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Single Machine Scheduling with Stochastic Processing Times or Stochastic Due-Dates to Minimize the Number of Early and Tardy Jobs

H. M. Soroush*

Department of Statistics and Operations Research, College of Science, Kuwait University, P.O. Box 5969, Safat 13060, Kuwait

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Abstract—We study a single machine scheduling problem in which processing times or due-dates are non-negative independent random variables and random weights (or penalties) are imposed on both early and tardy jobs. The objective is to find an optimal sequence that minimizes the expected total weighted number of early and tardy jobs. We explore three scenarios of the problem including a scenario with deterministic processing times and stochastic due-dates, a scenario with stochastic processing times and deterministic due-dates, and a scenario with stochastic processing times and stochastic due-dates. These problem scenarios are NP-hard to solve; however, when there are special structures on the stochasticity of processing times or due-dates, we establish certain conditions under which the various resulting cases are solvable exactly. We also approximate the solutions for the general versions of these cases. The proposed exact and approximate solution methods as well as our illustrative examples demonstrate that variations in processing times, due-dates, and earliness/tardiness penalties affect scheduling decisions. Furthermore, we show that the problem studied here is general in the sense that its special cases such as the stochastic problem of minimizing the expected weighted number of tardy jobs and the stochastic problem of minimizing the expected weighted number of early jobs are both solvable by the proposed exact or approximate methods.

Keywords—Scheduling, Single machine, Stochastic, Number of early and tardy jobs

1. INTRODUCTION

The single machine scheduling has been extensively studied in more than four decades for various performance measures (e.g., Baker, 1974, 1995; Conway et al., 1967; French, 1982; Morton and Pentico, 1993; Pinedo, 2002). The problem is concerned with finding a sequence among jobs as they proceed through a single machine in order to optimize some performance objectives. The significant of the problem is due to its importance in developing scheduling theory in more complex job shops, and its practical aspects in considering integrated processes as single machine systems.

Many researchers have studied the single machine scheduling problem with the objective of finding a sequence that minimizes the weighted number of tardy jobs. This problem, which we refer to as the "*T*" problem, is known to be NP-hard (e.g., Lenstra et al., 1977). Most of the available literature on the *T* problem deals with the deterministic case where job attributes (e.g., setup times, processing times, due dates) are known with certainty (e.g., Baptiste, 1999; Dauzere-Peres and Sevaux, 2004; Jolai, 2005; Moore, 1968).

In contrast to the deterministic *T* problem, the amount of literature on the stochastic *T* problem where some of job attributes are random variables is limited. These studies consider special cases of the problem; for example, Balut (1973) presents a chance-constrained formulation of a case where processing times (which may include setup times) are independent normal random variables. Boxma and Forst (1986) study a case where processing times and due dates have independent and identical distributions. De et al. (1991) examine a case with random processing times and an exponentially distributed common due date. Cai and Zhou (2005) consider a case with exponential processing times and random due dates. Assuming jobs have a common deterministic due-date and a common tardiness penalty, Pinedo (1983) analyzes a case with exponential processing times, while Jang (2002) and Seo et al. (2005) examine a case where processing times have normal distributions.

With the exception of Lann and Mosheiov (1996) and Soroush (2006), to the best of our knowledge, no attention is given to the single machine scheduling problem where the objective is to minimize the weighted number of both early and tardy jobs, which we refer to as the "*E-T*" problem. Lann and Mosheiov (1996) study the deterministic *E-T* problem by considering different early-tardy (*E-T*) penalty structures for jobs including job-independent (i.e., the *E-T* penalties for all jobs are

* Corresponding author's email: hsoroush@kuc01.kuniv.edu.kw

equal to one), job-dependent and symmetric (i.e., the *E-T* penalties for each job are identical), and job-dependent and asymmetric (i.e., the *E-T* penalties of each job are different). They show that the first two problem-classes are solvable in polynomial time, whereas the last class is NP-hard even when all jobs have a common due date. Soroush (2006) examines a stochastic *E-T* problem in which processing times are random variables but due dates and *E-T* penalties are known fixed quantities and the objective is to minimize the expected total weighted number of early and tardy jobs. He proposes certain conditions under which this problem is solvable exactly and also presents a very effective and efficient heuristic for the general case of the problem.

The stochastic *E-T* problem studied in this paper is a broad extension of that of Soroush (2006) and is defined as follows. There is a set of jobs that are simultaneously available to be processed sequentially on a continuously available single machine. Assume that no idle time insertion is allowed and once the processing begins no jobs can be pre-empted and the sequence remains unchanged until all jobs are finished. In this stochastic problem, processing times and/or due dates are stochastic, and each job is penalized by a random earliness weight (if the job is early) and a random tardiness weight (if the job is tardy). The random weights are independent of the amounts of time that jobs are early or tardy, that is, jobs missing their due dates by short or long periods are penalized by the same amounts. The objective is to find an optimal sequence that minimizes the expected total weighted number of early and tardy jobs on a single machine.

The importance of the proposed problem stems from the fact that in many real-world stochastic scheduling systems, each early/tardy job is penalized by the same penalty no matter how early/tardy the job is. For example, in various industries, raw materials or parts are often needed at specific times. Similarly, in air or space flight scheduling, tasks need to be performed on exact time points or during particular time windows in order to ensure the success of a flight. Also, the penalty functions in the production of perishable items such as food, drugs, etc., have similar structures. In addition, in pick-up and delivery systems, items should be picked up or delivered at certain times. Therefore, when jobs (e.g., raw material, tasks, items) are early or late, penalties are incurred no matter how early or late the jobs are (e.g., Lann and Mosheiov, 1996).

We formulate the stochastic *E-T* problem in Section 2. Three scenarios of the problem are explored in Section 3 including a scenario with deterministic processing times and stochastic due-dates, a scenario with stochastic processing times and deterministic due-dates, and a scenario with stochastic processing times and stochastic due dates. Under some structures on the stochasticity of processing times or due-dates, we present exact solution methods for the various resulting cases of the three scenarios. The solutions for the general versions of these cases are also approximated. Finally, a summary and a few concluding remarks are given in Section 4.

2. PROBLEM NOTATION AND FORMULATION

The stochastic *E-T* problem studied in this paper is as follows. A set $N = \{1, \ldots, n\}$ of jobs is available at time zero to be processed sequentially without preemption and no idle time insertions on a continuously available single machine. Let $r = [1], \ldots, [k], \ldots, [n]$ be a sequence among jobs in N where $[k]$, $k = 1, \ldots, n$, indicates the job occupying the k-th position in $r \in R$ and where R the set of all $n!$ sequences. The processing times $p_{[k]}$, $k = 1, \ldots, n$, are non-negative independent random variables with probability density functions (pdf) $f_{[k]}(.)$ (i.e., $p_{[k]} \sim f_{[k]}(.)$) and cumulative distribution functions (cdf) $F_{[k]}(.)$. Then, the completion time $t_{[k]}$ for job $[k]$, a random variable, is defined as

$$t_{[k]} = \sum_{\ell=1}^{k} p_{[\ell]}. \qquad (1)$$

The due dates $\xi_{[k]}$, $k = 1, \ldots, n$, are also non-negative independent random variables with pdfs $g_{[k]}(.)$ (i.e., $\xi_{[k]} \sim g_{[k]}(.)$) and cdfs $G_{[k]}(.)$. Let $w_{[k]}^{E}$ and $w_{[k]}^{T}$ denote the random *E-T* weights (penalties) for jobs $[k]$, $k = 1, \ldots, n$, where the expected values $\omega_{[k]}^{E} = E(w_{[k]}^{E})$ and $\omega_{[k]}^{T} = E(w_{[k]}^{T})$ exist. Moreover, $p_{[k]}$, $\xi_{[k]}$, $w_{[k]}^{E}$ and $w_{[k]}^{T}$, $k = 1, \ldots, n$, are statistically independent of each other. For each job $[k]$, $k = 1, \ldots, n$, let $X_{[k]}^{E}$ be an earliness indicator variable and $X_{[k]}^{T}$ be a tardiness indicator variable where

$$X_{[k]}^{E} = \begin{cases} 1, & \textit{if job } [k] \textit{ is early with probability } Pr(t_{[k]} < \xi_{[k]}), \\ 0, & \textit{otherwise}; \end{cases}$$

and

$$X_{[k]}^{T} = \begin{cases} 1, & \textit{if job } [k] \textit{ is tardy with probability } Pr(t_{[k]} > \xi_{[k]}), \\ 0, & \textit{otherwise}. \end{cases}$$

The expected total weighted number of early and tardy jobs in a sequence $r \in R$ (denoted by W_r), which we refer to as the expected weighted number of *E-T* jobs in r, is defined as

$$\begin{aligned} W_r &= E[\sum_{k=1}^{n} [w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{E}] \mid r] \\ &= \sum_{k=1}^{n} [\omega_{[k]}^{E} Pr(t_{[k]} < \xi_{[k]}) + \omega_{[k]}^{T} Pr(t_{[k]} > \xi_{[k]})] \mid r, \end{aligned}$$

or

$$W_r = \sum_{k=1}^{n} \omega_{[k]}^{T} + \sum_{k=1}^{n} \lambda_{[k]} Pr(t_{[k]} < \xi_{[k]}) \mid r, \quad -\infty < \lambda_{[k]} < \infty \qquad (2)$$

where $\lambda_{[k]} = \omega_{[k]}^{E} - \omega_{[k]}^{T}$ and $t_{[k]}$ is defined by (1). Note that the objective function (2) of the stochastic *E-T* problem is more general than that of (i) the stochastic T problem (i.e., when $\omega^E_{[k]} = 0$ and $\lambda_{[k]} = -\omega^T_{[k]}$, $k = 1, \ldots, n$) where only the expected weighted number of tardy jobs is minimized (e.g., Boxma and Forst, 1986; Cai and Zhou, 1997; Jang, 2002), and (ii) the stochastic earliness (E) problem (i.e., $\omega^T_{[k]} = 0$ and $\lambda_{[k]} = \omega^E_{[k]}$, $k=1, \ldots, n$) where only the expected weighted number of early jobs is minimized. Moreover, in our stochastic E-T problem, $\omega^E_{[k]}$ for all jobs $[k]$, $k = 1, \ldots, n$, are neither at most equal to nor at least equal to their $\omega^T_{[k]}$ (i.e., neither $\lambda_{[k]} \le 0$ nor $\lambda_{[k]} \ge 0$) for all jobs $[k]$, that is, $-\infty < \lambda_{[k]} < \infty$, $k = 1, \ldots, n$). Hence, in general, the problem can be neither formulated as a pure stochastic T problem (i.e., $\lambda_{[k]} \le 0$, $k = 1, \ldots, n$) or as a pure stochastic E problem (i.e., $\lambda_{[k]} \ge 0$, $k=1, \ldots, n$).

Utilizing the conventional notation, the proposed stochastic E-T problem can be represented by $1// E[\sum_{k=1}^{n} w^E_{[k]} X^E_{[k]} + w^T_{[k]} X^T_{[k]}]$.

Definition 1. For $1// E[\sum_{k=1}^{n} w^E_{[k]} X^E_{[k]} + w^T_{[k]} X^T_{[k]}]$ a sequence $r^* \in R$ is *optimal* if

$$W_{r^*} = \min_{r \in R}\{W_r\}, \tag{3}$$

where W_r is given by (2). Since $\sum_{k=1}^{n} \omega^T_{[k]} = \sum_{k=1}^{n} \omega^T_k$ is a constant and is independent of job ordering, using (2), r^* can be equivalently found as

$$\begin{aligned} &r^* = \arg\min_{r \in R}\{\sum_{k=1}^{n} \lambda_{[k]} Pr(t_{[k]} < \xi_{[k]}) \mid r\}, \\ &-\infty < \lambda_{[k]} < +\infty. \end{aligned} \tag{4}$$

Observe that $1// E[\sum_{k=1}^{n} w^E_{[k]} X^E_{[k]} + w^T_{[k]} X^T_{[k]}]$ is general in the sense that its limiting or special cases reduce to some classical single machine scheduling problems. For example, when $\omega^E_{[k]} \le \omega^T_{[k]}$, $k = 1, \ldots, n$, we get the stochastic T problem, i.e., $1// E[\sum_{k=1}^{n} w^T_{[k]} X^T_{[k]}]$, where $\omega^T_{[k]} = -\lambda_{[k]} \ge 0$ and $p_{[k]}$ and $\xi_{[k]}$ are random variables (e.g., Boxma and Forst, 1986; Cai and Zhou, 2005). In this case, if $\xi_{[k]} = d_{[k]}$, $k = 1, \ldots, n$, where $d_{[k]}$ are known fixed quantities, we have $1// E[\sum_{k=1}^{n} w^T_{[k]} X^T_{[k]}]$ where $p_{[k]}$ are random variables (e.g., Jang, 2002; Seo et al., 2005). When $\omega^E_{[k]} \ge \omega^T_{[k]}$, $k = 1, \ldots, n$, we get the stochastic E problem, that is, $1// E[\sum_{k=1}^{n} w^E_{[k]} X^E_{[k]}]$, where $\omega^E_{[k]} = \lambda_{[k]} \ge 0$ and $p_{[k]}$ and $\xi_{[k]}$ are random variables. When $p_{[k]}$, $\xi_{[k]}$, $w^E_{[k]}$, and $w^T_{[k]}$, $k = 1, ..., n$, are all known with certainty, we have the deterministic E-T problem, that is, $1// \sum_{k=1}^{n} w^E_{[k]} X^E_{[k]} + w^T_{[k]} X^T_{[k]}$ (e.g., Lann and Mosheiov, 1996) where $X^E_{[k]}$ and $X^T_{[k]}$ are defined as

$$X^E_{[k]} = \begin{cases} 1, & \textit{if } t_{[k]} < d_{[k]}, \\ 0, & \textit{otherwise}; \end{cases} \quad \text{and} \quad X^T_{[k]} = \begin{cases} 1, & \textit{if } t_{[k]} > d_{[k]}, \\ 0, & \textit{otherwise}. \end{cases}$$

In this case, if $w^E_{[k]} = 0$ with certainty, we get the deterministic T problem, i.e., $1// \sum_{k=1}^{n} w^T_{[k]} X^T_{[k]}$ (e.g., Baptiste, 1999; Lenstra et al., 1977; Moore, 1968). When $Pr(t_{[k]} < \xi_{[k]}) = 1$, $k = 1, \ldots, n$ (i.e., all jobs are early with certainty), $Pr(t_{[k]} > \xi_{[k]}) = 1$, $k = 1, \ldots, n$ (i.e., all jobs are tardy with certainty), or $\omega^E_k = \omega^T_k$, $k = 1, \ldots, n$, using (4), any sequence $r \in R$ is optimal for the proposed stochastic E-T problem.

A naive approach to exactly solve $1// E[\sum_{k=1}^{n} w^E_{[k]} X^E_{[k]} + w^T_{[k]} X^T_{[k]}]$ is to (i) enumerate all sequences $r \in R$, (ii) derive the joint cdf of $p_{[k]}$ for all jobs $[k]$, $k = 1, \ldots, n$ in each $r \in R$, (iii) use (1) to get the cdf of each $t_{[k]}$, $k = 1, \ldots, n$, in $r \in R$, (iv) compute $Pr(t_{[k]} < \xi_{[k]})$, $k = 1, \ldots, n$, in each $r \in R$, (v) apply (2) to compute W_r, $r \in R$, and then (vi) use (3) or (4) to find r^*. This approach may be the only one if there are no special structures on the stochasticity of processing times or due dates. However, since the general case of the deterministic problem is NP-hard (e.g., Lann and Mosheiov, 1996), the general case of the stochastic problem is even harder to solve due to the additional difficulty of computing $Pr(t_{[k]} < \xi_{[k]})$, $k = 1, \ldots, n$, which require complex integrations of multi-variate distributions.

