, in an optimal allocation, the inventory stock levels for components 3 and 4 are the same and those for components 5 and 6 are the same. The optimization formulation for Model N is

(P1) Minimize

$$EUS_N =\int_{S_{N3}}^\infty (x-S_{N3})f_X(x)dx+\int_{S_{N6}}^\infty (y-S_{N6}f_Y(y)dy, \tag{1}$$

such that $S_{N3} + S_{N6} \le T/2$, $S_{Ni} \ge 0$, i = 3, 6, and T is fixed. Since both demands X and Y are continuous and unbounded from above, all the available budget will be used up in an optimal allocation. So we have $S_{N3} + S_{N6} = T/2$. Substituting this into the objective function (1) we get Equation (2):

$$EUS_N =\int_{S_{N3}}^\infty (x-S_{N3})f_X(x)dx +\int_{T/2-S_{N3}}^\infty (y-T/2+S_{N3})f_Y(y)dy \tag{2}$$

It is easy to prove that EUS_N in Equation (2) is a convex function of S_{N3} , implying that any local minimum is also a global minimum.

$$\phi_1(S_{N3}^*)-\phi_3(T/2-S_{N3}^*)=0 \tag{3}$$

Note that from strict convexity, there is at most one solution for (3). Also, since $0<\phi_i(S)<1$ for $S>0$ and $\phi_i(S)=1$ for $S \le 0, i=1,3,$ a solution always exists and it will satisfy the constraint $0 \le S_{N3}^* \le T/2$. Given S_{N3}^*

$$S^*_{N6} = T/2 - S^*_{N3}, \tag{4}$$

$$EUS^*_N = \phi_2(S^*_{N3}) + \phi_4(T/2 - S^*_{N3}) - T/2\phi_1(S^*_{N3}) \tag{5}$$

For the mixtures of Erlang demand distributions, we have

$$\phi_1(S)=r_1\exp(-\beta_1 S)\sum_{m=0}^{\alpha_1-1}\frac{\beta_1^m S^m}{m!}+(1-r_1)\exp(-\beta_2 S)\sum_{m=0}^{a_2-1}\frac{\beta_2^m S^m}{m!},$$
$$\phi_2(S)=r_1\frac{\alpha_1}{\beta_1}\exp(-\beta_1 S)\sum_{m=0}^{\alpha_1}\frac{\beta_1^m S^m}{m!} +(1-r_1)\frac{\alpha_2}{\beta_2}\exp(-\beta_2 S)\sum_{m=0}^{a_2}\frac{\beta_2^m S^m}{m!},$$
$$\phi_3(S)=r_2\exp(-\beta_3 S)\sum_{m=0}^{\alpha_3-1}\frac{\beta_3^m S^m}{m!} +(1-r_2)\exp(-\beta_4 S)\sum_{m=0}^{a_4-1}\frac{\beta_4^m S^m}{m!},$$
$$\phi_4(S)=r_2\frac{\alpha_3}{\beta_3}\exp(-\beta_3 S)\sum_{m=0}^{\alpha_3}\frac{\beta_3^m S^m}{m!} +(1-r_2)\frac{\alpha_4}{\beta_4}\exp(-\beta_4 S)\sum_{m=0}^{a_4}\frac{\beta_4^m S^m}{m!}$$

The derivation is given in Appendix A. We solve Equation (3) using a subroutine called DZREAL from the IMSL Math/Library (Software Edition 1.1, pp. 773-775).

4. OPTIMAL INVENTORY LEVELS FOR MODEL *C*

Referring to Figure 1b, Component 7 is the common component. From Jönsson and Silver (1989), we want to

(P2) Minimize

$$EUS_C = \int_{S_3}^{\infty}\int_{S_7-S_3}^{\infty}(x+y-S_7)f_X(x)f_Y(y)dydx$$
$$+\int_{S_7-S_6}^{S_3}\int_{S_7-x}^{\infty}(x+y-S_7)f_X(x)f_Y(y)dydx$$
$$+\int_{0}^{S_7-S_6}\int_{S_6}^{\infty}(y-S_6)f_X(x)f_Y(y)dydx$$
$$+\int_{S_3}^{\infty}\int_{0}^{S_7-S_3}(x-S_3)f_X(x)f_Y(y)dydx \tag{6}$$

such that $S_3 + S_6 + S_7 \le T$, $S_3 \le S_7$, $S_6 \le S_7$, $S_3 + S_6 \ge S_7$, $S_i \ge 0$, $i = 3, 6, 7$, and T is fixed.

The four terms in the Equation (6) pertain to all different possible regions of the two-product demand space for any given set of component stocking levels. These regions are illustrated in Figure 2, in which the numbers designating the shortage region correspond to the terms in Equation (6).

