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On the Potentially Optimal Solutions of Classical Shop Scheduling Problems

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Abstract—In this paper, we consider an important class of *NP*-hard shop scheduling problems, where one of the major tasks is to minimize the makespan objective over the set of all sequences. We study the existence of minimal potentially optimal solution in classical shop scheduling problems. The concept of potentially optimal solution has proven one of the most important and fertile research topics as this solution set contains at least one optimal sequence for arbitrary processing times. Here, the potentially optimal solution of all irreducible sequences is surveyed and a new decomposition approach is presented in this class. The contribution of this paper is a brief survey of the existing results together with few new results. The research results obtained in past several years are presented along with open problems and possible extensions. Varieties of results and examples we analyzed provide useful structural insights and enough motivations for the developments of exact or heuristic algorithms.

Keywords—Shop scheduling problems, Computational complexity, Counting problem, Potential optimality, Irreducibility, Sequence decomposition

1. INTRODUCTION

We consider the strongly *NP*-hard shop scheduling problem $\alpha||C_{\max}$, where α represents a job shop ($\alpha = J$) or the open shop ($\alpha = O$). We refer to Graham et al. (1979) for the classification scheme $\alpha|\beta|\gamma$ of scheduling problems, where β describes the machine environment, γ gives some job characteristics and additional requirements and is the optimality criterion. In a nonpreemptive classical shop problem, each job i, $i \in I = \{1, 2, \ldots, n\}$ has to be processed on each machine j, $j \in J = \{1, 2, \ldots, m\}$ exactly once without preemption for the positive time. The sets of $m \geq 2$ machines and $n \geq 2$ jobs are denoted by $\{M_1, M_2, \ldots, M_m\}$ and $\{J_1, J_2, \ldots, J_n\}$, respectively. Let $SIJ = I \times J$, $P = [p_{ij}]$ and $C = [c_{ij}]$ be the sets of all operations o_{ij}, the matrix of processing times p_{ij} and the matrix of completion times c_{ij} with $i \in I$ and $j \in J$, respectively. The objective function $C_{\max} = \max_{i \in I} C_i$, where the completion time C_i is the time when the last operation of J_i is finished. The order in which a certain machine processes the corresponding jobs is called job order and the order in which a certain job is processed on the corresponding machines is called machine order. In the case of a job shop all machine orders are given in advance, whereas all machine orders and all job orders can be chosen arbitrarily in the open shop. All jobs have identical machine order in the flow shop ($\alpha = F$). We have to find a feasible (acyclic) combination of all machine orders and all job orders, called sequence, which minimizes the maximum completion time. A sequence is optimal if it generates a schedule with minimum objective value among all other sequences.

The problem $O2||C_{\max}$ is solvable in $O(n)$ time, but it is *NP*-hard for $m \geq 3$ (Gonzales and Sahni, 1976). Bräsel and Kleinau (1996) present an algorithm of the same complexity for $O2||C_{\max}$ by means of block-matrices model. The problem $F3||C_{\max}$ is *NP*-hard (Lenstra et al., 1977), however this problem with only two machines is $O(n\log n)$ time solvable (Johnson (1954)). The problem $J2||C_{\max}$ is *NP*-hard as the problem $J2|p_{ij} \in \{1, 2\}|C_{\max}$ is already in this class (Lenstra and Rinnooy Kan (1979)). The study concentrated either on the determination of polynomial solvable subproblems or on the development of an algorithm for an approximate solution plays a meaningful role. Counting sequences by considering the cardinality of special latin rectangles or the chromatic polynomial of the Hamming graph $K_n \times K_m$ is hard (see Harborth (1999)). A closed formula for this unsolved counting problem is unknown up to today and only bounds are available in general cases (Bräsel and Dhamala (2001a), Bräsel and Kleinau (1992a, 1992b), Dhamala (2002b)). Enumerative algorithms demonstrate a huge number of sequences (Bräsel et al. (1999a, 1999b), Harborth (1999)). This motivates the structural investigations and the potential optimality of shop scheduling problems.