3. PROBLEM SCENARIOS AND SOLUTIONS

To analyze $1// E[\sum_{k=1}^{n} w^E_{[k]} X^E_{[k]} + w^T_{[k]} X^T_{[k]}]$, consider a sequence $\theta ij\delta$ in R where θ is an arbitrary sub-sequence of jobs, excluding jobs i and j and jobs in δ, appearing in the first $q_1 - 1$ positions (i.e., $\theta = [1], ..., [q_1 - 1]$), jobs i and j are adjacent which respectively occupy positions q_1 and $q_1 + 1$ (i.e., $[q_1] = i$ and $[q_1 + 1] = j$), and δ is an arbitrary sub-sequence of jobs, excluding jobs i and j and jobs in θ, occupying positions $q_1 + 2$ to n (i.e., $\delta = [q_1 + 2], \ldots, [n]$). Then, the expected weighted number of E-T jobs in $\theta ij\delta$, using (1) and (2), is

$$\begin{aligned} W_{\theta ij\delta} = &\sum_{k=1}^{n} \omega^T_k + \sum_{k=1}^{q_1-1} \lambda_{[k]} Pr(\sum_{\ell=1}^{k} p_{[\ell]} < \xi_{[k]}) \\ &+ \lambda_i Pr(p_\theta + p_i < \xi_i) \\ &+ \lambda_j Pr(p_\theta + p_i + p_j < \xi_j) \\ &+ \sum_{k=q_1+2}^{n} \lambda_{[k]} Pr(p_\theta + p_i + p_j + \sum_{\ell=q_1+2}^{k} p_{[\ell]} < \xi_{[k]}), \end{aligned}$$

where $p_\theta = \sum_{k=1}^{q_1-1} p_{[k]}$. Interchanging jobs i and j in $\theta ij\delta$ produces another sequence $\theta ji\delta$ in R whose $W_{\theta ji\delta}$ can be similarly computed as $W_{\theta ij\delta}$. Then, $\theta ij\delta$ is *preferred to* $\theta ji\delta$, denoted by $\theta ij\delta \succ \theta ji\delta$, (i.e., job i immediately precedes job j) for every jobs $i \neq j \in N$ and every choice of θ and δ iff

$$\begin{aligned}\Delta W_{ij} &= W_{\theta ij\delta} - W_{\theta ji\delta}\\ &= \lambda_i[Pr(p_\theta + p_i < \xi_i) - Pr(p_\theta + p_j + p_i < \xi_i)]\\ &\quad - \lambda_j[Pr(p_\theta + p_j < \xi_j) - Pr(p_\theta + p_i + p_j < \xi_j)] \le 0, \end{aligned} \tag{5}$$

where $Pr(p_\theta + p_j + p_i < \xi_i) \le Pr(p_\theta + p_i < \xi_i)$ and $Pr(p_\theta + p_i + p_j < \xi_j) \le Pr(p_\theta + p_j < \xi_j)$ due to non-negative processing times and non-negative due dates. From (5), we observe that $\Delta W_{ij} = -\Delta W_{ji}$ does not depend on jobs in δ at all; however, it depends on jobs in θ but it is independent of their ordering. Hence, in some occasions, we replace ΔW_{ij} by $\Delta W_{ij}(\theta)$ to remind the reader of this dependence.

Inequality (5) is too general to allow the development of useful statements to establish the relation $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ. However, when there are special structures on the stochasticity of processing times or due-dates, we can use this inequality to investigate and solve exactly the various resulting cases for three scenarios of the problem including a scenario with deterministic processing times and stochastic due-dates, a scenario with stochastic processing times and deterministic due-dates, and a scenario with stochastic processing times and stochastic due-dates.

3.1 Deterministic processing times and stochastic due dates

Consider the scenario $1/p_k = \pi_k, \xi_k \sim g_k(.)/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ where $p_k = \pi_k$ and $\xi_k \sim g_k(.)$, $k = 1, \ldots, n$, and the non-negative quantities π_k are known with certainty. Using (5), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ (i.e., job i immediately precedes job j) iff

$$\begin{aligned}\lambda_i[G_i(\pi_\theta + \pi_i + \pi_j) - G_i(\pi_\theta + \pi_i)]&\\ \le \lambda_j[G_j(\pi_\theta + \pi_i + \pi_j) - G_j(\pi_\theta + \pi_j)],&\end{aligned} \tag{6}$$

where $\pi_\theta = \sum_{k=1}^{q_1-1} \pi_{[k]}$, $G_i(\pi_\theta + \pi_i + \pi_j) \ge G_i(\pi_\theta + \pi_i)$, and $G_j(\pi_\theta + \pi_i + \pi_j) \ge G_j(\pi_\theta + \pi_j)$. The sufficient conditions to satisfy (6) are then as follow.

(i) $\lambda_i \le 0 \le \lambda_j$; or (7)

(ii) $0 \le \lambda_i \le \lambda_j$, $G_i(\pi_\theta + \pi_i + \pi_j) - G_i(\pi_\theta + \pi_i) \le G_j(\pi_\theta + \pi_i + \pi_j) - G_j(\pi_\theta + \pi_j)$; or (8)

(iii) $\lambda_i \le \lambda_j \le 0$, $G_i(\pi_\theta + \pi_i + \pi_j) - G_i(\pi_\theta + \pi_i) \ge G_j(\pi_\theta + \pi_i + \pi_j) - G_j(\pi_\theta + \pi_j)$. (9)

Below, we use (7) − (9) to examine some cases of $1/p_k = \pi_k, \xi_k \sim g_k(.)/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$.

3.1.1 Identically distributed due-dates

Suppose that ξ_k, $k = 1, \ldots, n$, are independently and identically distributed (i.i.d) random variables with a general pdf $g(.)$ (i.e., $\xi_k \sim g(.)$).

Theorem 1. For $1/p_k = \pi_k, \xi_k \sim g(.)/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if

(i) $\lambda_i \le 0 \le \lambda_j$; or
(ii) $0 \le \lambda_i \le \lambda_j$ and $\pi_i \ge \pi_j$; or
(iii) $\lambda_i \le \lambda_j \le 0$ and $\pi_i \le \pi_j$.

Proof. It immediately follows from conditions (7) − (9).

Corollary 1. For $1/p_k = \pi_k, \xi_k \sim g(.)/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, a sequence $[1], \ldots, [\ell], [\ell + 1], \ldots, [n]$, $\ell \in \{0, 1, \ldots, n\}$, where jobs $[0]$ and $[n + 1]$ do not exist, is optimal if

(i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$, and
(ii) $\pi_{[1]} \le \ldots \le \pi_{[\ell]}$ and $\pi_{[\ell+1]} \ge \ldots \ge \pi_{[n]}$.

Proof. Consider a sequence r': $\theta i\pi j\rho$ in R, $i \neq j \in N$, where $\theta = [1], \ldots, [q_1 - 1]$, $i = [q_1]$, $\pi = [q_1 + 1], \ldots, [q_2 - 1]$, $j = [q_2]$, and $\rho = [q_2 + 1], \ldots, [n]$, that is, r': $\theta i\pi j\rho = \theta[q_1][q_1 + 1] \ldots [q_2 - 1][q_2]\rho$. Suppose that jobs in r' are arranged according to Corollary 1. Interchanging jobs i and j in r', $i \neq j \in N$, produces another sequence r: $\theta j\pi i\rho = \theta[q_2][q_1 + 1] \ldots [q_2 - 1][q_1]\rho$ in R. We show that switching jobs $i \neq j \in N$ in r' increases the expected number of E-T jobs, that is, $W_{r'} < W_r$. Using (5), $W_{r'}$ can be written as

$$\begin{aligned}W_{r'} &= W_{\theta[q_1+1][q_1]\ldots[q_2-1][q_2]\rho} + \Delta W_{[q_1][q_1+1]}\\ &\quad\ldots\\ &\quad\ldots\\ &\quad\ldots\\ &= W_{\theta[q_1+1]\ldots[q_2-1][q_1][q_2]\rho} + \Delta W_{[q_1][q_1+1]} + \ldots + \Delta W_{[q_1][q_2-1]}\\ &= W_{\theta[q_1+1]\ldots[q_2-1][q_2][q_1]\rho} + \Delta W_{[q_1][q_1+1]} + \ldots\\ &\quad + \Delta W_{[q_1][q_2-1]} + \Delta W_{[q_1][q_2]}\\ &= W_{\theta[q_1+1]\ldots[q_2][q_2-1][q_1]\rho} + \Delta W_{[q_1][q_1+1]} + \ldots\\ &\quad + \Delta W_{[q_1][q_2-1]} + \Delta W_{[q_1][q_2]} + \Delta W_{[q_2-1][q_2]}\\ &\quad\ldots\\ &\quad\ldots\\ &\quad\ldots\\ &= W_r + \sum_{(p,q)\in r'\to r} \Delta W_{pq},\end{aligned} \tag{10}$$

where $\sum_{(p,q)\in r'\to r} \Delta W_{pq}$, the sum of ΔW_{pq} for the adjacent jobs $p \neq q \in N$ interchanged in r' to get r, is given as

$$\sum_{(p,q)\in r'\to r} \Delta W_{pq} = \Delta W_{[q_1][q_1+1]} + \ldots + \Delta W_{[q_1][q_2-1]} + \Delta W_{[q_1][q_2]} + \Delta W_{[q_2-1][q_2]} + \ldots + \Delta W_{[q_1+1][q_2]}.$$

Since jobs in r' are arranged according to Corollary 1 (see also conditions (i) − (iii) of Theorem 1), based on (5), $\Delta W_{pq} \leq 0$ yielding $\sum_{(p,q)\in r'\to r} \Delta W_{pq} \leq 0$ where the strict inequality holds if and only if jobs p and q are not identical. Using (10), then $W_{r'} < W_r$ if and only if the pairs of adjacent jobs switched in r' to get r are not the same. In other words, the interchange of jobs p and q in r', $p \neq q \in N$, increases the expected number of E-T jobs. Therefore, since the interchange of any pair of jobs in a sequence found according to Corollary 1 increases the expected number of E-T jobs, then such a sequence itself must be optimal.

In general, since the optimality conditions of Corollary 1 do not hold among jobs, r^* cannot be found. However, if these conditions are satisfied among the jobs in a sequence $[1], \ldots, [\ell], [\ell+1], \ldots, [n]$, $\ell \in \{0, 1, \ldots, n\}$, then $\lambda_{[1]}/\pi_{[1]} \leq \ldots \leq \lambda_{[\ell]}/\pi_{[\ell]} \leq 0 \leq \lambda_{[\ell+1]}/\pi_{[\ell+1]} \leq \ldots \leq \lambda_{[n]}/\pi_{[n]}$, or $\lambda_{[1]}/\pi_{[1]} \leq \ldots \leq \lambda_{[n]}/\pi_{[n]}$. The converse of the latter may not be true, that is, it is possible to have $\lambda_{[p]}/\pi_{[p]} \leq \lambda_{[q]}/\pi_{[p]}$, $p < q = 1, \ldots, n$, such that the conditions of Theorem 1 do not hold for every job $[p]$ preceding job $[q]$. Hence, we can approximate the solution (i.e., find a candidate for r^*) for $1/p_k = \pi_k, \xi_k \sim g(.)/\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$ by arranging jobs in non-decreasing order of λ_k/π_k, $k = 1, \ldots, n$. In the case where $\lambda_i/\pi_i = \lambda_j/\pi_j$, $i \neq j \in N$, the job with smaller λ_k, $k \in \{i, j\}$, is placed before the other job. This is due to the fact that $\lambda_i/\pi_i = \lambda_j/\pi_j \leq 0$, $i \neq j \in N$, if only $\lambda_i \leq \lambda_j \leq 0$(i.e., $\omega^E_k \leq \omega^T_k$, $k = i, j$); thus, the job with smaller $\lambda_k \leq 0$, $k \in \{i, j\}$, is scheduled first to avoid large tardiness penalty. Also, $\lambda_i/\pi_i = \lambda_j/\pi_j \geq 0$, $i \neq j \in N$, if only $0 \leq \lambda_i \leq \lambda_j$ (i.e., $\omega^E_k \geq \omega^T_k$, $k = i, j$); hence, the job with smaller $\lambda_k \geq 0$, $k \in \{i, j\}$, is scheduled first to avoid large earliness penalty.

Remark 1. Since $1/p_k = \pi_k, \xi_k \sim g(.)/\ E[\sum_{k=1}^{n} w^T_{[k]}X^T_{[k]}]$ (i.e., the stochastic T problem) and $1/p_k = \pi_k, \xi_k \sim g(.)/\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]}]$ (i.e., the stochastic E problem) are special cases of $1/p_k = \pi_k, \xi_k \sim g(.)/\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$ (i.e., the stochastic E-T problem) (see Section 2), based on Corollary 1, a sequence $[1], \ldots, [n]$ is optimal for the stochastic T problem if $\lambda_{[1]} \leq \ldots \leq \lambda_{[n]}$ (i.e., $\omega^T_{[1]} \geq \ldots \geq \omega^T_{[n]}$) and $\pi_{[1]} \leq \ldots \leq \pi_{[n]}$, and is optimal for the stochastic E problem if $\lambda_{[1]} \leq \ldots \leq \lambda_{[n]}$ (i.e., $\omega^E_{[1]} \leq \ldots \leq \omega^E_{[n]}$) and $\pi_{[1]} \geq \ldots \geq \pi_{[n]}$. Also, in the stochastic T problem if $\omega^T_{[k]} = \omega$, $k = 1, \ldots, n$ (i.e., jobs have a common mean tardiness penalty), arranging jobs in non-decreasing order of π_k (i.e., according to shortest processing time (SPT) rule) yields r^*. In the stochastic E problem if $\omega^E_{[k]} = \omega$, $k = 1, \ldots, n$, (i.e., jobs have a common mean earliness penalty), arranging jobs in non-increasing order of π_k (i.e., based on longest processing time (LPT) rule) gives r^*.

3.1.2 Distinctly distributed due-dates

When $\xi_k \sim g_k(.)$, $k = 1, \ldots, n$, it is more difficult to develop useful statements to establish $\theta ij\delta \succ \theta ji\delta$ for $1/p_k = \pi_k, \xi_k \sim g_k(.)/\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$ Nevertheless, such statements can be derived for the case. where ξ_k, $k = 1, \ldots, n$, are exponentially distributed (i.e., $\xi_k \sim \exp(\gamma_k)$) with $G_k(x) = 1 - \exp(-\gamma_k x)$ and means $1/\gamma_k$. The use of exponential distribution in shop scheduling is justified by, for example, Boxma and Forst (1986), Cai and Zhou (1997, 2005), Jang (2002), and Pinedo (1983).

Theorem 2. For $1/p_k = \pi_k, \xi_k \sim \exp(\gamma_k)\ /\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if

(i) $\lambda_i \leq 0 \leq \lambda_j$; or
(ii) $0 \leq \lambda_i \leq \lambda_j$, $\gamma_i \geq \gamma_j$, and $\pi_i/\gamma_i \geq \pi_j/\gamma_j$; or
(iii) $\lambda_i \leq \lambda_j \leq 0$, $\gamma_i \leq \gamma_j$, and $\pi_i/\gamma_i \leq \pi_j/\gamma_j$.

Proof. Using $G_k(x) = 1 - \exp(-\gamma_k x)$ in (8), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if $0 \leq \lambda_i \leq \lambda_j$

and

$$[1 - \exp(-\gamma_i\pi_j)]\exp[-\gamma_i(\pi_\theta + \pi_i)] \leq [1 - \exp(-\gamma_j\pi_i)]\exp[-\gamma_j(\pi_\theta + \pi_j)]. \qquad (11)$$

From (11), for every $i \neq j \in N$ and every θ, $1 - \exp(-\gamma_i\pi_j) \leq 1 - \exp(-\gamma_j\pi_i)$ iff $\gamma_i\pi_j \leq \gamma_j\pi_i$ and $\exp[-\gamma_i(\pi_\theta + \pi_i)] \leq \exp[-\gamma_j(\pi_\theta + \pi_j)]$ iff $\gamma_i(\pi_\theta + \pi_i) \geq \gamma_j(\pi_\theta + \pi_j)$. The latter condition always holds if $\gamma_i \geq \gamma_j$ and $\gamma_i\pi_i \geq \gamma_j\pi_j$. Since $\gamma_i \geq \gamma_j$ and $\gamma_i\pi_j \leq \gamma_j\pi_i$ imply $\pi_i \geq \pi_j$, then $\gamma_i\pi_i \geq \gamma_j\pi_j$. Hence, using (8) and (11), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if $0 \leq \lambda_i \leq \lambda_j$, $\gamma_i \geq \gamma_j$ and $\pi_i/\gamma_i \geq \pi_j/\gamma_j$ (due to $\gamma_i\pi_j \leq \gamma_j\pi_i$). Similarly, using (9), we can show that $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if $\lambda_i \leq \lambda_j \leq 0$, $\gamma_i \leq \gamma_j$, and $\pi_i/\gamma_i \leq \pi_j/\gamma_j$. Therefore, $\theta ij\delta \succ \theta ji\delta$ holds for every $i \neq j \in N$ and every θ and δ if (i) $\lambda_i \leq 0 \leq \lambda_j$; or $0 \leq \lambda_i \leq \lambda_j$, $\gamma_i \geq \gamma_j$, and $\pi_i/\gamma_i \geq \pi_j/\gamma_j$; or (iii) $\lambda_i \leq \lambda_j \leq 0$, $\gamma_i \leq \gamma_j$, and $\pi_i/\gamma_i \leq \pi_j/\gamma_j$.