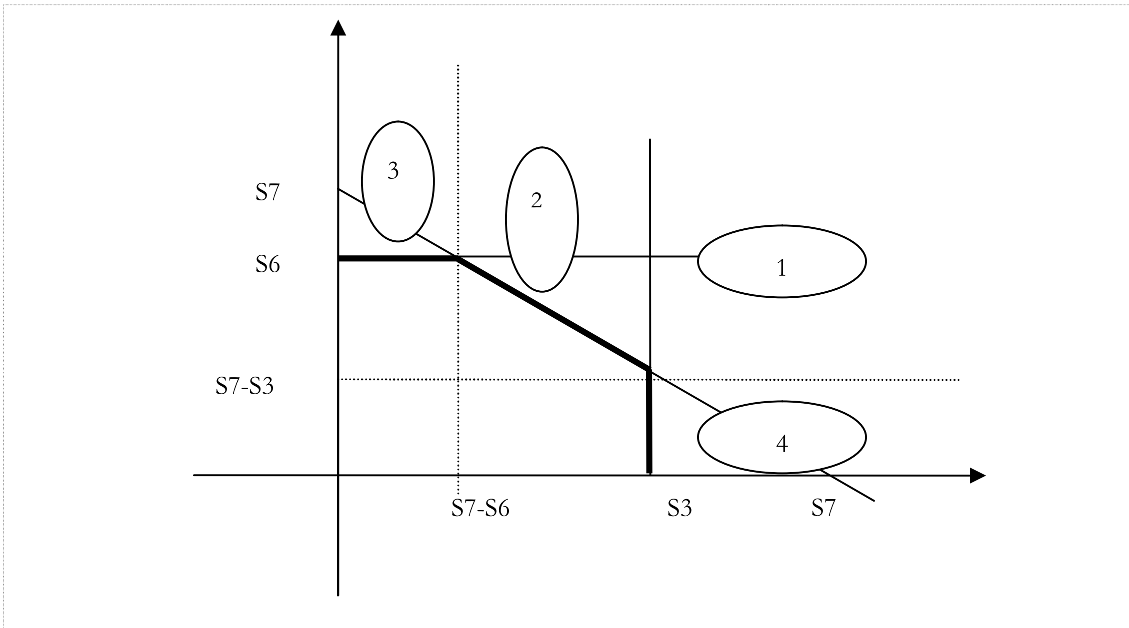

Figure 2. Shortage regions corresponding to the terms in (6).

Also, because X and Y are continuous and unbounded from above, all the available budget will be used up in an optimal allocation and so $S_7 = T - S_3 - S_6$. (Here we observe that if the demand is bounded from above, it would imply that the value of β would be large. When demands are independent and identical, Result 3 in Section 5 that follows indicates that the relative savings from commonality are independent of β. Consequently, the findings of our analysis hold, even when the demand is bounded from above.) Substituting $S_7 = T - S_3 - S_6$ into Equation (6) and simplifying, we get

(P3) Minimize

$$\begin{aligned}&EUS_C\\&=\phi_1(S_3)[\phi_4(T-2S_3-S_6)-(T-2S_3-S_6)\phi_3(T-2S_3-S_6)]\\&+[1-\phi_1(T-S_3-2S_6)][\phi_4(S_6)-S_6\phi_3(S_6)]\\&+\phi_2(S_3)-S_3\phi_1(S_3)+F_2(S_3,S_6,T-S_3-S_6)\end{aligned} \tag{7}$$

with $(S_3, S_6) \in K = \{(S_3, S_6) : 2S_3 + S_6 - T \le 0,\ S_3 + 2S_6 - T \le 0,\ T - 2S_3 - 2S_6 \le 0\}$.

To find the optimal solution to Problem P3, we first observe that the objective function (7) is convex. Then, we provide an algorithm to find the optimal solution.

Result 1 *The objective function (7) is a convex function for any unbounded continuous demand distributions over its feasible region K.*

For proof of this result, see Fu and Fong (1998).

Also, we have,

$$\frac{\partial EUS_C}{\partial S_3} = 2\phi_3(T-2S_3-S_6)\phi_1(S_3)-\phi_1(S_3) + F_1(S_3,S_6,T-S_3-S_6)$$
$$\frac{\partial EUS_C}{\partial S_6} = \phi_3(T-2S_3-S_6)\phi_1(S_3)-\phi_3(S_6) + \phi_1(T-S_3-2S_6)\phi_3(S_6) + F_1(S_3,S_6,T-S_3-S_6) \tag{8}$$

Result 2 *The optimal solution to Problem P3 can be obtained from the following steps:*

1. *Set the first order partial derivatives in Equation (8) to zero. If the root* (S_3^*, S_6^*) belongs *to the set K, then it is the optimal solution. Otherwise, go to step two.*
2. *Solve the following set of equations:* $\frac{\partial EUS_C}{\partial S_3} - 2\mu = 0$, $\frac{\partial EUS_C}{\partial S_6} - 2\mu = 0$, $T - 2S_3 - 2S_6 = 0$, and $\mu \geq 0$. *If a solution exists, it is the optimal solution. Otherwise, compare* EUS_C evaluated at ($S_3 = 0$, $S_6 = T/2$), .

$$EUS_{C1} = \phi_4(T/2) - T/2\phi_3(T/2) + E(X),$$

with that evaluated at ($S_3 = T/2$, $S_6 = 0$),

$$EUS_{C2} = \phi_2(T/2) - T/2\phi_1(T/2) + E(Y),$$

where $E(X) = \frac{r_1\alpha_1}{\beta_1} + \frac{(1-r_1)\alpha_2}{\beta_2}$ and $E(Y) = \frac{r_2\alpha 3}{\beta 3} + \frac{(1-r_2)\alpha_4}{\beta_4}$ are the means of X and Y, respectively. The one that gives the smaller value is the optimal solution.