A set S of sequences is called potentially optimal solution if it contains an optimal sequence with respect to the objective γ for arbitrary processing times. The elements of such a solution set are called potentially optimal. A careful analysis of potential optimality excludes unnecessary sequences from further considerations independent of the given processing times. It deals the question whether there exists a subset of the set of

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sequences with the property that for arbitrary processing times at least one optimal sequence belongs to this set. A study of shop problems for arbitrary processing times examine properties of a schedule which are a measure of its quality. The set of all sequences in a given shop scheduling is a potentially optimal solution thought not the minimal one. For the job shop $J|n = 2|C_{\max}$, Akers and Friedman (1955) give a criterion in terms of free machines to eliminate initially a great number of the sequences that are not going to be optimal, independent of the processing times. They show for this problem that the examination of sequences which do not contain a free machine is sufficient. Conway et al. (1967) show that in the flow shop problem it is sufficient to examine those sequences in which the job orders for the first two and the last two machines, respectively, are the same. Ashour (1972) presents different decompositions of sequences and schedules for the job shop problem.

Kleinau (1993), Bräsel and Kleinau (1996) investigate the irreducible sequences by introducing a dominance relation $\preceq$ on the set of all sequences with a fixed format $n \times m$. The irreducible elements are the minimal sequences with respect to this partial order independent of the given processing times. Clearly, the solution set of all these locally optimal sequences is a potentially optimal solution of smaller cardinality. Irreducibility analysis is interesting when the processing times are erroneous, difficult to find out in advance or simply unknown in manufacturing and service industries. For example, a car may require repairs on its engine, body and electrical circuit in a large automotive garage with specialized shop-centers. These operations may be processed in any order but it is not possible to perform any two of the same job simultaneously. Similar applications of open shop scheduling may arise in testing components of an electronic system, repairing parts of an airplane in a large aircraft garage and satellite communications (Prins (1994), Bräsel et al. (1999a)).

Bräsel and Kleinau (1996) investigate the irreducible sequences for the problem $O||C_{\max}$ on an operation set with spanning tree structure. For these operation sets, each combination of machine orders and job orders is feasible. They describe in detail the set of all locally minimal elements for the problem $O2||C_{\max}$. Bräsel et al. (1999a, 1999b) and Harborth (1999) extend the concept of irreducibility and reducibility for the open shop and job shop problems to minimize the maximum completion time. They present several sufficient conditions for reducibility of a sequence each of which can be tested in polynomial time. Enumeration algorithms for irreducible sequences have been proposed and number of all sequences and irreducible sequences for small formats have been computed. One of the most strong motivations of irreducibility analysis comes from the experiences that only a very small fraction of all sequences is irreducible.

Tautenhahn (2000) presents a necessary and sufficient condition for the open shop irreducibility on tree-like operation sets. This test can be performed in polynomial time. A study of the potentially (universally) optimal solution set and unavoidable sequences with respect to an arbitrary sequence set can be found in Tautenhahn and Willenius (2000) and Willenius (2000). Their study generalizes the concept of irreducibility by considering a dominance relation between a sequence and a set of sequences. A sequence A is unavoidable with respect to a sequence set if for all sequences in this set, there exists a matrix of processing times resulting in a better objective value on A. They give several necessary and sufficient conditions for a sequence to be dominated by a set of other sequences. They formulate the dominance relation as a mixed integer program and compute minimal potentially optimal solutions for certain small sized open shop problems. The results on irreducibility for other regular objectives are extended by Willenius (2000). A decomposition approach is introduced and some sufficient conditions for the irreducibility of sequences are presented by Dhamala (2002b), Bräsel and Dhamala (2002b).

More general way of dealing with arbitrary processing times is the stability analysis of single sequence. A lot of attention have been given to the stability and structural analysis of more general shop scheduling methods with uncertainty in the numerical data. These studies concentrate mainly on the stability analysis and scheduling methods for interval processing times. Stability analysis is used for the phase of an algorithm at which a schedule (sequence) of a scheduling problem has already been found and additional calculations are preformed in order to investigate how this schedule (sequence) depends on the numerical input data.