Corollary 2. For $1/p_k = \pi_k$, $\xi_k \sim \exp(\gamma_k)$/ $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, a sequence [1], …, [ℓ], [ℓ + 1], …, [n], $\ell \in \{0, 1, \ldots, n\}$, is optimal if
(i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$, and
(ii) $\gamma_{[1]} \le \ldots \le \gamma_{[\ell]}$, $\gamma_{[\ell+1]} \ge \ldots \ge \gamma_{[n]}$, $\pi_{[1]}/\gamma_{[1]} \le \ldots \le \pi_{[\ell]}/\gamma_{[\ell]}$, and $\pi_{[\ell+1]}/\gamma_{[\ell+1]} \ge \ldots \ge \pi_{[n]}/\gamma_{[n]}$.

Proof. Using an approach similar to that of the proof of Corollary 1, we show that a sequence obtained by arranging jobs according to Corollary 2 (see also conditions (i) − (iii) of Theorem 2) is optimal.

Let us relax the optimality conditions of Corollary 2 by removing $\gamma_{[1]} \le \ldots \le \gamma_{[\ell]}$ and $\gamma_{[\ell+1]} \ge \ldots \ge \gamma_{[n]}$. If the remaining conditions of the corollary hold among the jobs in a sequence [1], …, [ℓ], [ℓ + 1], …, [n], $\ell \in \{0, 1, \ldots, n\}$, then $\lambda_{[1]}\gamma_{[1]}/\pi_{[1]} \le \ldots \le \lambda_{[\ell]}\gamma_{[\ell]}/\pi_{[\ell]} \le 0 \le \lambda_{[\ell+1]}\gamma_{[\ell+1]}/\pi_{[\ell+1]} \le \ldots \le \lambda_{[n]}\gamma_{[n]}/\pi_{[n]}$ (i.e., $\lambda_{[1]}\gamma_{[1]}/\pi_{[1]} \le \ldots \le \lambda_{[n]}\gamma_{[n]}/\pi_{[n]}$). Hence, we can approximate the solution (i.e., find a candidate for r^*) for $1/p_k = \pi_k$, $\xi_k \sim \exp(\gamma_k)$/ $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ by arranging jobs in non-decreasing order of $\lambda_k\gamma_k/\pi_k$.

Remark 2. Based on Corollary 2, a sequence [1], …, [n] in which $\lambda_{[1]} \le \ldots \le \lambda_{[n]}$ is optimal for $1/p_k = \pi_k$, $\xi_k \sim \exp(\gamma_k)$/ $E[\sum_{k=1}^{n} w_{[k]}^T X_{[k]}^T]$ if $\gamma_{[1]} \le \ldots \le \gamma_{[n]}$ and $\pi_{[1]}/\gamma_{[1]} \le \ldots \le \pi_{[\ell]}/\gamma_{[n]}$, and is optimal for $1/p_k = \pi_k$, $\xi_k \sim \exp(\gamma_k)$ / $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E]$ if $\gamma_{[1]} \le \ldots \le \gamma_{[n]}$ and $\pi_{[1]}/\gamma_{[1]} \ge \ldots \ge \pi_{[n]}/\gamma_{[n]}$.

Remark 3. For $1/p_k = \pi_k$, $\xi_k \sim g_k(.)$/ $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ according to Corollaries 1 or 2, jobs [k], k= 1, …, ℓ, ℓ + 1, …, n, $\ell \in \{0, 1, \ldots, n\}$, are arranged in r^* in non-decreasing order of $\omega_{[k]}^E - \omega_{[k]}^T$ (i.e., $\lambda_{[k]}$) where there are additional conditions imposed on $\pi_{[k]}$ or $\gamma_{[k]}$ of jobs [k], k = 1, …, ℓ (i.e., jobs with $\omega_{[k]}^E \le \omega_{[k]}^T$ or $\lambda_{[k]} \le 0$) as well as on those of jobs [k], k = ℓ + 1, …, n (i.e., jobs with $\omega_{[k]}^E \ge \omega_{[k]}^T$ or $\lambda_{[k]} \ge 0$). Hence, $1/p_k = \pi_k$, $\xi_k \sim g_k(.)$/ $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ among jobs [1], …, [ℓ], [ℓ + 1], …, [n] where $-\infty < \lambda_{[k]} < +\infty$ (i.e., the stochastic E−T problem) is a mixture of $1/p_k = \pi_k$, $\xi_k \sim g_k(.)$/ $E[\sum_{k=1}^{n} w_{[k]}^T X_{[k]}^T]$ among jobs [1], …, [ℓ] where $\omega_{[k]}^T = -\lambda_{[k]} \ge 0$ (i.e., the stochastic T problem) and $1/p_k = \pi_k$, $\xi_k \sim g_k(.)$/ $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E]$ among jobs [ℓ + 1], …, [n] where $\omega_{[k]}^E = \lambda_{[k]} \ge 0$ (i.e., the stochastic E problem).

3.2 Stochastic processing times and deterministic due dates

Consider the scenario $1/p_k \sim f_k(.)$, $\xi_k = d_k$/ $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ where $p_k \sim f_k(.)$ and $\xi_k = d_k$, k = 1, ..., n, with d_k being known constants (see also Soroush (2006)). Then, using (5), $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ iff

$$\Delta W_{ij}(\theta) = \lambda_i[Pr(p_\theta + p_i < d_i) - Pr(p_\theta + p_j + p_i < d_i)] - \lambda_j[Pr(p_\theta + p_j < d_j) - Pr(p_\theta + p_i + p_j < d_j)] \le 0, \quad (12)$$

or

$$\lambda_i[\tilde{F}_\theta * F_i(d_i) - \tilde{F}_\theta * F_j * F_i(d_i)] \le \lambda_j[\tilde{F}_\theta * F_j(d_j) - \tilde{F}_\theta * F_i * F_j(d_j)] \le 0, \quad (13)$$

where $\tilde{F}_\theta(x) = F_{[1]} * \ldots * F_{[q_1-1]}(x)$, $\tilde{F}_\theta * F_i(x)$, $\tilde{F}_\theta * F_j(x)$, and $\tilde{F}_\theta * F_i * F_j(x)$ are the convolutions of the cdfs of $p_{[k]}$ for jobs [k], k = 1, ..., $q_1 - 1$, in θ, for jobs in θ and job i, for jobs in θ and job j, and for jobs in θ and jobs i and j, respectively. Moreover, $\tilde{F}_\theta * F_j * F_i(d_i) \le \tilde{F}_\theta * F_i(d_i)$ and $\tilde{F}_\theta * F_i * F_j(d_j) \le \tilde{F}_\theta * F_j(d_j)$ due to non-negative processing times.

Since (12) or (13) are too general to allow the development of practical statements to establish $\theta ij\delta \succ \theta ji\delta$, we analyze the following cases of $1/p_k \sim f_k(.)$, $\xi_k = d_k$ / $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$.

3.2.1 Identically distributed processing times

Assume that p_k, k = 1, …, n, are i.i.d. with a general pdf $f(.)$ (i.e., $p_k \sim f(.)$) and $\xi_k = d_k$ (known constant). Using (13), $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ iff

$$\lambda_i[F^{(q_1)*}(d_i) - F^{(q_1+1)*}(d_i)] \le \lambda_i[F^{(q_1)*}(d_j) - F^{(q_1+1)*}(d_j)] \quad (14)$$

where $F^{(q_1)*}(x)$ is the convolution of the cdfs $F_k(.)$ of job i and the $q_1 - 1$ jobs in θ, and $F^{(q_1+1)*}(x)$ is that of jobs i and j and the jobs in θ.

Inequality (14) is still difficult to be explored for i.i.d. processing times with a general $f(.)$. However, we can examine a case where $p_k \sim \exp(a)$ with $F_k(x) = 1 - \exp(-ax)$ and mean $1/a$, k = 1, …, n.

Theorem 3. For $1/p_k \sim \exp(a)$, $\xi_k = d_k$/ $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if

(i) $\lambda_i \le 0 \le \lambda_j$; or
(ii) $d_i \le d_j$ and either $0 \le \delta_i \le \delta_j$ or $\psi_i \le \psi_j \le 0$ where

$$\delta_k = \lambda_k d_k \exp(-ad_k), \tag{15}$$

and

$$\psi_k = \lambda_k d_k^{n-1} \exp(-ad_k); \text{ or} \tag{16}$$

(iii) $d_i \ge d_j$ and either $\delta_i \le \delta_j \le 0$ or $0 \le \psi_i \le \psi_j$.

Proof. For p_k~exp(a), $k = 1, \ldots, n$, using the relationship between Poisson and exponential distributions, (14) can be equivalently written as

$$\begin{aligned}\lambda_i\Big[\sum_{n=q_1}^{\infty}[N(d_i)=n]-\sum_{n=q_1+1}^{\infty}[N(d_i)=n]\Big]\\ \le \lambda_j\Big[\sum_{n=q_1}^{\infty}[N(d_j)=n]-\sum_{n=q_1+1}^{\infty}[N(d_j)=n]\Big],\end{aligned} \tag{17}$$

where $N(t)$ has a Poisson distribution with mean rate a. Simplifying (17), we have

$$\lambda_i[(\alpha d_i)^{q_1}\exp(-\alpha d_i)/q_1!] \le \lambda_j[(\alpha d_j)^{q_1}\exp(-\alpha d_j)/q_1!],$$

or

$$\lambda_i d_i^{q_1}\exp(-\alpha d_i) \le \lambda_j d_j^{q_1}\exp(-\alpha d_j), \tag{18}$$

where (18) depends on jobs $i \ne j \in N$ as well as the position q_1 of job i. For $\lambda_i \le 0 \le \lambda_j$, $i \ne j \in N$, (18) is always satisfied. For $d_i \le d_j$, $i \ne j \in N$, (18) holds for $q_1 = 1, \ldots, n-1$, (job j occupies position $q_1 + 1$) if either

(i) $0 \le \lambda_i d_i \exp(-\alpha d_i) \le \lambda_j d_j \exp(-\alpha d_j)$; or
(ii) $\lambda_i d_i^{n-1}\exp(-\alpha d_i) \le \lambda_j d_j^{n-1}\exp(-\alpha d_j) \le 0$.

For $d_i \ge d_j$, $i \ne j \in N$, (18) holds if either

(i) $\lambda_i d_i \exp(-\alpha d_i) \le \lambda_j d_j \exp(-\alpha d_j) \le 0$; or
(ii) $0 \le \lambda_i d_i^{n-1}\exp(-\alpha d_i) \le \lambda_j d_j^{n-1}\exp(-\alpha d_j)$.

Based on Theorem 3, the following corollary provides the conditions under which an optimal sequence can be found.

Corollary 3. For $1/p_k$~exp(a), $\xi_k = d_k/\ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, a sequence [1], …, [$\ell$], [$\ell$ + 1], …, [n], $\ell \in \{0, 1, \ldots, n\}$, is optimal if either

(i) $\delta_{[1]} \le \ldots \le \delta_{[\ell]} \le 0 \le \delta_{[\ell+1]} \le \ldots \le \delta_{[n]}$ where δ_k is given by (15), $d_{[1]} \ge \ldots \ge d_{[\ell]}$ and $d_{[\ell+1]} \le \ldots \le d_{[n]}$; or
(ii) $\psi_{[1]} \le \ldots \le \psi_{[\ell]} \le 0 \le \psi_{[\ell+1]} \le \ldots \le \psi_{[n]}$ where ψ_k is given by (16), $d_{[1]} \le \ldots \le d_{[\ell]}$ and $d_{[\ell+1]} \ge \ldots \ge d_{[n]}$.

Proof. We use an approach similar to that of the proof of Corollary 1 to show that a sequence found by arranging jobs according to Corollary 3 (see also conditions (i) − (iii) of Theorem 3) is optimal.

In general, the optimality conditions of Corollary 3 do not hold among jobs and thus r^* cannot be found. However, if these conditions are satisfied among the jobs in a sequence [1], …, [ℓ], [ℓ + 1], …, [n], $\ell \in \{0, 1, \ldots, n\}$, then either

(i) $\delta_{[1]}d_{[1]} \le \ldots \le \delta_{[n]}d_{[n]}$; or
(ii) $\psi_{[1]}/d_{[1]} \le \ldots \le \psi_{[\ell]}/d_{[n]}$.

Hence, the solution for $1/p_k$~exp(a), $\xi_k = d_k/\ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ an be approximated (i.e., can find a candidate for r^*) by arranging jobs in non-decreasing order of either $\delta_k d_k$ or ψ_k/d_k where δ_k and ψ_k are defined by (15) and (16).

Example 1. Consider the problem $1/p_k$~exp(a), $\xi_k = d_k/\ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ of Table 1 where $a_k = 0.5$ (i.e., $E(p_k) = 2$) and d_k are different $k = 1, ..., 5$.

Here, $\delta_4 < \delta_1 < 0 < \delta_3 < \delta_5 < \delta_2$ where, using (15), $\delta_k =$ −0.317, 1.195, 0.41, −0.439, and 1.055, $k = 1, \ldots, 5$, respectively, and $d_4 > d_1$ and $d_3 < d_5 < d_2$. Based on condition (i) of Corollary 3, r^*: 4 − 1 − 3 − 5 − 2. Since $t_{[k]}$ of each job [k], $k = 1, \ldots, 5$, has an Erlang pdf with parameters k and $a = 0.5$, then $Pr(t_{[k]} < d_{[k]}) = 1 - \exp(-ad_{[k]})\sum_{j=0}^{k-1}\frac{(\alpha d_{[k]})^j}{j!}$. Hence, processing jobs according to r^* results in jobs [k], $k = 1, \ldots, 5$, being early with $Pr(t_{[1]} < 8) = 0.982$, $Pr(t_{[2]} < 7) = 0.864$, $Pr(t_{[3]} < 5) = 0.456$, $Pr(t_{[4]} < 5.5) = 0.297$, and $Pr(t_{[5]} < 6) = 0.185$, respectively. Note that the candidate found by arranging jobs in non-decreasing order of $\delta_k d_k$ is also optimal.

Table 1. A stochastic E-T problem with p_k~exp(0.5) and $\xi_k = d_k$ (known constant)

Job k	d_k	ω_k^E	ω_k^T	λ_k
1	7.0	1.0	2.5	−1.5
2	6.0	6.0	2.0	4.0
3	5.0	2.0	1.0	1.0
4	8.0	3.0	6.0	−3.0
5	5.5	4.0	1.0	3.0

Remark 4. Based on Corollary 3, a sequence [1], …,[n] is optimal for $1/p_k$~exp(a), $\xi_k = d_k/\ E[\sum_{k=1}^{n} w_{[k]}^T X_{[k]}^T]$ if either $\delta_{[1]} \le \ldots \le \delta_{[n]}$ and $d_{[1]} \ge \ldots \ge d_{[n]}$, or $\psi_{[1]} \le \ldots \le \psi_{[n]}$ and $d_{[1]} \le \ldots \le d_{[n]}$. Also, a sequence [1], …, [n] is optimal for

$1/p_k$~exp(a), $\xi_k = d_k / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E]$ if either $\delta_{[1]} \leq \ldots \leq \delta_{[n]}$ and $d_{[1]} \leq \ldots \leq d_{[n]}$, or $\psi_{[1]} \leq \ldots \leq \psi_{[n]}$ and $d_{[1]} \geq \ldots \geq d_{[n]}$.