Proof. See Appendix B.

To solve the equations $\frac{\partial EUS_C}{\partial S_3} = 0$ and $\frac{\partial EUS_C}{\partial S_6} = 0$, a subroutine called DNEQNJ from the IMSL Math/Library (Software Edition 1.1, pp. 780-783) is used. For the mixtures of Erlang demand distributions, $\phi_i(S)$, $i = 1, 2, 3, 4$ are given in Section 3, $F_1(S_3, S_6, T - S_3 - S_6)$ and $F_2(S_3, S_6, T - S_3 - S_6)$ are given in Appendix C. All are in closed forms. Thus, we have closed form expressions for the objective function EUS_C as well as the first order partial derivatives in Equation (8).

5. REDUCING SHORTAGE USING COMMONALITY

We now explore the role of commonality in reducing expected units shortage under different demand patterns. We also examine how the benefits of employing commonality change when the budget availability is changed for the base system, i.e., the one without commonality. Since different demand patterns and budget levels may have different impact on the benefits from commonality, this exploration should provide insights into the effects of commonality under various situations, and guide practitioners with helpful guidelines on when and which unique components should be replaced by a common one.

For each demand pattern, suppose a service level π is achieved in Model N for a fixed budget T:

$p(X \leq S_{N3}) = \pi$, $p(Y \leq S_{N6}) = \pi$, and $T = 2S_{N3} + 2S_{N6}$.

We compute the expected units shortage for Model N (EUS_N^*) and Model C (EUS_C^*) for this T value. The relative reduction in expected units shortage is

$$\frac{EUS_N^* - EUS_C^*}{EUS_N^*}.$$

The following two demand scenarios are considered: independent and identical Erlang distributions (IIR) of which independent and identical exponential distributions (IIE) are a special case; and independent but non-identical Erlang distributions (INR) of which independent but non-identical exponential distributions (INE) are a special case.

5.1 Independent and identical erlang distributions (IIR)

There are two parameters for the IIR case, namely α and β of an Erlang distribution. To investigate the benefits from introducing one common component to the base model, we find that, for the same π and α, the relative reduction in expected units shortage is a constant for all β values.

Result 3 *For a given* π, *the relative reduction in expected units shortage achieved in Model C versus Model N, at fixed* α, *is a constant for all* β *values when the demands are IIR.*

Proof. See Appendix D.

The numerical results for different α values are summarized in Table 1 and they display similar patterns as observed in IIE case.

Table 1. Relative reductions (%) in expected units shortage for the IIE case

$\pi \backslash \alpha$	1	2	4	5	10
0.8	6.2	6.7	7.2	7.3	7.6
0.9	15.4	16.4	17.2	17.5	18.1
0.95	25.8	27.1	28.2	28.5	29.4
0.99	48.6	50.1	51.5	51.9	53.0

Result 3 indicates that when the demand for the products can be adequately represented by independent and identical Erlang (IIR) distributions, the relative reduction in expected units shortage is independent of the scale parameter of the distribution, β, which determines the level of demand. All we need is an estimate of α, the shape parameter. Once we are satisfied that the demands can be adequately represented by identical and independent Erlang distributions, and an estimate of the parameter α, we can ascertain the percentage relative reduction in the expected units shortage (which is a surrogate for the cost savings from the commonality strategy), corresponding to any specified service level, π. Since higher values of α imply a distribution with a smaller coefficient of variation, it is interesting to observe (from Table 2) that benefits from a commonality strategy (i.e., relative reduction in the expected unit shortage) are higher when the demand distributions have smaller coefficients of variation. We offer an intuitive explanation for this as follows: When the demand coefficient of variation is lower, the stringency imposed on the optimization models by the uncertain demand environment will be less. This in turn implies that the two optimization models will be operating at a level where the diminishing marginal returns to commonality will be lower than otherwise. Therefore, the relative benefits from commonality would be higher.

Special Case: Independent and Identical Exponential demand distributions – For a given π, the relative reductions in expected units shortage are not dependent on the parameter of the independent and identical exponential demand distributions. The relative reductions in expected units shortage for different π ranging from 0.8 to 0.99 are given in the first column of Table 1, which shows that the relative reduction rises from 6.2% to 48.6% as π increases. This implies that a substantial relative reduction is possible when the inventory budget is big (the larger the value of T, the higher the value of π). We provide a formal proof of this observation in Appendix E.