Sotskov (1991) considers the problem of optimal scheduling n uninterrupted operations on m machines with respect to a regular objective and reduces the problem of calculating the stability radius to the solution of a non-linear programming. He gives the necessary and sufficient conditions for the stability radius $\rho_s(P)$ (the largest quantity of independent variations of the processing times of the operations such that an optimal schedule remains optimal) to be positive and to be infinite in case when S is the optimal makespan schedule. Kravchenko et al. (1995) present the necessary and sufficient conditions for minimizing the makespan or maximum lateness to have at least one optimal schedule with infinite stability radius and show that there does not exist such a schedule for other regular criteria. Bräsel et al. (1996) calculate the stability radius of an optimal schedule for general shop scheduling with mean flow time objective. They derive formulas for calculating the stability radius and give necessary and sufficient conditions when the radius is zero. By generalizing the dominance relation, Lai et al. (1997) present a characterization of $\rho_s(P) = 0$ and $\rho_s(P) = \infty$ and give the exact value of the relative stability radius for the general shop scheduling problem $G|a_{ij} \leq p_{ij} \leq b_{ij}|C_{\max}$. By considering randomly generated job shop instances, Sotskov et al. (1997) study the influence of errors and possible changes of the processing times on the optimality of a schedule. Sotskova (2001) deals with the stability analysis and computations of the stability radius for the objectives $\gamma \in \{C_{\max}, \sum_i C_i\}$ on the basis of the existence of more practical problems with uncertain input data under strict uncertainty.

The theory of irreducibility generalizes the earlier concept of potential optimality. There are relations between the stability and irreducibility analysis. The works of Harborth (1999), Sotskov (1991) and Sotskov et al. (1998) illustrates this relation. Clearly irreducibility analysis deals the problems with $0 \le p_{ij} < \infty$. Lai and Sotskov (1999) and Sotskova (2001) link the stability analysis to the potential optimality by characterizing a minimal set of optimal schedule describing two-stages. We refer to Sotskova (2001) for a survey and the literature therein for the know approaches of the earlier results. Up to today, no polynomial time algorithm is known for the decision whether a sequence is irreducible (stable) in the general case. An existence of such an algorithm seems unlikely.

Section 2 describes some basic notions of shop graphs and their properties. Sections 3 and 4 are devoted to the study of potential optimal solution sets of different cardinalities. Section 5 gives a decomposition of sequences. A number of new results are contained in Sections 3 and 5. All other sections present a survey of the existing literature. Conclusions are contained at the end.

2. BASIC CONCEPTS

Instead of the disjunctive graph model (Sussman, 1972) and the polyhedral approach, we use the block-matrices model introduced by Bräsel (1990) which is equivalent to the disjunctive graph model. In the block-matrices model all graph theoretical structures of shop problems are basically described by means of special latin rectangles, also called sequences. In this section, we describe the easy block-matrices model in compact form, give definitions and state basic properties of irreducibility, similarity and isomorphism to be used here. We use notation well understood in the block-matrices model and irreducibility theory in shop problems (e.g., Bräsel et al. (1999a, 1999b), Dhamala (2002b)). The $n \times m$ matrices of all machine orders and job orders are denoted by MO and JO, respectively. Clearly, $mo_{ij} = k$ ($jo_{ij} = k$) means that o_{ij} is the k-th operation of job i (machine j) in the machine (job) order. For any pair (MO, JO), we recall the shop graph $GMO, JO = (SIJ, E_{MO,JO})$ where the arc set reflects the union of all machine orders and all job orders (Dhamala (2002b)). A shop graph is known as a sequence graph (non-sequence graph) if it is acyclic (cyclic). Note that the sequence graph is an acyclic orientation of the disjunctive graph. The decision problem whether a given connected digraph is a sequence graph is efficiently solved by Bräsel et al. (2001) and Harborth (1999). Bräsel and Dhamala (2001a), and Dhamala (2002a) present an efficient algorithm to decide whether a given digraph is a shop graph. Both algorithms have time complexity $O(\max\{mn^2, m^2n\})$.