Assuming jobs have a common known fixed due date (i.e., $\xi_k = d_k = d$, $k = 1, \ldots, n$), we have the following corollary.

Corollary 4. For $1/p_k$~exp(a), $\xi_k = d/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, an optimal sequence is found by arranging jobs in a non-decreasing order of λ_k.

Proof. It follows from Corollary 3.

3.2.2 Distinctly distributed processing times

Suppose that $p_k \sim f_k(.)$ and $\xi_k = d$ (known constant), $k = 1, \ldots, n$. Using (13), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ iff

$$\begin{aligned} \lambda_i[\tilde{F}_\theta * F_i(d) - \tilde{F}_\theta * F_j * F_i(d)] \\ \leq \lambda_j[\tilde{F}_\theta * F_j(d) - \tilde{F}_\theta * F_i * F_j(d)], \end{aligned} \tag{19}$$

where $\tilde{F}_\theta * F_j * F_i(d) \leq \min\{\tilde{F}_\theta * F_i(d), \tilde{F}_\theta * F_j(d)\}$.

Theorem 4. For $1/p_k \sim f_k(.)$, $\xi_k = d/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if
(i) $\lambda_i \leq 0 \leq \lambda_j$; or
(ii) $0 \leq \lambda_i \leq \lambda_j$ and $F_i(y) \leq F_j(y)$, $0 \leq y \leq d$; or
(iii) $\lambda_i \leq \lambda_j \leq 0$ and $F_i(y) \geq F_j(y)$, $0 \leq y \leq d$.

Proof. Using (19), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if

(i) $\lambda_i \leq 0 \leq \lambda_j$; or (20)

(ii) $0 \leq \lambda_i \leq \lambda_j$ and $\tilde{F}_\theta * F_i(d) \leq \tilde{F}_\theta * F_j(d)$; or (21)

(iii) $\lambda_i \leq \lambda_j \leq 0$ and $\tilde{F}_\theta * F_i(d) \geq \tilde{F}_\theta * F_j(d)$. (22)

Thus, if (20), (21), or (22) are satisfied, then (19) holds; however, the converse may not be true. The condition $\tilde{F}_\theta * F_i(d) \leq \tilde{F}_\theta * F_j(d)$ in (21) can be written as

$$\int_0^d F_i(d-x)\, \tilde{f}_\theta(x)dx \leq \int_0^d F_j(d-x)\, \tilde{f}_\theta(x)dx, \tag{23}$$

where $\tilde{f}_\theta(x) = f_{[1]} * \ldots * f_{[q_1-1]}(x)$ is be the convolution of the pdfs of $p_{[k]}$ for jobs $[k]$, $k = 1, \ldots, q_1 - 1$, in θ.
Letting $y = d - x$, $0 \leq y \leq d$, we can write (23) as

$$\int_0^d F_i(y)\, \tilde{f}_\theta(d-y)dy \leq \int_0^d F_j(y)\, \tilde{f}_\theta(d-y)dy,$$

or

$$\int_0^d [F_i(y) - F_j(y)]\tilde{f}_\theta(d-y)dy \leq 0. \tag{24}$$

If $F_i(y) \leq F_j(y)$, $0 \leq y \leq d$, then (24) holds and thus $\tilde{F}_\theta * F_i(d) \leq \tilde{F}_\theta * F_j(d)$. Similarly, we can write $\tilde{F}_\theta * F_i(d) \geq \tilde{F}_\theta * F_j(d)$ in (22) as

$$\int_0^d [F_i(y) - F_j(y)]\tilde{f}_\theta(d-y)dy \geq 0. \tag{25}$$

If $F_i(y) \geq F_j(y)$, $0 \leq y \leq d$, then (25) holds and $\tilde{F}_\theta * F_i(d) \geq \tilde{F}_\theta * F_j(d)$. Accordingly, using (20) − (22), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if

(i) $\lambda_i \leq 0 \leq \lambda_j$; or
(ii) $0 \leq \lambda_i \leq \lambda_j$ and $F_i(y) \leq F_j(y)$, $0 \leq y \leq d$; or
(iii) $\lambda_i \leq \lambda_j \leq 0$ and $F_i(y) \geq F_j(y)$, $0 \leq y \leq d$.

Theorem 4 can be used to examine $1/p_k \sim f_k(.)$, $\xi_k = d/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ for any general processing time pdfs. Below, we investigate cases with, for example, exponential, weibull, or uniform distributions. The use of these distributions in shop scheduling is justified by, e.g., Boxma and Forst (1986), Cai and Zhou (1997, 2005), Jang (2002), and Pinedo (1983). (Even though the exponential case is a special situation of the weibull case, it is analyzed due to the development of a general optimality condition.)

Exponential processing times

Suppose that p_k~exp(a_k) with cdf $F_k(x) = 1 - \exp(-a_k x)$ and means $1/a_k$, $k = 1, \ldots, n$.

Corollary 5. For $1/p_k$~exp(a_k), $\xi_k = d/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if

(i) $\lambda_i \leq 0 \leq \lambda_j$; or
(ii) $0 \leq \lambda_i \leq \lambda_j$ and $a_i \leq a_j$; or
(iii) $\lambda_i \leq \lambda_j \leq 0$ and $a_i \geq a_j$.

Proof. For p_k~exp(a_k), $k = 1, \ldots, n$, using condition (ii) of Theorem 4, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if $0 \leq \lambda_i \leq \lambda_j$ and $\exp(-a_i y) \geq \exp(-a_j y)$, $0 \leq y \leq d$. However, this inequality holds for $0 \leq y \leq d$ if $a_i \leq a_j$. Similarly, using condition (iii) of Theorem 4, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if $\lambda_i \leq \lambda_j \leq 0$ and $\alpha_i \geq a_j$.

Corollary 6. For $1/p_k \sim \exp(a_k)$, $\xi_k = d/ \ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{T}]$, a sequence [1], …, [ℓ], [ℓ + 1], …, [n], ℓ ∈ {0, 1, …, n}, is optimal if

(i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$, and
(ii) $a_{[1]} \ge \ldots \ge a_{[\ell]}$ and $a_{[\ell+1]} \le \ldots \le a_{[n]}$.

Proof. We use an approach similar to that of the proof of Corollary 1 to show that a sequence obtained by arranging jobs according to Corollary 6 (see also conditions (i) − (iii) of Corollary 5) is optimal.

For the exponential processing times, as the following theorem shows, we can provide a more general optimality condition.

Theorem 5. For $1/p_k \sim \exp(a_k)$, $\xi_k = d/ \ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{T}]$, an optimal sequence is found by arranging jobs in a non-decreasing order of $\lambda_k a_k$.

Proof. Since $p_k \sim \exp(a_k)$ with $F_k(x) = 1 - \exp(-a_k x)$, $k = 1, \ldots, n$, then

$$\begin{aligned} & F_i * F_j(d) \\ & = \int_0^d F_i(d-x)\, f_j(x) dx \\ & = \alpha_j \int_0^d [1-\exp(-\alpha_i(d-x))] \exp(-\alpha_j x) dx \\ & = 1-\exp(-\alpha_j d) - \frac{\alpha_j [\exp(-\alpha_j d) - \exp(-\alpha_i d)]}{(\alpha_i - \alpha_j)}, \\ & \alpha_i \ne \alpha_j. \end{aligned} \tag{26}$$

Substituting (26) into (19), we get

$$(\lambda_i \alpha_i - \lambda_j \alpha_j)\big[[\exp(-\alpha_j d) - \exp(-\alpha_i d)]/(\alpha_i - \alpha_j)\big] \tilde{F}_\theta(d) \le 0,\ \alpha_i \ne \alpha_j,$$

which holds if

$$(\lambda_i \alpha_i - \lambda_j \alpha_j)\big[[\exp(-\alpha_j d) - \exp(-\alpha_i d)]/(\alpha_i - \alpha_j)\big] \le 0,\ \alpha_i \ne \alpha_j. \tag{27}$$

Since the fraction in the brackets of inequality (27) is non-negative, then the inequality holds if $\lambda_i a_i \le \lambda_j a_j$, $i \ne j \in N$. Using an approach similar to that of the proof of Corollary 1, we can show that a sequence [1], …, [n] found by arranging jobs according to $\lambda_i a_i \le \lambda_j a_j$, $i \ne j \in N$ (i.e., $\lambda_{[1]} a_{[1]} \le \ldots \le \lambda_{[n]} a_{[n]}$) is optimal. That is, r^* can be identified by arranging jobs in a non-decreasing order of $\lambda_k a_k$.

Remark 5. Using Theorem 5, r^* for $1/p_k \sim \exp(a_k)$, $\xi_k = d / \ E[\sum_{k=1}^{n} w_{[k]}^{T} X_{[k]}^{T}]$ can be found by arranging jobs in non-increasing order of $\alpha_k \omega_k^T$ or in non-decreasing order of $E(p_k)/\ \omega_k^T$ (i.e., according to the weighted shortest expected processing time (WSEPT) rule) (e.g., Pinedo, 1983), and for $1/p_k \sim \exp(a_k)$, $\xi_k = d/ \ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E}]$ by arranging jobs in non-decreasing order of $\alpha_k \omega_k^E$ or in non-increasing order of $E(p_k)/\ \omega_k^E$ (i.e., according to the weighted longest expected processing time (WLEPT) rule).

Weibull processing times

Suppose that p_k, $k = 1, \ldots, n$, have Weibull distributions with shape and scale parameters a_k and β_k (i.e., $p_k \sim W(a_k, \beta_k)$) and cdfs $F_k(x) = 1 - \exp[-(\alpha_k x)^{\beta_k}]$.

Corollary 7. For $1/p_k \sim W(a_k, \beta_k)$, $\xi_k = d/ \ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{T}]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if

(i) $\lambda_i \le 0 \le \lambda_j$; or
(ii) $0 \le \lambda_i \le \lambda_j$ and $F_i(d) \le F_j(d)$ where $F_k(d) = 1 - \exp[-(\alpha_k d)^{\beta_k}]$ and $\beta_i \ge \beta_j$; or
(iii) $\lambda_i \le \lambda_j \le 0$ and $F_i(d) \ge F_j(d)$ where $\beta_i \le \beta_j$.

Proof. For $p_k \sim W(a_k, \beta_k)$, $k = 1, \ldots, n$, using condition (ii) of Theorem 4, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if $0 \le \lambda_i \le \lambda_j$ and $\exp[-(\alpha_i y)^{\beta_i}] \ \ge \ \exp[-(\alpha_j y)^{\beta_j}]$. This inequality simplifies to $y^{\beta_i - \beta_j} \le \ \alpha_j^{\beta_j} / \alpha_i^{\beta_i}$ which holds for all $0 \le y \le d$ if $\beta_i \ge \beta_j$ and $d^{\beta_i - \beta_j} \le \alpha_j^{\beta_j} / \alpha_i^{\beta_i}$ (i.e., $(\alpha_i d)^{\beta_i} \le (\alpha_j d)^{\beta_j}$ or $F_i(d) \le F_j(d)$). Similarly, using condition (iii) of Theorem 4, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if $\lambda_i \le \lambda_j \le 0$ and $\exp[-(\alpha_i y)^{\beta_i}] \ \le \ \exp[-(\alpha_j y)^{\beta_j}]$, which reduces to $y^{\beta_i - \beta_j} \ge \alpha_j^{\beta_j} / \alpha_i^{\beta_i}$, $0 \le y \le d$. This holds if $\beta_i \le \beta_j$ and $(\alpha_i d)^{\beta_i} \ge (\alpha_j d)^{\beta_j}$ or $F_i(d) \ge F_j(d)$. Therefore, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if (i) $\lambda_i \le 0 \le \lambda_j$; or (ii) $0 \le \lambda_i \le \lambda_j$ and $F_i(d) \le F_j(d)$ where $\beta_i \ge \beta_j$; or (iii) $\lambda_i \le \lambda_j \le 0$ and $F_i(d) \ge F_j(d)$ where $\beta_i \le \beta_j$.

Corollary 8. For $1/p_k \sim W(a_k, \beta_k)$, $\xi_k = d/ \ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{T}]$, a sequence [1], …, [ℓ], [ℓ + 1], …, [n], ℓ ∈ {0, 1, …, n}, is optimal if

(i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$, and
(ii) $F_{[1]}(d) \ge \ldots \ge F_{[\ell]}(d)$ and $F_{[\ell+1]}(d) \le \ldots \le F_{[n]}(d)$ where $F_k(d) = 1 - \exp[-(\alpha_k d)^{\beta_k}]$, $\beta_{[1]} \le \ldots \le \beta_{[\ell]}$, and $\beta_{[\ell+1]} \ge \ldots \ge \beta_{[n]}$.

Proof. We use an approach similar to that of the proof of Corollary 1 to show that a sequence obtained by arranging jobs according to Corollary 8 (see also conditions (i)-(iii) of Corollary 7) is optimal.

In general, the optimality conditions of Corollary 8 do not hold among jobs and thus r^* cannot be identified. However, if these conditions are satisfied among the jobs in a sequence [1], …, [ℓ], [ℓ + 1],…,[n], $\ell \in \{0, 1, \ldots, n\}$, then $\lambda_{[1]}F_{[1]}(d) \leq \ldots \leq \lambda_{[n]}F_{[n]}(d)$. Therefore, the solution for $1/p_k{\sim}W(a_k, \beta_k)$, $\xi_k = d/\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$ can be approximated (i.e., can find a candidate for r^*) by arranging jobs in non-decreasing order of $\lambda_k F_k(d)$.

Example 2. Consider the problem $1/p_k{\sim}W(a_k,\beta_k)$, $\xi_k = d/\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$ of Table 2 where $d_k = 3$, $k = 1, \ldots, 5$.

Table 2. A stochastic *E-T* problem with $p_k{\sim}W(a_k, \beta_k)$ and $\xi_k = d_k = 3$

Job k	a_k	β_k	ω^E_k	ω^T_k	λ_k
1	1.0	0.6	2.0	1.0	1.0
2	0.5	0.2	5.0	6.5	-1.5
3	2.0	0.4	2.5	0.5	2.0
4	4.0	0.1	1.0	3.0	-2.0
5	0.4	0.3	4.0	5.0	-1.0

Here, $F_k(3)$ = 0.855, 0.662, 0.871, 0.722, and 0.652, $k = 1, \ldots, 5$, respectively. Since $\lambda_4 < \lambda_2 < \lambda_5 < 0 < \lambda_1 < \lambda_3$, $F_4(3) > F_2(3) > F_5(3)$, and $F_1(3) < F_3(3)$ where $\beta_4 < \beta_2 < \beta_5$ and $\beta_1 > \beta_3$, based on Corollary 8, r^*: 4 − 2 − 5 − 1 − 3.The candidate obtained by arranging jobs in non-decreasing order of $\lambda_k F_k(d)$ is also optimal.

Remark 6. According to Corollary 8, a sequence [1], …, [n] in which $\lambda_{[1]} \leq \ldots \leq \lambda_{[n]}$ is optimal for $1/p_k{\sim}W(a_k, \beta_k)$, $\xi_k = d/\ E[\sum_{k=1}^{n} w^T_{[k]}X^T_{[k]}]$ if $F_{[1]}(d) \geq \ldots \geq F_{[n]}(d)$ where $F_k(d) = 1 - \exp[-(\alpha_k d)^{\beta_k}]$ and $\beta_{[1]} \geq \ldots \geq \beta_{[n]}$, and is optimal for $1/p_k{\sim}W(a_k, \beta_k)$, $\xi_k = d/\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]}]$ if $F_{[1]}(d) \leq \ldots \leq F_{[n]}(d)$ where $\beta_{[1]} \leq \ldots \leq \beta_{[n]}$.