A practical implication of the above observation is that if a firm wants to improve an already high service level (in terms of expected units shortage) for an item, commonality may be a viable option. Also, when the demands are IIE, since the relative reduction in expected units shortage is independent of the demand parameter, β, it implies that, to get an idea of the economic benefits of commonality, we don't need an estimate of the parameter β of the distribution. From column 1 of Table 1, we can estimate the percentage relative reduction in the expected units shortage, corresponding to any desired target service level, π.

5.2 Independent, non-identical Erlang distributions

Suppose $X \sim Er(\alpha_1,\beta_1)$ and $Y \sim Er(\alpha_2,\beta_2)$. We fix $\alpha_1 = \alpha_2 = 5$, $\beta_1 = 1$ and vary β_2 for our calculation here. The relative reductions in expected units shortage for Model C over Model N are summarized in Table 2.

Table 2. Relative reductions (%) in expected units shortage for the IIR case ($\alpha_1 = \alpha_2 = 5$, and $\beta_1 = 1$)

$\pi \setminus \beta_2$	0.2	0.5	1	2	5
0.8	3.9	6.4	7.3	6.4	3.9
0.9	9.0	15.2	17.5	15.2	9.0
0.95	14.6	24.8	28.5	24.8	14.6
0.99	27.4	45.3	51.9	45.3	27.4

Numerical results from Table 2 also suggest that the case of independent and identically distributed demands (IIR) produces the largest relative reductions compared to cases with independent but non-identical demands (INR). The relative reduction drops quickly when β_2 is moving away from the fixed β_1 (= 1).

Special Case: Independent, non-identical exponential distributions - Let $X \sim Exp(\beta_1)$ and $Y \sim Exp(\beta_2)$. We fix the mean of X at 5 (i.e., $\beta_1 = 0.2$) and vary the mean of Y from 0.5 to 100. The relative reductions in expected units shortage for Model C over Model N are summarized in Table 3.

Table 3. Relative reductions (%) in expected units shortage for the INE case ($\beta_1 = 0.2$)

$\pi \setminus \beta_2$	0.01	0.02	0.1	0.2	0.5	1	2
0.8	1.0	1.9	5.4	6.2	4.9	3.3	1.9
0.9	2.4	4.4	13.3	15.4	12.0	7.8	4.4
0.95	3.9	7.2	22.1	25.8	19.9	12.7	7.2
0.99	7.5	13.9	41.5	48.6	37.3	24.1	13.9

In both INE and INR cases, we find that the benefits from a commonality strategy (defined as the percentage relative reduction in the expected units shortage), are highest when the demand distributions for the two products have the same β (i.e., $\beta_1 = \beta_2$). Since we have set $\alpha_1 = \alpha_2$, in our numerical example, this situation and the results are the same as for IIE and IIR. The finding that the benefits from commonality decreases as β_2 moves away from β_1, indicates that as the product demand distributions become non identical, the relative savings from commonality will reduce.

We provide an intuitive explanation for this observation as follows. When the demand distributions for the two products are identical, the commonality model has the best scope to exploit the benefits from the pooling effect, and offers the best relative reduction in the expected units short compared to the non-commonality model. When the demand distributions for the two products are not identical, the structural uncertainty in the demand process will be greater, by which we mean operating in a more heterogeneous environment. This situation is analogous to the reduction in the flow rate or throughput capacity in physical systems, when the flow units are of dissimilar size. While this imposes a greater stringency on the optimal solutions in both models, the deterioration in the commonality model will be relatively less compared to the non-commonality model due to the pooling benefits from commonality. Thus, while we do derive relative savings from commonality, it is lower compared to the case in which the distributions are identical.

6. CONCLUSIONS

In this paper we provide analytical results for the product commonality problem in the context of a proto-typical, single-period two-product model when the demand distributions are mixtures of Erlang distributions. We propose efficient algorithms to compute the optimal stock levels. Relative reductions in expected units shortage by employing commonality under different demand patterns for a fixed budget are computed and compared. In general, we find that the relative reduction is small when the budget level is low relative to the demand requirements for the end products but it is substantial when the budget level is high. The case of independent and identically distributed demands appears to produce the largest relative reduction compared to cases with independent but non-identical demands. Also, the relative reduction drops quickly when the two demands vary further away from each other. In our simple model, we have assumed that the costs of the components are all the same. In practice, the cost of the common component is likely to be higher, and in such cases, we anticipate that the relative benefits from commonality would be lower.

We note that our analysis of the simple model with mixtures of Erlang distributions provides an impetus for several interesting avenues for further research. For example, from the cost perspective, it would be interesting to consider (a) different costs for the unique and the common components, and (b) different shortage costs or different margins for the two products. Also, from the demand perspective, it would be interesting to consider (a) correlated demands and (b) non-unimodal demand distributions.