A latin rectangle $LR[n, m, q] = [l_{ij}]$ is a matrix of size $n \times m$ with $l_{ij} \in \{1, 2, \dots, q\}$ such that each integer of the symbol set occurs at most once in each row and in each column of LR. If $n = m = q$ holds, then the matrix is a latin square $LS[n]$ (see Dhamala (2002b)). For each sequence graph $G_{MO,JO}$ we can describe the sequence (MO, JO) by a special latin rectangle $A = [a_{ij}]$, where $a_{ij} = rank(o_{ij})$, with *sequence property* - for each integer $a_{ij} > 1$ there exists $a_{ij} - 1$ in row i or in column j. The rank of a vertex o_{ij} is the number of vertices on a longest path from a source to this vertex in the sequence graph. Thus, Bräsel (1990) establishes a one-to-one correspondence between the sets of all sequences and all sequence graphs for the open shop. This transformation of an individual element can be performed in $O(nm)$ time (Bräsel, 1990). In this correspondence, an arc from o_{ij} to o_{kl} exists if and only if $i = k$ or $j = l$ is satisfied and $a_{ij} < a_{kl}$ holds. The terms used in the sequence graph are used in the corresponding sequence as well.

For empty β field, a sequence (non-sequence) is feasible (infeasible) w.r.t. a given shop. We denote these solution sets by $S^{nm}(\alpha)$ and $\underline{S}^{nm}(\alpha)$, respectively. The main difficulty on the complexity of these shop problems lies to the construction of appropriate sequence because determining the associated semiactive schedule $C = [c_{ij}]$ for a given sequence A and the matrix $P = [p_{ij}]$, where c_{ij} is the completion time of the operation o_{ij} , is a polynomially solvable problem. A schedule is semiactive if each operation is started as early as possible with respect to the given processing orders. For a regular objective function (i.e., monotonously nondecreasing in each job completion time), one can restrict the investigation to semiactive schedules. Clearly, the quality of the schedule depends on the good structure of a sequence. Note that an infinite set of schedules can be assigned to each sequence. However, we can define an equivalence relation on the set of all schedules decomposing the set into finite number of equivalence classes. In order to find a set of distinct representatives, we may use the semiactive schedules under unit processing times, i.e., all sequences.

We denote the objective value of schedule $C = (A, P)$ corresponding to the sequence A, the matrix of processing times P and objective γ by $\gamma(A)$. The set of all instances of processing times $P = [p_{ij}]$ is denoted by P_{nm}. A sequence $A \in S^{nm}(\alpha)$ is called reducible to $B \in S^{nm}(\alpha)$ if $C_{\max}(B) \le C_{\max}(A)$ for all $P \in P_{nm}$, we write $B \preceq A$. We may exclude any reducible sequence from the sequence set without loosing the potential optimality. In general, an optimal solution of a shop problem is not unique. However, $B \in S^{nm}(\alpha)$ is optimal for $\alpha||C_{\max}$ if $B \preceq A$ for all $A \in S^{nm}(\alpha)$. In the open shop, $C_{\max}(A) = C_{\max}(A_{-1})$ holds for the reversed sequence A_{-1} constructed from A by reversing the orientation of all arcs in the sequence graph G_A, and $B \preceq A$ implies $B \preceq A_{-1}$, $B_{-1} \preceq A$ and $B_{-1} \preceq A_{-1}$. A sequence $A \in S^{nm}(\alpha)$ is called strongly reducible to $B \in S^{nm}(\alpha)$, denoted by $B \prec A$, if $B \preceq A$ but not $A \preceq B$.

Two sequences $A, B \in S^{nm}(\alpha)$ are called similar, denoted by $A \simeq B$ if $B \preceq A$ and $A \preceq B$. A sequence $A \in$

$S^{nm}(\alpha)$ is called irreducible if there exists no other non-similar $B \in S^{nm}(\alpha)$ to which A can be reduced. The set of all irreducible sequences is denoted by $S_I^{nm}(\alpha)$. The irreducible elements are the minimal sequences with respect to the partial order $\prec$ and hence are locally optimal. This relation drastically reduces the set of all sequences which must be considered. The similarity relation $\simeq$ is an equivalence relation on $S^{nm}(\alpha)$ decomposing the set into disjoint equivalence classes. The set of all pairwise non-similar irreducible sequences $S_{I_*}^{nm}(\alpha)$ is potentially optimal for $O||C_{\max}$. Clearly, $|S_{I_*}^{nm}(\alpha)| \leq \frac{1}{2}|S_I^{nm}(\alpha)|$ for $O||C_{\max}$ since a sequence is similar to its reversed sequence.

A path w_A with vertex set $V(w_A)$ in the sequence A (equivalently in the corresponding sequence graph G_A) is called maximal if there does not exist another path w_A^* with $V(w_A) \subset V(w_A^*)$. Note that there exists an exponential number of maximal paths in a sequence. Denoting the set of all maximal paths in A by W_A, one of the paths in W_A becomes the longest depending on the processing times.