Uniform processing times

Suppose that p_k, $k = 1, \ldots, n$, are uniform random variables defined in the intervals $[a_k, b_k]$, $b_k > a_k \geq 0$ (i.e., $p_k{\sim}U[a_k, b_k]$). A uniformly distributed processing time provides a time window (i.e., $[a_k, b_k]$) during which the job is processed with equal probability. The cdf of $p_k{\sim}U[a_k, b_k]$ is defined as

$$F_k(x) = \begin{cases} 0, \ if\ x \leq a_k, \\ \dfrac{x - a_k}{b_k - a_k}, \ if\ a_k \leq x \leq b_k, \\ 1, \ if\ x \geq b_k. \end{cases} \quad (28)$$

Corollary 9. For $1/p_k{\sim}U[a_k, b_k]$, $\xi_k = d/\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if

(i) $\lambda_i \leq 0 \leq \lambda_j$; or
(ii) $0 \leq \lambda_i \leq \lambda_j$ and $F_i(d) \leq F_j(d)$ where $F_k(x)$ is defined by (28); or
(iii) $\lambda_i \leq \lambda_j \leq 0$ and $F_i(d) \geq F_j(d)$.

Proof. For $p_k{\sim}U[a_k,b_k]$, $k = 1, \ldots, n$, $k = 1, \ldots, n$, using condition (ii) of Theorem 4, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if $0 \leq \lambda_i \leq \lambda_j$ and $(y-a_i)/(b_i-a_i) \leq (y - a_j)/(b_j - a_j)$ or

$$y[(b_j - a_j) - (b_i - a_i)] \leq a_i(b_j - a_j) - a_j(b_i - a_i),\ 0 \leq y \leq d \quad (29)$$

If $b_i - a_i \geq b_j - a_j$ and $a_i(b_j - a_j) \geq a_j(b_i - a_i)$ (i.e., $(b_i - a_i)/a_i \leq (b_j - a_j)/a_j$), then (29) holds for any $0 \leq y \leq d$; hence, $F_i(d) \leq F_j(d)$ where $F_k(x)$ is defined by (28). If $b_i - a_i < b_j - a_j$, (29) can be rewritten as $y \leq [a_i(b_j - a_j) - a_j(b_i - a_i)]/[(b_j - a_j) - (b_i - a_i)]$, $0 \leq y \leq d$, which holds as long as $d \leq [a_i(b_j - a_j) - a_j(b_i - a_i)]/[(b_j - a_j) - (b_i - a_i)]$ or $F_i(d) \leq F_j(d)$. Thus, when $0 \leq \lambda_i \leq \lambda_j$, (29) holds if either (1) $b_i - a_i \geq b_j - a_j$ and $(b_i - a_i)/a_i \leq (b_j - a_j)/a_j$, or (2) $b_i - a_i < b_j - a_j$ and $F_i(d) \leq F_j(d)$. Similarly, using condition (iii) of Theorem 4, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if $\lambda_i \leq \lambda_j \leq 0$ and

$$y[(b_j - a_j) - (b_i - a_i)] \geq a_i(b_j - a_j) - a_j(b_i - a_i),\ 0 \leq y \leq d. \quad (30)$$

If $b_i - a_i \leq b_j - a_j$ and $(b_i - a_i)/a_i \geq (b_j - a_j)/a_j$, then (30) holds for any $0 \leq y \leq d$; thus, $F_i(d) \geq F_j(d)$. If $b_i - a_i > b_j - a_j$, we can write (30) $y \leq [a_i(b_j - a_j) - a_j(b_i - a_i)]/[(b_j - a_j) - (b_i - a_i)]$, $0 \leq y \leq d$, which holds as long as $d \leq [a_i(b_j - a_j) - a_j(b_i - a_i)]/[(b_j - a_j) - (b_i - a_i)]$ or $F_i(d) \geq F_j(d)$. Therefore, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if (i) $\lambda_i \leq 0 \leq \lambda_j$; or (ii) $0 \leq \lambda_i \leq \lambda_j$ and $F_i(d) \leq F_j(d)$; or (iii) $\lambda_i \leq \lambda_j \leq 0$ and $F_i(d) \geq F_j(d)$.

Corollary 10. For $1/p_k{\sim}U[a_k,b_k]$, $\xi_k = d/\ E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$, a sequence [1], …, [$\ell$], [$\ell$ + 1], …, [n], $\ell \in \{0, 1, \ldots, n\}$, is optimal if

(i) $\lambda_{[1]} \leq \ldots \leq \lambda_{[\ell]} \leq 0 \leq \lambda_{[\ell+1]} \leq \ldots \leq \lambda_{[n]}$, and
(ii) $F_{[1]}(d) \geq \ldots \geq F_{[\ell]}(d)$ and $F_{[\ell+1]}(d) \leq \ldots \leq F_{[n]}(d)$ where $F_k(x)$ is defined by (28).

Proof. We use an approach similar to that of the proof of Corollary 1 to show that a sequence obtained by arranging jobs according to Corollary 10 (see also conditions (i) − (iii) of Corollary 9) is optimal.

Similar to the case with weibull $f_k(.)$, we can approximate the solution (or find a candidate for r^*) for $1/p_k{\sim}U[a_k, b_k]$,

$\xi_k = d / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ by arranging jobs in non-decreasing order of $\lambda_k F_k(d)$.

Remark 7. Based on Corollary 10, a sequence [1], …, [n] in which $\lambda_{[1]} \le \ldots \le \lambda_{[n]}$ is optimal for $1/p_k \sim U[a_k, b_k]$, $\xi_k = d / E[\sum_{k=1}^{n} w_{[k]}^T X_{[k]}^T]$ if $F_{[1]}(d) \ge \ldots \ge F_{[n]}(d)$, and is optimal for $1/p_k \sim U[a_k, b_k]$, $\xi_k = d / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E]$ if $F_{[1]}(d) \le \ldots \le F_{[n]}(d)$.

Remark 8. Based on Corollaries 6, 8, and 10, for $1/p_k \sim f_k(.)$, $\xi_k = d_k / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, jobs [$k$], $k = 1, \ldots, \ell, \ell + 1, \ldots, n$, $\ell \in \{0, 1, \ldots, n\}$, are arranged in r^* in non-decreasing order of $\omega_{[k]}^E - \omega_{[k]}^T$ (i.e., $\lambda_{[k]}$) where the $F_{[k]}(d)$ of jobs with $\omega_{[k]}^E \le \omega_{[k]}^T$, $k = 1, \ldots, \ell$, are in non-increasing order while the $F_{[k]}(d)$ of jobs with $\omega_{[k]}^E \ge \omega_{[k]}^T$, $k = \ell + 1, \ldots, n$, are in non-decreasing order. Thus, $1/p_k \sim f_k(.)$, $\xi_k = d_k / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ among jobs [1], …, [ℓ], [$\ell + 1$], …, [n] where $-\infty < \lambda_{[k]} < +\infty$ (i.e., the stochastic E-T problem) is a mixture of $1/p_k \sim f_k(.)$, $\xi_k = d_k / E[\sum_{k=1}^{n} w_{[k]}^T X_{[k]}^T]$ among jobs [1], …, [ℓ] where $\omega_{[k]}^T = -\lambda_{[k]} \ge 0$ (i.e., the stochastic T problem) and of $1/p_k \sim f_k(.)$, $\xi_k = d_k / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E]$ among jobs [$\ell + 1$], …, [n] where $\omega_{[k]}^E = \lambda_{[k]} \ge 0$ (i.e., the stochastic E problem).

3.3 Stochastic processing times and stochastic due dates

Consider the scenario $1/p_k \sim f_k(.)$, $\xi_k \sim g_k(.) / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$. Then, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ, using (5), iff

$$\begin{aligned} &\lambda_i \int_0^\infty [\tilde{F}_\theta * F_i(x) - \tilde{F}_\theta * F_i * F_j(x)] dG_i(x) \\ &\quad \le \lambda_j \int_0^\infty [\tilde{F}_\theta * F_j(x) - \tilde{F}_\theta * F_i * F_j(x)] dG_j(x). \end{aligned} \tag{31}$$

where $\tilde{F}_\theta * F_j * F_i(x) \le \tilde{F}_\theta * F_i(x)$ and $\tilde{F}_\theta * F_i * F_j(x) \le \tilde{F}_\theta * F_j(x)$.

Inequality (31) is too general to allow the development of useful statements to establish $\theta ij\delta \succ \theta ji\delta$. However, we can utilize this inequality to explore the following cases.

3.3.1 Identically distributed due-dates

Suppose that $\xi_k \sim g(.)$ and $p_k \sim f_k(.)$, $k = 1, ..., n$. Using (31), $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ iff

$$\begin{aligned} &\int_0^\infty \tilde{F}_\theta(x) * \left[\lambda_i [F_i(x) - F_i * F_j(x)]\right] dG(x) \\ &\le \int_0^\infty \tilde{F}_\theta(x) * \left[\lambda_j [F_j(x) - F_i * F_j(x)]\right] dG(x). \end{aligned} \tag{32}$$

We use (32) to analyze different cases of $1/p_k \sim f_k(.)$, $\xi_k \sim g(.) / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$.

Exponential processing times

Consider the case where $p_k \sim \exp(a_k)$ with $F_k(x) = 1 - \exp(-a_k x)$ and $\xi_k \sim g(.)$, $k = 1, \ldots, n$.

Theorem 6. For $1/p_k \sim \exp(a_k)$, $\xi_k \sim g(.) / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, an optimal sequence can be found by arranging jobs in a non-decreasing order of $\lambda_k a_k$.

Proof. Using $F_k(x) = 1 - \exp(-a_k x)$ and (26) in (32), $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ iff

$$\int_0^\infty \tilde{F}_\theta(x) \left[(\lambda_i \alpha_i - \lambda_j \alpha_j) \left(\frac{\exp(-\alpha_j x) - \exp(-\alpha_i x)}{\alpha_i - \alpha_j}\right)\right] dG(x) \le 0, \alpha_i \ne \alpha_j. \tag{33}$$

Since the fraction inside the parentheses of (33) is non-negative, then (33) holds if $\lambda_i a_i \le \lambda_j a_j$, $i \ne j \in N$. Using an approach similar to that of the proof of Corollary 1, we can show that a sequence [1], …, [n] obtained by arranging jobs according to $\lambda_i a_i \le \lambda_j a_j$, $i \ne j \in N$ (i.e., $\lambda_{[1]} a_{[1]} \le \ldots \le \lambda_{[n]} a_{[n]}$) is optimal. That is, r^* can be identified by arranging jobs in non-decreasing order of $\lambda_k a_k$.

Remark 9. Based on Theorem 6, r^* for $1/p_k \sim \exp(a_k)$, $\xi_k \sim g(.) / E[\sum_{k=1}^{n} w_{[k]}^T X_{[k]}^T]$ can be found by arranging jobs in non-increasing order of $\alpha_k \omega_k^T$ or in non-decreasing order of $E(p_k) / \omega_k^T$ (i.e., WSEPT rule) (e.g., Cai and Zhou, 2005), and r^* for $1/p_k \sim \exp(a_k)$, $\xi_k \sim g(.) / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E]$ can be found by arranging jobs in non-decreasing order of $\alpha_k \omega_k^E$ or in non-increasing order of $E(p_k) / \omega_k^E$ (i.e., WLEPT rule).

Exponential due-dates

Consider the case where $\xi_k \sim \exp(\gamma)$ with $G_k(x) = 1 - \exp(-\gamma x)$ and $p_k \sim f_k(.)$, $k = 1, ..., n$.

Theorem 7. For $1/p_k \sim f_k(.)$, $\xi_k \sim \exp(\gamma) / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, an optimal sequence can be found by arranging jobs in a non-decreasing order of $\lambda_k / [1/L_k(\gamma) - 1]$

where $L_k(\gamma)$ denotes the Laplace-Stieltjes transform (LST) of $f_k(.)$ evaluated at γ.

Proof. Substituting $G(x) = 1 - \exp(-\gamma x)$ into (32), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ iff

$$\lambda_i \int_0^\infty \exp(-\gamma x)\tilde{F}_\theta(x) * [F_i(x) - F_i * F_j(x)]dx$$
$$\leq \lambda_j \int_0^\infty \exp(-\gamma x)\tilde{F}_\theta(x) * [F_j(x) - F_i * F_j(x)]dx,$$

or

$$\lambda_i L_\theta(\gamma) L_i(\gamma)[1 - L_j(\gamma)] \leq \lambda_j L_\theta(\gamma) L_j(\gamma)[1 - L_i(\gamma)], \quad (34)$$

where

$$L_k(\gamma) = \int_0^\infty \exp(-\gamma x) f_k(x)dx = \gamma \int_0^\infty \exp(-\gamma x) F_k(x)dx,$$

is the LST of $f_k(.)$ evaluated at γ, and $L_\theta(\gamma) = \prod_{k=1}^{q_1-1} L_{[k]}(\gamma)$. Inequality (34) simplifies to $\lambda_i L_i(\gamma)[1 - L_j(\gamma)] \leq \lambda_j L_j(\gamma)[1 - L_i(\gamma)]$, or $\lambda_i/[1/L_i(\gamma) - 1] \leq \lambda_j/[1/L_j(\gamma) - 1]$.

Using an approach similar to that of the proof of Corollary 1, we can show that a sequence [1], …, [n] obtained by arranging jobs according to $\lambda_1/[1/L_1(\gamma) - 1] \leq \ldots \leq \lambda_n/[1/L_n(\gamma) - 1]$ is optimal; thus, r^* is obtained by arranging jobs in a non-decreasing order of $\lambda_k/[1/L_k(\gamma) - 1]$.

Example 3. Consider the problem $1/p_k \sim f_k(.)$, $\xi_k \sim \exp(\gamma)/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ of Table 3 where p_1 has a normal pdf with mean 5 and variance 2 (i.e., $p_1 \sim N(5, 2)$), p_2 has a gamma pdf with shape and slope parameters 3 and 1 (i.e., $p_2 \sim G(3, 1)$), $p_3 \sim \exp(4)$, $p_4 \sim U[2, 10]$, and p_5 has a chi-square pdf with degree of freedom 8 (i.e., $p_5 \sim \chi^2(8)$). Also, let $\xi_k \sim \exp(\gamma)$, $k = 1, \ldots, 5$, with $\gamma = 1$. The use of normal, exponential, and uniform processing times in shop scheduling is justified by, e.g., Balut (1973), Bertrand (1983), Cai and Zhou (1997, 2005), Jang (2002), Kise and Ibaraki (1983), Sarin et al. (1991), Soroush (1999), and Soroush and Allahverdi (2005). Since p_1 has a normal distribution, its variance must be small enough relative to the mean such that $Pr(p_1 \leq 0) \approx 0$. To accomplish this, a coefficient of variation (CV) of at most 0.32 is considered for p_1 (i.e., $CV_k = \sigma_k/\mu_k \leq 0.32$) to insure $Pr(p_1 > 0) \geq 0.999$.

Table 3. A stochastic E-T problem with $p_k \sim f_k(.)$ and $\xi_k \sim \exp(1)$

Job k	$f_k(.)$	ω_k^E	ω_k^T	λ_k
1	$N(5,2)$	2.0	5.0	-3.0
2	$G(3,1)$	6.0	1.0	5.0
3	Exp(4)	4.0	3.0	1.0
4	$U[2, 10]$	0.0	7.0	-7.0
5	$\chi^2(8)$	1.0	5.0	-4.0

Using the pdfs of p_k, $k = 1, \ldots, 5$, we respectively have $L_1(1) = \exp(-4)$, $L_2(1) = 0.125$, $L_3(1) = 0.8$, $L_4(1) = \exp(-2)[1 - \exp(-8)]/8$, and $L_5(1) = 0.197$. Then, $\lambda_k/[1/L_k(\gamma) - 1] = -0.056, 0.714, 4.0, -0.12$, and -0.981, $k = 1, \ldots, 5$, respectively. Arranging jobs in non-decreasing order of $\lambda_k/[1/L_k(\gamma) - 1]$ provides r^*: 5 − 4 − 1 − 2 − 3 (see Theorem 7).

Remark 10. Based on Theorem 7, r^* for $1/p_k \sim f_k(.)$, $\xi_k \sim \exp(\gamma)/ E[\sum_{k=1}^{n} w_{[k]}^T X_{[k]}^T]$ can be found by arranging jobs in non-increasing order of $\omega_k^T/[1/L_k(\gamma) - 1]$ (e.g. Boxma and Forst, 1986), and r^* for $1/p_k \sim f_k(.)$, $\xi_k \sim \exp(\gamma)$ / $E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E]$ can be found by arranging jobs in non-decreasing order of $\omega_k^E / [1/L_k(\gamma) - 1]$.