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APPENDIX A: ϕ_i (*S*) FOR A MIXTURE OF TWO ERLANG DISTRIBUTIONS

Because of similarities, it suffices to derive the results for $\phi_1(S)$, and $\phi_2(S)$:

$$\phi_1(S) = \int_S^\infty f_X(x)dx = \int_S^\infty \left(\frac{r_1\beta_1^{\alpha_1}x^{\alpha_1-1}\ \exp(-\beta_1 x)}{(\alpha_1-1)!} + \frac{(1-r_1)\beta_2^{\alpha_2}x^{\alpha_2-1}\ \exp(-\beta_2 x)}{(\alpha_2-1)!}\right)dx$$

$$= r_1\ \exp(-\beta_1 S)\sum_{m=0}^{\alpha_1-1}\frac{\beta_1^m S^m}{m!} + (1-r_1)\ \exp\ (-\beta_2 S)\sum_{m=0}^{\alpha_2-1}\frac{\beta_2^m S^m}{m!},$$

$$\phi_2(S) = \int_S^\infty x f_X(x)dx = \int_S^\infty x\left(\frac{r_1\beta_1^{\alpha_1}x^{\alpha_1-1}\ \exp(-\beta_1 x)}{(\alpha_1-1)!} + \frac{(1-r_1)\beta_2^{\alpha_2}x^{\alpha_2-1}\ \exp)(-\beta_2 x)}{(\alpha_2-1)!}\right)dx$$

$$= r_1\frac{\alpha_1}{\beta_1}\ \exp(-\beta_1 S)\sum_{m=0}^{\alpha_1}\frac{\beta_1^m S^m}{m!} + (1-r_1)\frac{\alpha_2}{\beta_2}\ \exp\ (-\beta_2 S)\sum_{m=0}^{\alpha_2}\frac{\beta_2^m S^m}{m!}.$$

To obtain the above equations, we make use of the following result about the complementary incomplete gamma function

$$\Gamma(\alpha, s) = \int_s^\infty \exp\ (-u)u^{\alpha-1}du = (\alpha-1)!e^{-s}\sum_{m=0}^{\alpha-1}\frac{s^m}{m!}$$

and that

$$\int_{s}^{\infty} z^{\alpha-1} \exp(-\beta z) dz = \int_{\beta S}^{\infty} \frac{u^{\alpha-1}}{\beta^{\alpha}} \exp(-u) du = \frac{(\alpha-1)! \exp(-\beta S)}{\beta^{\alpha}} \sum_{m=0}^{\alpha-1} \frac{\beta^{m} S^{m}}{m!}.$$

APPENDIX B: PROOF OF RESULT 2

From Result 1, the objective function (7) is convex. Since the constraints of P3 are all linear, the Karush-Kuhn-Tucker (KKT) conditions are both necessary and sufficient (Bazaraa, Sherali, and Shetty (1993, pp. 162-164). Thus, for any optimal solution (S_3, S_6), there exist parameters μ_1, μ_2, μ_3 such that

$$\frac{\partial EUS_C}{\partial S_3} + 2\mu_1 + \mu_2 - 2\mu_3 = 0, \ \frac{\partial EUS_C}{\partial S_6} + \mu_1 + 2\mu_2 - 2\mu_3 = 0,$$
$$\mu_1(2S_3 + S_6 - T) = 0, \ \mu_2(S_3 + 2S_6 - T) = 0, \ \mu_3(T - 2S_3 - 2S_6) = 0,$$
$$2S_3 + S_6 - T \le 0, \ S_3 + 2S_6 - T \le 0, \ T - 2S_3 - 2S_6) \le 0, \ \mu_i \ge 0, \ i = 1, 2, 3.$$

Because the objective function (7) is continuous over the feasible region K, which is a compact set, there exists a minimizing solution to Problem P3 (Bazaraa, Sherali, and Shetty (1993, p. 41). Noting that the objective function is strictly convex, the optimal solution (S_3^*, S_6^*) is actually unique, which should occur either inside or on the boundary of triangle ABC in Figure 3.

If the optimal solution (S_3^*, S_6^*) is on the line BC (excluding points B and C), then (S_3^*, S_6^*) can be obtained by solving

$$\frac{\partial EUS_C}{\partial S_3} + 2\mu_1 = 0, \ \frac{\partial EUS_C}{\partial S_6} + \mu_1 = 0, 2S_3 + S_6 - T = 0, \text{ and } \mu_1 \ge 0.$$

However, when $2S_3 + S_6 - T = 0$, $\frac{\partial EUS_C}{\partial S_3} = \phi_1(S_3) + F_1(S_3, S_6, T - S_3 - S_6) > 0$, and it is impossible to have $\frac{\partial EUS_C}{\partial S_3} + 2\mu_1 = 0$ with $\mu_1 \ge 0$. So the optimal solution could not be on the line BC (excluding points B and C).
Using the same argument, the optimal solution could not be on the line AC (excluding point A). If the optimal solution (S_3^*, S_6^*) occurs inside the triangle ABC, then $\mu_i = 0$, $i = 1, 2, 3$, and (S_3^*, S_6^*) is obtained by solving