$B \preceq A$ if and only if for all $w_B \in W_B$, there exists $w_A \in W_A$ such that $V(w_B) \subseteq V(w_A)$. If $B \prec A$, then there exists $w_B \in W_B$ such that $V(w_B) \subset V(w_A)$ for some $w_A \in W_A$. Note that $B \prec A$ does not necessarily imply $C_{\max}(B) < C_{\max}(A)$ for arbitrary p_{ij}. The strict inequality remains true if there exists a unique maximal path in A. Consider the matrix $P = [p_{ij}]$ with $p_{ij} \in Z_+$ such that $p_{ij} = \begin{cases} k_o & \text{if } o_{ij} \in V(w_A), \\ 1 & \text{otherwise,} \end{cases}$ where $k_o > nm$, then $C_{\max}(A) > C_{\max}(B)$ whenever $A \succ B$. For further explanation of this concept of path structure, we consider the following matrices.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 3 \end{pmatrix},\ B = \begin{pmatrix} 4 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$
$$C = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ D = \begin{pmatrix} 1 & 1 & 10 \\ 1 & 1 & 1 \end{pmatrix}$$

We have $A \succ B$ (respectively, $B \succ C$) since $\{o_{11}, o_{21}, o_{22}\}$ (respectively, $\{o_{23}, o_{13}, o_{11}\}$) belong to a common path in A (respectively, in B) but not in B (respectively, in C), and whenever certain operations belong to a common path in the latter sequence these operations also belong to a common path in the former. Moreover, $C_{\max}(A) = C_{\max}(B) = 13$ but $C_{\max}(C) = 12$ for given $P = [p_{ij}]$.

An undirected graph $G = (V, E)$ is called a comparability graph if there exists an acyclic orientation E^{tr} of E such that the corresponding digraph $G^{tr} = (V, E^{tr})$ is transitive closure. The comparability graph of a sequence graph G_A is denoted by $[G_A^{tr}]$, where $[G]$ stands for the underlying undirected graph of a digraph G and $G^{tr} = (V, E^{tr})$ denotes the transitive closure of G (Golumbic, 1980).

A sufficient condition for irreducibility has been presented in terms of implication classes (Bräsel et al., 1999a). An arc (o_{ij}, o_{kj}) in A is said to directly imply an arc (o_{ij}, o_{il}) in the same sequence if and only if $\{o_{kj}, o_{il}\} \notin [E_A^{tr}]$. Similarly, we have $(o_{il}, o_{kl})\ \Gamma\ (o_{kj}, o_{kl})$ in sequence A if and only if $\{o_{il}, o_{kj}\} \notin [E_A^{tr}]$. If an edge $\{o_{il}, o_{kj}\} \notin [E_A^{tr}]$, then $\{o_{il}, o_{kj}\} \notin [E_B^{tr}]$ whenever $B \preceq A$. An arc (o_{ij}, o_{uv}) in a sequence A is said to imply an arc (o_{kl}, o_{xy}) in the same sequence, denoted by $(o_{ij}, o_{uv})\ \Gamma^{tr}\ (o_{kl}, o_{xy})$, if there exists a chain of arcs $e_1, e_2, \ldots, e_k$ in A such that $(o_{ij}, o_{uv})\ \Gamma e_1\ \Gamma e_2\ \ldots\ \Gamma e_k\ \Gamma(o_{kl}, o_{xy})$ holds. The relation Γ^{tr} is an equivalence relation partitioning the arc set of sequence graph into disjoint equivalence implication classes in $O(n^2m^2)$ time and space (Golumbic, 1980). We call these classes by sequence implication classes. Recall that a graph is a comparability graph if and only if there is no implication class containing both an arc and its reverse.