Identically distributed processing times

Consider the case where $p_k \sim f(.)$ and $\xi_k \sim g(.)$, $k = 1, \ldots, n$. Using (32), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ iff

$$(\lambda_i - \lambda_j)\int_0^\infty [F^{(q_1)*}(x) - F^{(q_1+1)*}(x)]dG(x) \leq 0. \quad (35)$$

Theorem 8. For $1/p_k \sim f(.)$, $\xi_k \sim g(.)/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, an optimal sequence can be found by arranging jobs in a non-decreasing order of λ_k.

Proof. Since $F^{(q_1)*}(x) \geq F^{(q_1+1)*}(x)$, $x \geq 0$, using (35), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if $\lambda_i \leq \lambda_j$. We can use an approach similar to that in the proof of Corollary 1 to show that a sequence [1], …, [n] found based on $\lambda_{[1]} \leq \ldots \leq \lambda_{[n]}$ is optimal, that is, r^* can be found by arranging jobs in a non-decreasing order of λ_k.

Common expected difference between earliness and tardiness penalties

Consider the case where $\lambda = E(w_k^E - w_k^T) = \omega_k^E - \omega_k^T$, $-\infty < \lambda < \infty$ (i.e., job expected earliness penalties differ from their expected tardiness penalties by a common constant), $p_k \sim f_k(.)$, and $\xi_k \sim g(.)$, $k = 1, \ldots, n$. Then, using (32), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ iff

$$\lambda \int_0^\infty \tilde{F}_\theta(x) * [F_i(x) - F_j(x)]dG(x) \leq 0. \quad (36)$$

Let the stochastic ordering $p_i \leq_{st} p_j$ ($p_i \geq_{st} p_j$) denote $F_i(x) \geq F_j(x)$ ($F_i(x) \leq F_j(x)$) for all $x \geq 0$.

Theorem 9. For $1/p_k \sim f_k(.)$, $\xi_k \sim g(.)$, $\lambda_k = \lambda/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, an optimal sequence can be found by arranging jobs in non-increasing stochastic ordering of p_k (i.e., $p_{[1]} \geq_{st} \dots \geq_{st} p_{[n]}$) if $\lambda > 0$, and in non-decreasing stochastic ordering of p_k (i.e., $p_{[1]} \leq_{st} \dots \leq_{st} p_{[n]}$) if $\lambda < 0$.

Proof. When $\lambda = 0$, any sequence $r \in R$ is optimal (see Section 2). When $\lambda > 0$, inequality (36) holds for every $i \neq j \in N$ and every θ if $F_i(x) \leq F_j(x)$ for all $x \geq 0$ (i.e., $p_i \geq_{st} p_j$). Similarly, when $\lambda < 0$, the inequality is satisfied for every $i \neq j \in N$ and every θ if $F_i(x) \geq F_j(x)$ for all $x \geq 0$ (i.e., $p_i \leq_{st} p_j$). Using an approach similar to that of the proof of Corollary 1, we can show that a sequence [1], …, [n] obtained according to $p_{[1]} \geq_{st} \dots \geq_{st} p_{[n]}$ when $\lambda > 0$, or according to $p_{[1]} \leq_{st} \dots \leq_{st} p_{[n]}$ when $\lambda < 0$ is optimal. Hence, r^* can be found by arranging jobs in non-increasing stochastic ordering of p_k if $\lambda > 0$, and non-decreasing stochastic ordering of p_k if $\lambda < 0$.

Example 4. Consider the problem $1/p_k \sim N(\mu_k,\sigma_k^2)$, $\xi_k \sim g(.)$, $\lambda_k = \lambda / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$ of Table 4 where $\lambda = \omega_k^E - \omega_k^T = -2$, $k = 1, ..., 5$(i.e., $1/p_k \sim N(\mu_k, \sigma_k^2)$, $\xi_k \sim g(.)$, $\lambda_k = -2/ E[\sum_{k=1}^{n} w_{[k]}^T X_{[k]}^T]$). Since $p_k \sim N(\mu_k, \sigma_k^2)$, $k = 1, \dots, 5$, p_k are such that $CV_k \leq 0.32$ to insure $Pr(p_k > 0) \geq 0.999$.

Here, $F_k(x) = Pr(Z \leq z_k)$ where Z is a standard normal random variable and $z_k = (x - \mu_k)/\sigma_k$, $k = 1, \dots, 5$. If $Pr[Z \leq (x - \mu_i)/\sigma_i] \geq Pr[Z \leq (x - \mu_j)/\sigma_j]$ for all $x \geq 0$, then $p_i \leq_{st} p_j$. That is, $p_i \leq_{st} p_j$ if $(x - \mu_i)/\sigma_i \geq (x - \mu_j)/\sigma_j$ or $x(\sigma_j - \sigma_i) \geq \mu_i\sigma_j - \mu_j\sigma_i$ for all $x \geq 0$. Sufficient conditions to satisfy this inequality (i.e., $p_i \leq_{st} p_j$) are $\sigma_i \leq \sigma_j$ and $\mu_i\sigma_j \leq \mu_j\sigma_i$ (i.e., $CV_i \geq CV_j$). Since $\sigma_1 \leq \sigma_2$ and $CV_1 \geq CV_2$, then $p_1 \leq_{st} p_2$. Similarly, $p_1 \leq_{st} p_3$, $p_1 \leq_{st} p_4$, and $p_1 \leq_{st} p_5$ because $\sigma_1 \leq \sigma_3$ and $CV_1 \geq CV_3$, $\sigma_1 \leq \sigma_4$ and $CV_1 \geq CV_4$, and $\sigma_1 \leq \sigma_5$ and $CV_1 \geq CV_5$, respectively. Analogously, we can show that $p_4 \leq_{st} p_2$, $p_4 \leq_{st} p_3$, and $p_4 \leq_{st} p_5$; $p_2 \leq_{st} p_5$ and $p_2 \leq_{st} p_3$; and $p_5 \leq_{st} p_3$. Hence, $p_1 \leq_{st} p_4 \leq_{st} p_2 \leq_{st} p_5 \leq_{st} p_3$ which implies r^*: 1-4-2-5-3 (see Theorem 9).

Note that for $1/p_k \sim f_k(.)$, $\xi_k \sim g(.)$, $\lambda_k = \lambda / E[\sum_{k=1}^{n} w_{[k]}^T X_{[k]}^T]$ and $1/p_k \sim f_k(.)$, $\xi_k \sim g(.)$, $\lambda_k = \lambda / E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E]$, the optimal sequences are respectively found by arranging jobs in non-decreasing and non-increasing stochastic ordering of p_k.

Table 4. A stochastic E-T problem with $p_k \sim N(\mu_k,\sigma_k^2)$, $\xi_k \sim g(.)$, and $\lambda_k = \omega_k^E - \omega_k^T = -2$.

Job k	μ_k	σ_k^2	ω_k^E	ω_k^T	λ_k
1	2.0	0.25	0.0	2.0	-3.0
2	11.0	5.30	5.0	7.0	5.0
3	20.0	16.00	1.0	3.0	1.0
4	5.0	1.44	2.0	4.0	-7.0
5	17.0	12.25	3.0	5.0	-4.0

3.3.2 Identically distributed processing times

Suppose that $p_k \sim f(.)$ and $\xi_k \sim g_k(.)$, $k = 1, ..., n$. Using (31), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ iff

$$\int_0^\infty [F^{(q_1)*}(x) - F^{(q_1+1)*}(x)][\lambda_i dG_i(x) - \lambda_j dG_j(x)] \leq 0. \quad (37)$$

Since it is difficult to analyze (37) for general $g_k(.)$, we consider the case where $\xi_k \sim \exp(\gamma_k)$ with $G_k(x) = 1 - \exp(-\gamma_k x)$.

Theorem 10. For $1/p_k \sim f(.)$, $\xi_k \sim \exp(\gamma_k)/ E[\sum_{k=1}^{n} w_{[k]}^E X_{[k]}^E + w_{[k]}^T X_{[k]}^T]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if

(i) $\lambda_i \leq 0 \leq \lambda_j$; or
(ii) $\gamma_i \geq \gamma_j$ and either $0 \leq \eta_i \leq \eta_j$ or $\varphi_i \leq \varphi_j \leq 0$ where

$$\eta_k = \lambda_k L(\gamma_k)[1 - L(\gamma_k)], \quad (38)$$

and

$$\varphi_k = \lambda_k L^{n-1}(\gamma_k)[1 - L(\gamma_k)]; \quad (39)$$

or

(iii) $\gamma_i \leq \gamma_j$ and either $\eta_i \leq \eta_j \leq 0$ or $0 \leq \varphi_i \leq \varphi_j$.

Proof. Using (37), $\theta ij\delta \succ \theta ji\delta$ for every $i \neq j \in N$ and every θ and δ if

$$\int_0^\infty [F^{(q_1)*}(x) - F^{(q_1+1)*}(x)][\lambda_i \gamma_i \exp(-\gamma_i x) - \lambda_j \gamma_j \exp(-\gamma_j x)] \leq 0,$$

that can be equivalently written as

$$\lambda_i L^{q_1}(\gamma_i)[1 - L(\gamma_i)] - \lambda_j L^{q_1}(\gamma_j)[1 - L(\gamma_j)] \leq 0,$$

or

$$\lambda_i L^{q_1}(\gamma_i)[1 - L(\gamma_i)] \leq \lambda_j L^{q_1}(\gamma_j)[1 - L(\gamma_j)], \quad (40)$$

where $L(\gamma_k) = \gamma_k \int_0^\infty \exp(-\gamma_k x) F(x) dx$ is the LST of $F(.)$ evaluated at γ_k. Inequality (40) shows that the relation $\theta ij\delta \succ \theta ji\delta$ depends on every job $i \neq j \in N$ as well as the position q_1 of job i. (Note that $L(\gamma_i) \leq (\geq) L(\gamma_j)$ iff $\gamma_i \geq (\leq) \gamma_j$.) For $\lambda_i \leq 0 \leq \lambda_j$, $i \neq j \in N$, (40) is always satisfied. For $L(\gamma_i) \leq L(\gamma_j)$ (i.e., $\gamma_i \geq \gamma_j$), $i \neq j \in N$, (40) holds for $q_1 = 1, ..., n - 1$(job j occupies position $q_1 + 1$) if

(i) $0 \leq \lambda_i L(\gamma_i)[1 - L(\gamma_i)] \leq \lambda_j L(\gamma_j)[1 - L(\gamma_j)]$, or
(ii) $\lambda_i L^{n-1}(\gamma_i)[1 - L(\gamma_i)] \leq \lambda_j L^{n-1}(\gamma_j)[1 - L(\gamma_j)] \leq 0$.

For $L(\gamma_i) \geq L(\gamma_j)$ (i.e., $\gamma_i \leq \gamma_j$), (40) holds if

(i) $\lambda_i L(\gamma_i)[1 - L(\gamma_i)] \le \lambda_j L(\gamma_j)[1 - L(\gamma_j)] \le 0$, or
(ii) $0 \le \lambda_i L^{n-1}(\gamma_i)[1 - L(\gamma_i)] \le \lambda_j L^{n-1}(\gamma_j)[1 - L(\gamma_j)]$.

Corollary 11. For $1/p_k\sim f(.)$, $\xi_k\sim\exp(\gamma_k)$/ $E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$, a sequence $[1], \ldots, [\ell], [\ell + 1], \ldots, [n]$, $\ell \in \{0, 1, \ldots, n\}$, is optimal if

(i) $\eta_{[1]} \le \ldots \le \eta_{[\ell]} \le 0 \le \eta_{[\ell+1]} \le \ldots \le \eta_{[n]}$ where η_k is given by (38), $\gamma_{[1]} \le \ldots \le \gamma_{[\ell]}$ and $\gamma_{[\ell+1]} \ge \ldots \ge \gamma_{[n]}$; or
(ii) $\varphi_{[1]} \le \ldots \le \varphi_{[\ell]} \le 0 \le \varphi_{[\ell+1]} \le \ldots \le \varphi_{[n]}$ where φ_k is given by (39), $\gamma_{[1]} \ge \ldots \ge \gamma_{[\ell]}$ and $\gamma_{[\ell+1]} \le \ldots \le \gamma_{[n]}$.

Proof. We use an approach similar to that of the proof of Corollary 1 to show that a sequence obtained by arranging jobs according to Corollary 11 (see also conditions (i) − (iii) of Theorem 10) is optimal.

In general, the optimality conditions of Corollary 11 do not hold among jobs and hence r^* cannot be found. However, if these conditions are satisfied among the jobs in a sequence $[1], \ldots, [\ell], [\ell + 1], \ldots, [n]$, $\ell \in \{0, 1, \ldots, n\}$, then either

(i) $\eta_{[1]}/\gamma_{[1]} \le \ldots \le \eta_{[n]}/\gamma_{[n]}$; or
(ii) $\varphi_{[1]}\gamma_{[1]} \le \ldots \le \varphi_{[\ell]}\gamma_{[n]}$.

Hence, the solution for $1/p_k\sim f(.)$, $\xi_k\sim\exp(\gamma_k)$ / $E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$ can be approximated (i.e., can find a candidate for r^*) by arranging jobs in non-decreasing order of either η_k/γ_k or $\varphi_k\gamma_k$ where η_k and φ_k are defined by (38) and (39).

Example 5. Consider the problem $1/p_k\sim G(0.5, 2)$, $\xi_k\sim\exp(\gamma_k)$/ $E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$ of Table 5.

For $p_k\sim G(0.5,2)$, $L(\gamma_k) = [2/(2 + \gamma_k)]^{0.5}$, $k = 1, \ldots, 5$. Then, using (38), $\eta_k = -0.215$, 0.149, -0.378, 0.378, and 0.262, $k = 1, \ldots, 5$, respectively. Since $\eta_3 < \eta_1 < 0 < \eta_2 < \eta_5 < \eta_4$, $\gamma_3 < \gamma_1$ and $\gamma_2 > \gamma_5 > \gamma_4$, based on condition (i) of Corollary 11, r^*: 3-1-2-5-4. Here, the candidate found by arranging jobs in non-decreasing order of η_k/γ_k is also optimal.

Table 5. A stochastic E-T problem with $p_k\sim G(0.5, 2)$ and $\xi_k\sim\exp(\gamma_k)$

Job k	γ_k	ω^E_k	ω^T_k	λ_k	λ_k
1	0.6	1.0	3.0	-2.0	-3.0
2	1.0	2.0	1.0	1.0	5.0
3	0.5	1.0	5.0	-4.0	1.0
4	0.5	4.0	0.0	4.0	-7.0
5	0.8	3.0	1.0	2.0	-4.0

Remark 11. A sequence $[1], \ldots, [n]$ is optimal for $1/p_k\sim f(.)$, $\xi_k\sim\exp(\gamma_k)$/ $E[\sum_{k=1}^{n} w^T_{[k]}X^T_{[k]}]$ if either $\eta_{[1]} \le \ldots \le \eta_{[n]}$ and $\gamma_{[1]} \le \ldots \le \gamma_{[n]}$, or $\varphi_{[1]} \le \ldots \le \varphi_{[n]}$ and $\gamma_{[1]} \ge \ldots \ge \gamma_{[\ell]}$ (see Corollary 11). Also, a sequence $[1], \ldots, [n]$ is optimal for $1/p_k\sim f(.)$, $\xi_k\sim\exp(\gamma_k)$/ $E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]}]$ if either $\eta_{[1]} \le \ldots \le \eta_{[n]}$ and $\gamma_{[1]} \ge \ldots \ge \gamma_{[n]}$, or $\varphi_{[1]} \le \ldots \le \varphi_{[n]}$ and $\gamma_{[1]} \le \ldots \le \gamma_{[n]}$.

3.3.3 Distinctly distributed processing times and due-dates

Assuming $p_k\sim f_k(.)$ and $\xi_k\sim g_k(.)$, $k = 1, \ldots, n$, we analyze the following cases.