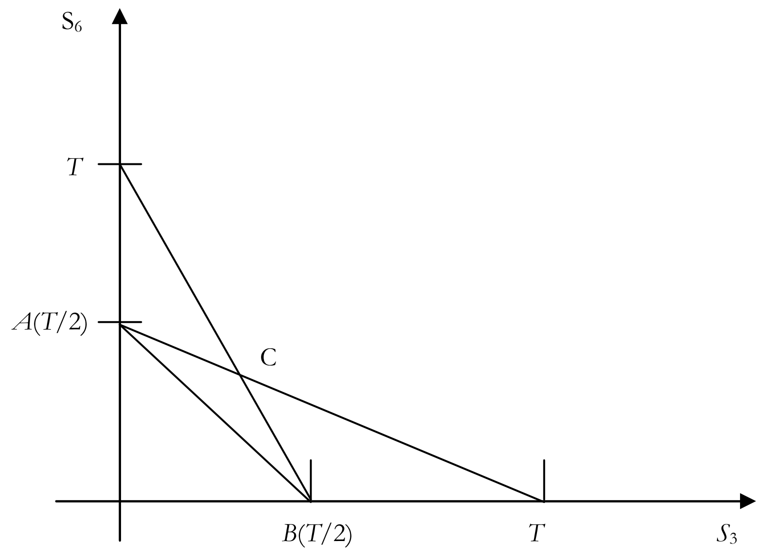

Figure 3. Feasible Region for Model *C*.

$\frac{\partial EUS_C}{\partial S_3} = 0$ and $\frac{\partial EUS_C}{\partial S_6} = 0$; if it is on the line AB (excluding points A and B), then (S_3^*, S_6^*) can be solved from $\frac{\partial EUS_C}{\partial S_3} - 2\mu_3 = 0$, $\frac{\partial EUS_C}{\partial S_6} - 2\mu_3 = 0$, $T - 2S_3 - 2S_6 = 0$, and $\mu_3 \ge 0$; otherwise, it is either point A or point B.

APPENDIX C: DISTRIBUTION FUNCTIONS OF ERLANG MIXTURES

$$F_1(a_1, a_2, a_3) = \int_{a_3-a_2}^{a_1} f_X(x)\,\phi_3(a_3-x)dx$$

$$= \int_{a_3-a_2}^{a_1} \left\{ \frac{r_1\beta_1^{\alpha_1}x^{\alpha_1-1}\ \exp(-\beta_1 x)}{(\alpha_1-1)!} + \frac{(1-r_1)\beta_2^{\alpha}x^{\alpha_2-1}\ \exp(-\beta_2 x)}{(\alpha_2-1)!} \right\} \left\{ r_2\ \exp(-\beta_3(a_3-x)) \times \sum_{m=0}^{\alpha_3-1} \frac{\beta_3^m (a_3-x)^m}{m!} \right.$$

$$\left. +(1-r_2)\ \exp\ (-\beta_4(a_3-x)\sum_{m=0}^{\alpha_4-1}\frac{\beta_4^m(a_3-x)^m}{m!} \right\} dx$$

$$= PF_1(r_1,\ r_2, \alpha_1, \beta_1, \alpha_3, \beta_3, a_1, a_2, a_3) + PF_1(r_1, 1-r_2, \alpha_1, \beta_1, \alpha_4, \beta_4, a_1, a_2, a_3)$$
$$+PF_1(1-r_1, r_2, \alpha_2, \beta_2, \alpha_3, \beta_3, a_1, a_2, a_3) + PF_1(1-r_1, 1-r_2, \alpha_2, \beta_2, \alpha_4, \beta_4, a_1, a_2, a_3);$$

$$F_2(a_1, a_2, a_3) = \int_{a_3-a_2}^{a_1} f_X(x)[(x-a_3)\,\phi_3(a_3-x) + \phi_4(a_3-x)]dx$$

$$= PF_2(r_1, r_2, \alpha_1, \beta_1, \alpha_3, \beta_3, a_1, a_2, a_3) + PF_2(r_1, 1-r_2, \alpha_1, \beta_1, \alpha_4, \beta_4, a_1, a_2, a_3)$$
$$+PF_2(1-r_1, r_2, \alpha_2, \beta_2, \alpha_3, \beta_3, a_1, a_2, a_3) + PF_2(1-r_1, 1-r_2, \alpha_2, \beta_2, \alpha_4, \beta_4, a_1, a_2, a_3)$$
$$+PF_3(r_1, r_2, \alpha_1, \beta_1, \alpha_3, \beta_3, a_1, a_2, a_3) + PF_3(r_1, 1-r_2, \alpha_1, \beta_1, \alpha_4, \beta_4, a_1, a_2, a_3)$$
$$+PF_3(1-r_1, r_2, \alpha_2, \beta_2, \alpha_3, \beta_3, a_1, a_2, a_3) + PF_3(1-r_1, 1-r_2, \alpha_2, \beta_2, \alpha_4, \beta_4, a_1, a_2, a_3)$$

where

$$PF_1(r, r', \alpha, \beta, \alpha', \beta', a_1,\ a_2, a_3)$$
$$= \int_{a_3-a_2}^{a_1} \frac{r\beta^{\alpha}x^{\alpha-1}\exp(-\beta x)}{(\alpha-1)!} r'\exp(-\beta'(a_3-x))\sum_{m=0}^{\alpha'-1}\frac{\beta'^m(a_3-x)^m}{m!}dx$$
$$= \frac{rr'\beta^{\alpha}\exp(-\beta' a_3)}{(\alpha-1)!}\sum_{m=0}^{\alpha'-1}\sum_{i=0}^{m}\left\{ \frac{\beta'^m a_3^i(-1)^{m-i}}{i!(m-i)!}\int_{a_3-a_2}^{a_1} x^{m-i+\alpha-1}\exp((\beta'-\beta)x)dx \right\},$$

$$PF_2(r, r', \alpha, \beta, \alpha', \beta', a_1, a_2, a_3)$$
$$= \int_{a_3-a_2}^{a_1} \frac{r\beta^{\alpha}x^{\alpha-1}\ \exp(-\beta x)}{(\alpha-1)!} r'\exp(-\beta'(a_3-x))\sum_{m=0}^{\alpha'-1}\frac{\beta'^m(a_3-x)^{m+1}(\alpha'-m-1)}{(m+1)!}dx$$
$$= \frac{rr'\beta^{\alpha}\ \exp(-\beta' a_3)}{(\alpha-1)!}\sum_{m=0}^{\alpha'-1}\sum_{i=0}^{m+1}\left\{ \frac{\beta'^m(\alpha'-m-1)a_3^i(-1)^{m+1-i}}{i!(m+1-i)!}\int_{a_3-a_2}^{a_1} x^{m-i+\alpha}\ \exp((\beta'-\beta)x)dx \right\},$$

$$PF_3(r, r', \alpha, \beta, \alpha', \beta', a_1, a_2, a_3) = \int_{\alpha_3-\alpha_2}^{\alpha_1} \frac{r\beta^{\alpha}x^{a-1}\ \exp(-\beta x)}{(\alpha-1)!} r'\frac{\alpha'}{\beta'}\exp(-\beta'(a_3-x))dx.$$

Note that $\int_{a_2}^{a_1} x^{\alpha-1}\exp(-\beta x)dx = \frac{(\alpha-1)!}{\beta^{\alpha}}[\exp(-\beta a_2)\sum_{m=0}^{\alpha-1}\frac{\beta^m a_2{}^m}{m!} - \exp(-\beta a_1)\sum_{m=0}^{\alpha-1}\frac{\beta^m a_1{}^m}{m!}]$ if $\beta \neq 0$, therefore F_1 and F_2 can be computed in closed form.

APPENDIX D: PROOF OF RESULT 3

Let X, Y ~ Er (α, β), independently, then $p(X \le S) = 1-\phi_1(S) = \pi$. Because $1-\phi_1(S_{N3}) = 1-\phi_3(S_{N6}) = \pi$ and $T = 2S_{N3} + 2S_{N6}$, it is easy to check that $S_{N3}\beta$, $S_{N6}\beta$, and thus $T\beta$, are independent of β

For Model N, $S_{N3}^* = S_{N6}^* = T/4$ and

$$EUS_N^* = 2\frac{\alpha}{\beta}\ \exp(-T\beta/4)\sum_{m=0}^{\alpha}\frac{\beta^m(T/4)^m}{m!} - T/2\ \exp(-T\beta/4)\sum_{m=0}^{\alpha-1}\frac{\beta^m(T/4)^m}{m!}.$$

So

$$\beta EUS_N^* = 2\alpha\ \ \exp(-T\beta/4)\sum_{m=0}^{\alpha}\frac{(T\beta/4)^m}{m!} - T\beta/2\ \ \exp(-T\beta/4)\sum_{m=0}^{\alpha-1}\frac{(T\beta/4)^m}{m!},$$

which is independent of β as $T\beta$ is independent of β.

For Model *C*, we expect $S_3^* = S_6^*$ because of identical demands. In this case,

$$\frac{\partial EUS_C}{\partial S_3} = 2\ \exp(-\beta(T-2S_3))\sum_{m=0}^{\alpha-1}\frac{\beta^m (T-3S_3)^m}{m!}\sum_{m=0}^{\alpha-1}\frac{\beta^m S_3^m}{m!} - \exp(-\beta S_3)\sum_{m=0}^{\alpha-1}\frac{\beta^m S_3^m}{m!} + \Xi,$$

where

$$\Xi = \frac{\beta^\alpha \exp(-\beta(T-2S_3))}{(\alpha-1)!}\sum_{m=0}^{\alpha-1}\sum_{i=0}^{m}\left\{\frac{\beta^m (T-2S_3)^i (-1)^{m-i}}{i!(m-i)!(m-i+\alpha)}[S_3^{m-i+\alpha} - (T-3S_3)^{m-i+\alpha}]\right\}.$$

If S_{31} is the optimal solution when $\beta = \beta_1$, it is easy to check that S_{32} is the optimal solution when $\beta = \beta_2$ where $S_{32} = \beta_1 S_{31}/\beta_2$. Thus the optimal solution (S_3^*, S_6^*) has the following property: $S_3^*\beta = S_6^*\beta$ is independent of β.