Exponential processing times and uniform due-dates

Consider the case where $p_k\sim\exp(\alpha_k)$ and $\xi_k\sim U[a_k, b_k]$, $k = 1, \ldots, n$. Using $F_k(x) = 1 - \exp(-\alpha_k x)$, $g_k(x) = 1/(b_k - a_k)$, and (26) in (31), $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ iff

$$\frac{\lambda_i\alpha_i}{(\alpha_i - \alpha_j)(b_i - a_i)}\int_{a_i}^{b_i}[\exp(-\alpha_j x) - \exp(-\alpha_i x)]\tilde{F}_\theta(x)dx \le \frac{\lambda_j\alpha_j}{(\alpha_i - \alpha_j)(b_j - a_j)}\int_{a_j}^{b_j}[\exp(-\alpha_j x) - \exp(-\alpha_i x)]\tilde{F}_\theta(x)dx \tag{41}$$

Theorem 11. For $1/p_k\sim\exp(\alpha_k)$, $\xi_k\sim U[a_k, b_k]$/ $E[\sum_{k=1}^{n} w^E_{[k]}X^E_{[k]} + w^T_{[k]}X^T_{[k]}]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if

(i) $\lambda_i < 0 < \lambda_j$; or
(ii) $\lambda_i a_i/(b_i - a_i) \le \lambda_j a_j/(b_j - a_j)$ and either
  (1) $0 < \lambda_i < \lambda_j$, $a_i \ge a_j$, and $b_i \le b_j$, or
  (2) $\lambda_i < \lambda_j < 0$, $a_i \le a_j$, and $b_i \ge b_j$.

Proof. We have $[\exp(-a_j x) - \exp(-a_i x)]/(a_i - a_j) \ge 0$ for $x \ge 0$. Then, for $\lambda_i < 0 < \lambda_j$, (41) always holds. For $0 < \lambda_i < \lambda_j$, using (41), $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if

$$\frac{\lambda_i\alpha_i(b_j - a_j)}{\lambda_j\alpha_j(b_i - a_i)} \le \frac{\int_{a_j}^{b_j}\left[[\exp(-\alpha_j x) - \exp(-\alpha_i x)]/(\alpha_i - \alpha_j)\right]\tilde{F}_\theta(x)dx}{\int_{a_i}^{b_i}\left[[\exp(-\alpha_j x) - \exp(-\alpha_i x)]/(\alpha_i - \alpha_j)\right]\tilde{F}_\theta(x)dx}. \tag{42}$$

Inequality (42) is satisfied if its left hand side (LHS) is at most equal to one (i.e., $\lambda_i a_i(b_j - a_j) \le \lambda_j a_j(b_i - a_i)$ or $\lambda_i a_i/(b_i - a_i) \le \lambda_j a_j/(b_j - a_j)$) and its right hand side (RHS) is at least equal to one (i.e., $a_i \ge a_j$ and $b_i \le b_j$). (By definition, (41) also holds if $\lambda_i = \lambda_j = 0$ or $a_i = a_j$.) Hence, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if $0 < \lambda_i < \lambda_j$, $a_i \ge a_j$, $b_i \le b_j$, and $\lambda_i a_i/(b_i - a_i) \le \lambda_j a_j/(b_j - a_j)$. For $\lambda_i < \lambda_j < 0$, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if an inequality similar to (42) but with direction "≥" holds. However, such an inequality is satisfied if its LHS is at least equal to one (i.e., $\lambda_i a_i/(b_i - a_i) \le \lambda_j a_j/(b_j - a_j)$ where $\lambda_i < \lambda_j < 0$) and its RHS is at most equal to one (i.e., $a_i \le a_j$ and $b_i \ge b_j$). Thus, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if $\lambda_i < \lambda_j < 0$, $a_i \le a_j$, $b_i \ge b_j$, and $\lambda_i a_i/(b_i - a_i) \le \lambda_j a_j/(b_j - a_j)$.

Corollary 12. For $1/p_k \sim \exp(a_k)$, $\xi_k \sim U[a_k, b_k]/\ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{T}]$, a sequence $[1], \ldots, [\ell], [\ell + 1], \ldots, [n]$, $\ell \in \{0, 1, \ldots, n\}$, is optimal if

(i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$ and $\lambda_{[1]}a_{[1]}/(b_{[1]} - a_{[1]}) \le \ldots \le \lambda_{[n]}a_{[n]}/(b_{[n]} - a_{[n]})$,
and
(ii) $a_{[1]} \le \ldots \le a_{[\ell]}$, $a_{[\ell+1]} \ge \ldots \ge a_{[n]}$, $b_{[1]} \ge \ldots \ge b_{[\ell]}$, and $b_{[\ell+1]} \le \ldots \le b_{[n]}$.

Proof. Using an approach similar to that of the proof of Corollary 1, we show that a sequence found by arranging jobs based on Corollary 12 (see also conditions (i) – (iii) of Theorem 11) is optimal.

The optimality conditions of Corollary 12 can be relaxed by removing $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$ and condition (ii) leaving behind only $\lambda_{[1]}a_{[1]}/(b_{[1]} - a_{[1]}) \le \ldots \le \lambda_{[n]}a_{[n]}/(b_{[n]} - a_{[n]})$. This condition can be used to approximate the solution (i.e., find a candidate for r^*) for $1/p_k \sim \exp(a_k)$, $\xi_k \sim U[a_k, b_k]/\ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{T}]$, that is, arranging jobs in non-decreasing order of $\lambda_k a_k/(b_k - a_k)$ provides a candidate for r^*.

Remark 12. Based on Corollary 12, a sequence $[1], \ldots, [n]$ where $\lambda_{[1]} \le \ldots \le \lambda_{[n]}$ and $\lambda_{[1]}a_{[1]}/(b_{[1]} - a_{[1]}) \le \ldots \le \lambda_{[n]}a_{[n]}/(b_{[n]} - a_{[n]})$ is optimal for $1/p_k \sim \exp(a_k)$, $\xi_k \sim U[a_k, b_k]/\ E[\sum_{k=1}^{n} w_{[k]}^{T} X_{[k]}^{T}]$ if $a_{[1]} \le \ldots \le a_{[n]}$ and $b_{[1]} \ge \ldots \ge b_{[n]}$, and is optimal for $1/p_k \sim \exp(a_k)$, $\xi_k \sim U[a_k, b_k]/\ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E}]$ if $a_{[1]} \ge \ldots \ge a_{[n]}$ and $b_{[1]} \le \ldots \le b_{[n]}$.

Corollary 13. For $1/p_k \sim \exp(a_k)$, $\xi_k \sim U[a, b]/\ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{T}]$, an optimal sequence is found by arranging jobs in non-decreasing ordering of $\lambda_k a_k$, $k = 1, \ldots, n$.

Proof. It immediately follows from Corollary 12 (see also inequality (41)).

Note that Corollary 13 and Theorem 6 provide the same results.

Exponential processing times and due-dates

Consider the case where $p_k \sim \exp(a_k)$ and $\xi_k \sim \exp(\gamma_k)$, $k = 1, \ldots, n$. Then, using $F_k(x) = 1 - \exp(-a_k x)$, $G_k(x) = 1 - \exp(-\gamma_k x)$, and (26) into (31), $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ iff

$$\frac{\lambda_i \alpha_i \gamma_i}{(\alpha_i - \alpha_j)} \int_0^\infty \left[\exp[-(\alpha_j + \gamma_i)x] - \exp[-(\alpha_i + \gamma_i)x]\right] \tilde{F}_\theta(x)dx$$
$$\le \frac{\lambda_j \alpha_j \gamma_j}{(\alpha_i - \alpha_j)} \int_0^\infty \left[\exp[-(\alpha_j + \gamma_j)x] - \exp[-(\alpha_i + \gamma_j)x]\right] \tilde{F}_\theta(x)dx. \quad (43)$$

Theorem 12. For $1/p_k \sim \exp(a_k)$, $\xi_k \sim \exp(\gamma_k)/\ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{T}]$, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if

(i) $\lambda_i < 0 < \lambda_j$; or
(ii) $\lambda_i a_i \gamma_i \le \lambda_j a_j \gamma_j$ and either
  (1) $0 < \lambda_i < \lambda_j$ and $\gamma_i \ge \gamma_j$, or
  (2) $\lambda_i < \lambda_j < 0$ and $\gamma_i \le \gamma_j$.

Proof. We have $[\exp[-(a_j + \gamma_k)x] - \exp[-(a_i + \gamma_k)x]]/(a_i - a_j) \ge 0$ for all $x \ge 0$, $k \in \{i, j\}$. When $\lambda_i < 0 < \lambda_j$, (43) is satisfied. When $0 < \lambda_i < \lambda_j$, using (43), $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if

$$\frac{\lambda_i \alpha_i \gamma_i}{\lambda_j \alpha_j \gamma_j} \le \frac{\int_0^\infty \left[\frac{[\exp(-(\alpha_j + \gamma_j)x) - \exp(-(\alpha_i + \gamma_j)x)]}{(\alpha_i - \alpha_j)}\right] \tilde{F}_\theta(x)dx}{\int_0^\infty \left[\frac{[\exp(-(\alpha_j + \gamma_i)x) - \exp(-(\alpha_i + \gamma_i)x)]}{(\alpha_i - \alpha_j)}\right] \tilde{F}_\theta(x)dx}. \quad (44)$$

Inequality (44) holds if its LHS is at most equal to one (i.e., $\lambda_i a_i \gamma_i \le \lambda_j a_j \gamma_j$) and its RHS is at least equal to one (i.e., $\gamma_i \ge \gamma_j$). (By definition, (43) also holds when $\lambda_i = \lambda_j = 0$, or $a_i = a_j$, or $\gamma_i = \gamma_j$ and $\lambda_i a_i \le \lambda_j a_j$.) Hence, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if $0 < \lambda_i < \lambda_j$, $\gamma_i \ge \gamma_j$ and $\lambda_i a_i \gamma_i \le \lambda_j a_j \gamma_j$. For $\lambda_i < \lambda_j < 0$, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if an inequality similar to (44) but with direction "≥" is satisfied. But, such an inequality holds if its LHS is at least equal to one (i.e., $\lambda_i a_i \gamma_i \le \lambda_j a_j \gamma_j$ where $\lambda_i < \lambda_j < 0$) and its RHS is at most equal to one (i.e., $\gamma_i \le \gamma_j$). Hence, $\theta ij\delta \succ \theta ji\delta$ for every $i \ne j \in N$ and every θ and δ if $\lambda_i < \lambda_j < 0$, $\gamma_i \le \gamma_j$ and $\lambda_i a_i \gamma_i \le \lambda_j a_j \gamma_j$.

Corollary 14. For $1/p_k \sim \exp(a_k)$, $\xi_k \sim \exp(\gamma_k)/\ E[\sum_{k=1}^{n} w_{[k]}^{E} X_{[k]}^{E} + w_{[k]}^{T} X_{[k]}^{T}]$, a sequence $[1], \ldots, [\ell], [\ell + 1], \ldots, [n]$, $\ell \in \{0, 1, \ldots, n\}$, is optimal if

(i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$ and $\lambda_{[1]}a_{[1]}\gamma_{[1]} \le \ldots \le \lambda_{[n]}a_{[n]}\gamma_{[n]}$, and
(ii) $\gamma_{[1]} \le \ldots \le \gamma_{[\ell]}$ and $\gamma_{[\ell+1]} \ge \ldots \ge \gamma_{[n]}$.

Proof. We use an approach similar to that of the proof of Corollary 1 to show that a sequence found by arranging jobs according to Corollary 14 (see also conditions (i) − (iii) of Theorem 12) is optimal.

We can relax the optimality conditions of Corollary 14 by removing $\lambda_{[1]} \leq \ldots \leq \lambda_{[\ell]} \leq 0 \leq \lambda_{[\ell+1]} \leq \ldots \leq \lambda_{[n]}$ and condition (ii) leaving behind only $\lambda_{[1]}a_{[1]}\gamma_{[1]} \leq \ldots \leq \lambda_{[n]}a_{[n]}\gamma_{[n]}$. This condition can be used to approximate the solution (i.e., find a candidate for r^*) for $1/p_k\sim\exp(a_k)$, $\xi_k\sim\exp(\gamma_k)$ / $E[\sum_{k=1}^{n} w_{[k]}^{E}X_{[k]}^{E} + w_{[k]}^{T}X_{[k]}^{T}]$, that is, arranging jobs in non-decreasing order of $\lambda_k a_k \gamma_k$ can provide a candidate for r^*.

Remark 13. According to Corollary 14, a sequence [1], …, [n] in which $\lambda_{[1]} \leq \ldots \leq \lambda_{[n]}$ and $\lambda_{[1]}a_{[1]}\gamma_{[1]} \leq \ldots \leq \lambda_{[n]}a_{[n]}\gamma_{[n]}$ is optimal for $1/\ p_k\sim\exp(a_k)$, $\xi_k\sim\exp(\gamma_k)/\ E[\sum_{k=1}^{n} w_{[k]}^{T}X_{[k]}^{T}]$ if $\gamma_{[1]} \leq \ldots \leq \gamma_{[n]}$, and is optimal for $1/p_k\sim\exp(a_k)$, $\xi_k\sim\exp(\gamma_k)$ / $E[\sum_{k=1}^{n} w_{[k]}^{E}X_{[k]}^{E}]$ if $\gamma_{[1]} \geq \ldots \geq \gamma_{[n]}$.

Corollary 15. For $1/p_k\sim\exp(a_k)$, $\xi_k\sim\exp(\gamma)/\ E[\sum_{k=1}^{n} w_{[k]}^{E}X_{[k]}^{E} + w_{[k]}^{T}X_{[k]}^{T}]$, an optimal sequence is found by arranging jobs in non-decreasing ordering of $\lambda_k a_k$, $k = 1, \ldots, n$.

Proof. It immediately follows from Corollary 14 (see also inequality (43)).

Observe that Corollary 15 and Theorem 6 provide the same results.

Remark 14. For $1/p_k\sim f_k(.)$, $\xi_k\sim g_k(.)/\ E[\sum_{k=1}^{n} w_{[k]}^{E}X_{[k]}^{E} + w_{[k]}^{T}X_{[k]}^{T}]$, based on this subsection's discussion, jobs [k], $k = 1, \ldots, \ell, \ell + 1, \ldots, n$, $\ell \in \{0, 1, \ldots, n\}$, are arranged in r^* in non-decreasing order of $\omega_{[k]}^{E} - \omega_{[k]}^{T}$ (i.e., $\lambda_{[k]}$) where there are additional conditions imposed on some other characteristics of jobs [k], $k = 1, \ldots, \ell$ (i.e., jobs with $\omega_{[k]}^{E} \leq \omega_{[k]}^{T}$) as well as on those of jobs [k], $k = \ell + 1, \ldots, n$ (i.e., jobs with $\omega_{[k]}^{E} \geq \omega_{[k]}^{T}$). Hence, $1/p_k\sim f_k(.)$, $\xi_k\sim g_k(.)$ / $E[\sum_{k=1}^{n} w_{[k]}^{E}X_{[k]}^{E} + w_{[k]}^{T}X_{[k]}^{T}]$ among jobs [1], …, [ℓ], [ℓ + 1], …, [n] where $-\infty < \lambda_{[k]} < \infty$ (i.e., the stochastic E-T problem) is a mixture of $1/p_k\sim f_k(.)$, $\xi_k\sim g_k(.)/\ E[\sum_{k=1}^{n} w_{[k]}^{T}X_{[k]}^{T}]$ among jobs [1], …, [ℓ] where $\omega_{[k]}^{T} = -\lambda_{[k]} \geq 0$ (i.e., the stochastic T problem) and of $1/p_k\sim f_k(.)$, $\xi_k\sim g_k(.)$ / $E[\sum_{k=1}^{n} w_{[k]}^{E}X_{[k]}^{E}]$ among jobs [ℓ + 1], …, [n] where $\omega_{[k]}^{E} = \lambda_{[k]} \geq 0$ (i.e., the stochastic E problem).