The optimal value of expected units shortage for the IIR case is

$$\begin{aligned}
EUS_C^* = {} & \sum_{m=0}^{\alpha-1}\frac{\beta^m S_3^{*m}}{m!}\left\{\frac{\alpha}{\beta}\ \exp(-\beta(T-2S_3^*))\sum_{m=0}^{\alpha}\frac{\beta^m (T-3S_3^*)^m}{m!} - (T-3S_3^*)\exp(-\beta(T-2S_3^*))\times\sum_{m=0}^{\alpha-1}\frac{\beta^m (T-3S_3^*)^m}{m!}\right\} \\
& + \left[2\exp\ (-\beta S_3^*) - \ \exp\ (\text{-}\beta(T\ \text{-}\ 2S_3^*))\sum_{m=0}^{\alpha-1}\frac{\beta^m (T-3S_3^*)^m}{m!}\right]\times\left[\frac{\alpha}{\beta}\sum_{m=0}^{\alpha}\frac{\beta^m S_3^{*m}}{m!} - S_3^*\sum_{m=0}^{\alpha-1}\frac{\beta^m S_3^{*m}}{m!}\right] + \frac{\beta^\alpha \exp\ (\text{-}\beta(T\ \text{-}\ 2S_3^*))}{(\alpha-1)!}\sum_{m=0}^{\alpha-1}\sum_{i=0}^{m+1} \\
& \times\left\{\frac{\beta^m (\alpha-m-1)(T-2S_3)^i (-1)^{m+1-i}}{i!(m+1-i)!(m-i+\alpha+1)}[S_3^{*(m-i+\alpha+1)} - (T-3S_3^*)^{m-i+\alpha+1}]\right\} + \frac{\beta^{\alpha-1}\ \exp(-\beta(T-2S_3^*))}{(\alpha-1)!}[S_3^{*\alpha} - (T-3S_3^*)^\alpha].
\end{aligned}$$

The expression for $(EUS_C^*)\beta$ as a function of β only has terms like $S_3^*\beta$ and $T\beta$, which are independent of β. Since $(EUS^*)\beta$ is independent of β for both Model *N* and Model *C*, the ratio $\dfrac{EUS_N^* - EUS_C^*}{EUS_N^*}$ is independent of β.

APPENDIX E: PROOF THAT UNDER IIE, LARGER THE BUDGET, BIGGER THE SAVINGS

When demands are IIE distributed, with parameter β, then, for Model *N*, it is easy to get the optimal value

$$EUS_N^* = \frac{2}{\beta}\exp(-\beta T/4) \tag{E1}$$

For Model *C*, due to symmetry, we have $S_3^* = S_6^*$. By taking the first order derivative when calculating optimal value *EUS*, we can have the following relationship for optimal value S_3^* and *T* as follows:

$$\exp(-3\beta S_3^* + \beta T) = 2 + \beta(4S_3^* - T) \tag{E2}$$

and the corresponding objective function is:

$$\begin{aligned}
EUS_C^* &= \frac{2}{\beta}\exp(-\beta S_3^*) + \exp(-\beta(T-2S_3^*))(4S_3^* - T) \\
&= \frac{1}{\beta}\exp(-\beta S_3^*)(2 + \beta\exp(-\beta(T-3S_3^*))(4S_3^* - T)) \\
&= \frac{1}{\beta}(3\exp(-\beta S_3^*) - 2\exp(-\beta(T-3S_3^*)))
\end{aligned}$$

Let $A = \dfrac{EUS_n^* - EUS_c^*}{EUS_N^*} = 1 - \dfrac{1}{2}(3\exp(-\beta(S_3^* - T/4)) - 2\exp(-\beta(3T/4 - 3S_3^*)))$

Taking the first order derivative of *A* with respect to T, we get

$$\begin{aligned}\frac{dA}{dT} &= \frac{1}{2}\beta\left[3\exp(-\beta(S_3^* - T/4))(\frac{dS_3^*}{dT} - 1/4) - 2\exp(-\beta(3T/4 - 3S_3^*))(3/4 - 3\frac{dS_3^*}{dT})\right] \\ &= \frac{3}{2}\beta(\frac{dS_3^*}{dT} - \frac{1}{4})\left[\exp(-\beta(S_3^* - T/4)) + 2\exp(-\beta(3T/4 - 3S_3^*))\right]\end{aligned} \quad \text{(E3)}$$

Taking derivatives of both sides of Equation (E2) with respect to T, we get

$$-\beta\exp(-3\beta S_3^* + \beta T)(3\frac{dS_3^*}{dT} - 1) = \beta(4\frac{dS_3^*}{dT} - 1)$$

Thus

$$\frac{dS_3^*}{dT} = \frac{\exp(-3\beta S_3^* + \beta T) + 1}{3\exp(-3\beta S_3^* + \beta T) + 4} > \frac{1}{4} \quad \text{(E4)}$$

Now, from (E3) and (E4), we obtain $\frac{dA}{dT} > 0$ which implies that A increases with T, and hence larger the budget T, bigger will be the relative benefit from using commonalty t.