4. SUMMARY AND SOME CONCLUDING REMARKS

In this paper, we have studied a stochastic single machine scheduling problem in which processing times or due-dates are non-negative independent random variables and random weights (penalties) are imposed on both early and tardy (E-T) jobs. These random weights do not depend on the amount of deviations of job completion times from their due dates, that is, the penalty for missing a due date by a short or long period is the same. The objective is to find an optimal sequence that minimizes the expected total weighted number of early and tardy jobs. We have examined three scenarios of the proposed stochastic E-T problem including a scenario with deterministic processing times and stochastic due-dates, a scenario with stochastic processing times and deterministic due-dates, and a scenario with stochastic processing times and stochastic due-dates. These problem scenarios are NP hard to solve; however, based on some structures on the stochasticity of processing times or due dates, we have solved exactly various resulting cases of the three scenarios (see Table 6). We have also presented methods to approximate the solutions for the general versions of these cases. It is demonstrated that in the proposed stochastic E-T problem those jobs whose mean earliness penalties are at most equal to their mean tardiness penalties appear in the optimal sequence before those whose mean earliness penalties are greater than their mean tardiness penalties. Moreover, the problem studied here is shown to be general in the sense that its special or limiting cases reduce to some classical single machine scheduling problems including the stochastic problem of minimizing the expected weighted number of tardy jobs and the stochastic problem of minimizing the expected weighted number of early which both are solvable by the proposed exact or approximate methods. This research validates one of the principles of synchronous manufacturing that statistical fluctuations in job characteristics such as processing times, due dates, and earliness and tardiness penalties affect scheduling decisions. An immediate extension of this study is to explore the most general version of the problem when processing times and due dates have distinct arbitrary distributions. In addition, due to the importance of research in scheduling with setup times (e.g., Allahverdi et al., 1999; Allahverdi et al., 2006; Allahverdi and Soroush, 2006), it is highly recommended to examine the proposed stochastic E-T problem by incorporating explicitly job setup times.

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Table 6. The exact solution methods for various cases for the three problem scenarios of the stochastic E-T problem, stochastic T problem, and stochastic E problem

Problem Scenario	Case	Stochastic E-T Problem	Stochastic T Problem	Stochastic E Problem
Deterministic Processing Times π_k & Stochastic Due Dates ξ_k	$p_k = \pi_k$ & $\xi_k \sim g(.)$	Sequence $[1],\ldots,[\ell],[\ell+1],\ldots,[n]$, $\ell \in \{0,1,\ldots,n\}$ is optimal if (i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$ and (ii) $\pi_{[1]} \le \ldots \le \pi_{[\ell]}, \pi_{[\ell+1]} \ge \ldots \ge \pi_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if $\omega^T_{[1]} \ge \ldots \ge \omega^T_{[n]}$ and $\pi_{[1]} \le \ldots \le \pi_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if $\omega^E_{[1]} \le \ldots \le \omega^E_{[n]}$ and $\pi_{[1]} \ge \ldots \ge \pi_{[n]}$.
	$p_k = \pi_k$ & $\xi_k \sim \exp(\gamma_k)$	Sequence $[1],\ldots,[\ell],[\ell+1],\ldots,[n]$, $\ell \in \{0,1,\ldots,n\}$, is optimal if (i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$, (ii) $\gamma_{[1]} \le \ldots \le \gamma_{[\ell]}, \gamma_{[\ell+1]} \ge \ldots \ge \gamma_{[n]}$, and (iii) $\pi_{[1]}/\gamma_{[1]} \le \ldots \le \pi_{[\ell]}/\gamma_{[\ell]}, \pi_{[\ell+1]}/\gamma_{[\ell+1]} \ge \ldots \ge \pi_{[n]}/\gamma_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if $\lambda_{[1]} \le \ldots \le \lambda_{[n]}$, $\gamma_{[1]} \le \ldots \le \gamma_{[n]}$, and $\pi_{[1]}/\gamma_{[1]} \le \ldots \le \pi_{[\ell]}/\gamma_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if $\lambda_{[1]} \le \ldots \le \lambda_{[n]}$, $\gamma_{[1]} \ge \ldots \ge \gamma_{[n]}$, and $\pi_{[1]}/\gamma_{[1]} \ge \ldots \ge \pi_{[n]}/\gamma_{[n]}$.
Stochastic Processing Times p_k & Deterministic Due Dates d_k	$p_k \sim \exp(a)$ & $\xi_k = d_k$	Sequence $[1],\ldots,[\ell],[\ell+1],\ldots,[n]$, $\ell \in \{0,1,\ldots,n\}$, is optimal if either (i) $\delta_{[1]} \le \ldots \le \delta_{[\ell]} \le 0 \le \delta_{[\ell+1]} \le \ldots \le \delta_{[n]}$, $d_{[1]} \ge \ldots \ge d_{[\ell]}$, and $d_{[\ell+1]} \le \ldots \le d_{[n]}$ where $\delta_k = \lambda_k d_k \exp(-ad_k)$; or (ii) $\psi_{[1]} \le \ldots \le \psi_{[\ell]} \le 0 \le \psi_{[\ell+1]} \le \ldots \le \psi_{[n]}$, $d_{[1]} \le \ldots \le d_{[\ell]}$, and $d_{[\ell+1]} \ge \ldots \ge d_{[n]}$ where $\psi_k = \lambda_k d_k^{n-1} \exp(-ad_k)$.	Sequence $[1],\ldots,[n]$ is optimal if either (i) $\delta_{[1]} \le \ldots \le \delta_{[n]}$ and $d_{[1]} \ge \ldots \ge d_{[n]}$ where $\delta_k = \lambda_k d_k \exp(-ad_k)$; or (ii) $\psi_{[1]} \le \ldots \le \psi_{[n]}$ and $d_{[1]} \le \ldots \le d_{[n]}$ where $\psi_k = \lambda_k d_k^{n-1} \exp(-ad_k)$.	Sequence $[1],\ldots,[n]$ is optimal if either (i) $\delta_{[1]} \le \ldots \le \delta_{[n]}$ and $d_{[1]} \le \ldots \le d_{[n]}$ where $\delta_k = \lambda_k d_k \exp(-ad_k)$; or (ii) $\psi_{[1]} \le \ldots \le \psi_{[n]}$ and $d_{[1]} \ge \ldots \ge d_{[n]}$. where $\psi_k = \lambda_k d_k^{n-1} \exp(-ad_k)$.
	$p_k \sim \exp(a_k)$ & $\xi_k = d$	Arrange jobs in non-decreasing order of $\lambda_k a_k$.	Arrange jobs in non-increasing order of $\alpha_k \omega^T_k$ or in non-decreasing order of $E(p_k)/\omega^T_k$ (i.e., WSEPT rule).	Arrange jobs in non-decreasing order of $\alpha_k \omega^E_k$ or in non-increasing order of $E(p_k)/\omega^E_k$ (i.e., WLEPT rule).
	$p_k \sim W(a_k, \beta_k)$ & $\xi_k = d$	Sequence $[1],\ldots,[\ell],[\ell+1],\ldots,[n]$, $\ell \in \{0,1,\ldots,n\}$, is optimal if (i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$ and (ii) $F_{[1]}(d) \ge \ldots \ge F_{[\ell]}(d), F_{[\ell+1]}(d) \le \ldots \le F_{[n]}(d)$ where $F_k(d) =$ 1- $\exp[-(\alpha_k d)^{\beta_k}]$ and $\beta_{[1]} \le \ldots \le \beta_{[\ell]}, \beta_{[\ell+1]} \ge \ldots \ge \beta_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if $\lambda_{[1]} \le \ldots \le \lambda_{[n]}$ and $F_{[1]}(d) \ge \ldots \ge F_{[n]}(d)$ where $F_k(d) =$ 1- $\exp[-(\alpha_k d)^{\beta_k}]$ and $\beta_{[1]} \ge \ldots \ge \beta_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if $\lambda_{[1]} \le \ldots \le \lambda_{[n]}$ and $F_{[1]}(d) \le \ldots \le F_{[n]}(d)$ where $F_k(d) =$ 1- $\exp[-(\alpha_k d)^{\beta_k}]$ and $\beta_{[1]} \le \ldots \le \beta_{[n]}$.
	$p_k \sim U[a_k, b_k]$ & $\xi_k = d$	Sequence $[1],\ldots,[\ell],[\ell+1],\ldots,[n]$, $\ell \in \{0,1,\ldots,n\}$, is optimal if (i) $\lambda_{[1]} \le \ldots \le \lambda_{[\ell]} \le 0 \le \lambda_{[\ell+1]} \le \ldots \le \lambda_{[n]}$ and (ii) $F_{[1]}(d) \ge \ldots \ge F_{[\ell]}(d), F_{[\ell+1]}(d) \le \ldots \le F_{[n]}(d)$ where $F_k(x) =$ $(x - a_k)/(b_k - a_k)$.	Sequence $[1],\ldots,[n]$ is optimal if $\lambda_{[1]} \le \ldots \le \lambda_{[n]}$ and $F_{[1]}(d) \ge \ldots \ge F_{[n]}(d)$ where $F_k(x) = (x - a_k)/(b_k - a_k)$.	Sequence $[1],\ldots,[n]$ is optimal if $\lambda_{[1]} \le \ldots \le \lambda_{[n]}$ and $F_{[1]}(d) \le \ldots \le F_{[n]}(d)$ where $F_k(x) = (x - a_k)/(b_k - a_k)$.

Table 6. Continued

Problem Scenario	Case	Stochastic E-T Problem	Stochastic T Problem	Stochastic E Problem
Stochastic Processing Times p_k & Stochastic Due Dates ξ_k	$p_k \sim \exp(a_k)$ & $\xi_k \sim g(.)$	Arrange jobs in a non-decreasing order of $\lambda_k a_k$.	Arrang jobs according to WSEPT.	Arrange jobs according to WLEPT.
	$p_k \sim f_k(.)$ & $\xi_k \sim \exp(\gamma)$	Arrange jobs in a non-decreasing order of $\lambda_k/[1/L_k(\gamma)$-1] where $L_k(\gamma)$ is the LSTof $f_k(.)$ evaluated at γ.	Arrange jobs in non-increasing order of $\omega_k^T/[1/L_k(\gamma) - 1]$.	Ararng jobs in non-decreasing order of $\omega_k^E / [1/L_k(\gamma) - 1]$.
	$p_k \sim f(.)$ & $\xi_k \sim g(.)$	Arrange jobs in a non-decreasing order of λ_k.	Arrange jobs in a non-increasing order of ω_k^T.	Arrange jobs in a non-decreasing order of ω_k^E.
	$p_k \sim f_k(.)$ & $\xi_k \sim g(.)$ & $\lambda_k = \lambda$	Arrange jobs in non-increasing stochastic ordering of p_k (i.e., $p_{[1]} \geq_{st} \ldots \geq_{st} p_{[n]}$) if $\lambda > 0$, and in non-decreasing stochastic ordering of p_k (i.e., $p_{[1]} \leq_{st} \ldots \leq_{st} p_{[n]}$) if $\lambda < 0$.	Arrange jobs in non-decreasing stochastic ordering of p_k.	Arrange jobs in non-increasing stochastic ordering of p_k.
	$p_k \sim f(.)$ & $\xi_k \sim \exp(\gamma_k)$	Sequence $[1],\ldots,[\ell],[\ell+1],\ldots,[n]$, $\ell \in \{0,1,\ldots,n\}$, is optimal if either (i) $\eta_{[1]} \leq \ldots \leq \eta_{[\ell]} \leq 0 \leq \eta_{[\ell+1]} \leq \ldots \leq \eta_{[n]}$, $\gamma_{[1]} \leq \ldots \leq \gamma_{[\ell]}$, and $\gamma_{[\ell+1]} \geq \ldots \geq \gamma_{[n]}$ where $\eta_k = \lambda_k L(\gamma_k)[1 - L(\gamma_k)]$; or (ii) $\varphi_{[1]} \leq \ldots \leq \varphi_{[\ell]} \leq 0 \leq \varphi_{[\ell+1]} \leq \ldots \leq \varphi_{[n]}$, $\gamma_{[1]} \geq \ldots \geq \gamma_{[\ell]}$, and $\gamma_{[\ell+1]} \leq \ldots \leq \gamma_{[n]}$ where $\varphi_k = \lambda_k L^{n-1}(\gamma_k)[1 - L(\gamma_k)]$.	Sequence $[1],\ldots,[n]$ is optimal if either (i) $\eta_{[1]} \leq \ldots \leq \eta_{[n]}$ and $\gamma_{[1]} \leq \ldots \leq \gamma_{[n]}$ where $\eta_k = \lambda_k L(\gamma_k)[1 - L(\gamma_k)]$; or (ii) $\varphi_{[1]} \leq \ldots \leq \varphi_{[n]}$ and $\gamma_{[1]} \geq \ldots \geq \gamma_{[\ell]}$ where $\varphi_k = \lambda_k L^{n-1}(\gamma_k)[1 - L(\gamma_k)]$.	Sequence $[1],\ldots,[n]$ is optimal if either (i) $\eta_{[1]} \leq \ldots \leq \eta_{[n]}$ and $\gamma_{[1]} \geq \ldots \geq \gamma_{[n]}$ where $\eta_k = \lambda_k L(\gamma_k)[1 - L(\gamma_k)]$; or (ii) $\varphi_{[1]} \leq \ldots \leq \varphi_{[n]}$ and $\gamma_{[1]} \leq \ldots \leq \gamma_{[n]}$ where $\varphi_k = \lambda_k L^{n-1}(\gamma_k)[1 - L(\gamma_k)]$.
	$p_k \sim \exp(a_k)$ & $\xi_k \sim U[a_k, b_k]$	Sequence $[1],\ldots,[\ell],[\ell+1],\ldots,[n]$, $\ell \in \{0,1,\ldots,n\}$, is optimal if (i) $\lambda_{[1]} \leq \ldots \leq \lambda_{[\ell]} \leq 0 \leq \lambda_{[\ell+1]} \leq \ldots \leq \lambda_{[n]}$, (ii) $\lambda_{[1]}a_{[1]}/(b_{[1]} - a_{[1]}) \leq \ldots \leq \lambda_{[n]}a_{[n]}/(b_{[n]} - a_{[n]})$, (iii) $a_{[1]} \leq \ldots \leq a_{[\ell]}$, $a_{[\ell+1]} \geq \ldots \geq a_{[n]}$, and (iv) $b_{[1]} \geq \ldots \geq b_{[\ell]}$, $b_{[\ell+1]} \leq \ldots \leq b_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if $\lambda_{[1]} \leq \ldots \leq \lambda_{[n]}$, $\lambda_{[1]}a_{[1]}/(b_{[1]} - a_{[1]}) \leq \ldots \leq \lambda_{[n]}a_{[n]}/(b_{[n]} - a_{[n]})$, $a_{[1]} \leq \ldots \leq a_{[n]}$, and $b_{[1]} \geq \ldots \geq b_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if $\lambda_{[1]} \leq \ldots \leq \lambda_{[n]}$, $\lambda_{[1]}a_{[1]}/(b_{[1]} - a_{[1]}) \leq \ldots \leq \lambda_{[n]}a_{[n]}/(b_{[n]} - a_{[n]})$, $a_{[1]} \geq \ldots \geq a_{[n]}$, and $b_{[1]} \leq \ldots \leq b_{[n]}$.
	$p_k \sim \exp(a_k)$ & $\xi_k \sim \exp(\gamma_k)$	Sequence $[1],\ldots,[\ell],[\ell+1],\ldots,[n]$, $\ell \in \{0,1,\ldots,n\}$, is optimal if (i) $\lambda_{[1]} \leq \ldots \leq \lambda_{[\ell]} \leq 0 \leq \lambda_{[\ell+1]} \leq \ldots \leq \lambda_{[n]}$, (ii) $\lambda_{[1]}a_{[1]}\gamma_{[1]} \leq \ldots \leq \lambda_{[n]}a_{[n]}\gamma_{[n]}$, and (iii) $\gamma_{[1]} \leq \ldots \leq \gamma_{[\ell]}$, $\gamma_{[\ell+1]} \geq \ldots \geq \gamma_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if (i) $\lambda_{[1]} \leq \ldots \leq \lambda_{[n]}$, (ii) $\lambda_{[1]}a_{[1]}\gamma_{[1]} \leq \ldots \leq \lambda_{[n]}a_{[n]}\gamma_{[n]}$, and (iii) $\gamma_{[1]} \leq \ldots \leq \gamma_{[n]}$.	Sequence $[1],\ldots,[n]$ is optimal if (i) $\lambda_{[1]} \leq \ldots \leq \lambda_{[n]}$, (ii) $\lambda_{[1]}a_{[1]}\gamma_{[1]} \leq \ldots \leq \lambda_{[n]}a_{[n]}\gamma_{[n]}$, and (iii) $\gamma_{[1]} \geq \ldots \geq \gamma_{[n]}$.