Consider a row permutation $\pi_r \in S_n$, a column permutation $\pi_c \in S_m$, a transposition $\Psi \in Z_2$, and a reversion $\Psi \in Z_2$ of a matrix, respectively, where Z_2 is the cyclic group of order two. Two given sequences A and B are called structure isomorphic, graph isomorphic or permutation isomorphic, denoted by $A \cong_s B$, $A \cong_g B$ or $A \cong_p B$, if there exists a mapping such that $(\pi_r, \pi_c, \Phi, \Psi)A = B$, $(\pi_r, \pi_c, \Phi)A = B$ or $(\pi_r, \pi_c)A = B$, respectively. Each of these isomorphism relations yields an equivalence relation decomposing the set of all sequences into disjoint isomorphism classes. Given two $n \times m$ sequences A and B, the isomorphism of A and B is decidable in $O(\min\{mn^2, m^2n\})$ time (see Bräsel et al (2001), Bräsel et al. (1999a), Dhamala (2002b)).

The set of all isomorphisms of the same type under the same formats form a group $\mathcal{G}$, namely the groups, $S_n \times S_m$, $S_n \times S_m \times Z_2$ and $S_n \times S_m \times Z_2 \times Z_2$ for permutation isomorphism, graph isomorphism and structure isomorphism, respectively, where S_t is the symmetric group on t letters. For $n = m$, the order of the group $\mathcal{G}$ is equal to $n!m!$, $n!m!$, and $2n!m!$ according to permutation isomorphism, to graph isomorphism and to structure isomorphism, respectively. For $n = m$, these orders are $n!m!$, $2n!m!$, and $4n!m!$, respectively. For each sequence A the sets $\{\phi \in \mathcal{G} : \phi(A) = A\}$ and $\{\phi(A) : \phi \in \mathcal{G}\}$ are the stabilizer and orbit of the sequence A with the property

$$|\{\phi \in \mathcal{G}: \ \phi(A) = A\}| \times |\{\phi(A): \ \phi \in \mathcal{G}\}| = |\mathcal{G}|.$$

Clearly, the orbits are the isomorphism classes and the elements of the stabilizer of sequence A are the automorphisms of A. Therefore, given a system of distinct representatives SDR for each isomorphism classes the total number of sequences is given by the following class equation

$$|S^{nm}| = \sum_{A \in SDR} |\{\phi(A): \phi \in \mathcal{G}\}| = \sum_{A \in SDR} \frac{|\mathcal{G}|}{|\{\phi \in \mathcal{G}: \phi(A) = A\}|}.$$

Thus, given the number of distinct automorphisms for a sequence A, the cardinality of its isomorphism class can be calculated. Therefore, given a system of distinct representatives for the isomorphism classes, the total number of sequences can be calculated. But the number of distinct automorphisms for a sequence A is not known in general.

The properties of sequence isomorphisms play important roles for the enumeration and classification of shop problems. Given two sequences A and B in the same isomorphism class, one sequence is os-irreducible if and only if the other is os-irreducible (Bräsel et al. (1999b) Harborth (1999)).

3. ON THE FEASIBLE REGION

Obviously, $|S^{nm}(O)| + |\underline{S}^{nm}(O)| = (m!)^n(n!)^m$, and $|S^{nm}(J)| + |\underline{S}^{nm}(J)| = (n!)^m$ hold, whereas each combination is feasible in the flow shop. The block-matrices model intends of counting sequences by the cardinality of latin rectangles with sequence property but the latter problem is also hard (see Dhamala (2002b), Harborth (1999)). Another possibility to count sequences is the chromatic polynomial of the Hamming graph $K_n \times K_m$ which yields an answer, however, the calculation is hard (see Harborth (1999)). A closed formula for this unsolved counting problem is unknown to date and only upper and lower bounds are available in general.

The exact number of sequences for the job shop $J|n = 2|\gamma$ are given by a method of mathematical induction by Akers and Friedman (1955). A logic is extended to the $n \times m$ job shop. Following the characterization by the existence of a cycle among operations, three different rules for detecting non-sequences are presented (e.g., Ashour (1972)). If $OM(k)$ denotes the set of all ordered k-tuples of machines standing in the same order, then $|S^{2m}(J)| = 1 + m + \sum_{k=2}^{m} |OM(K)|$ (Akers and Friedman (1955)). Bräsel and Kleinau (1992a) and Kleinau (1993) give a new approach to this result and extend for $O|n = 2|\gamma$: it holds $|S^{n2}(O)| = n!(n! + \sum_{k=1}^{n} \binom{n}{k} \frac{n!}{k!}$. An estimation of the lower bound is given by exact enumeration for fixed n and m (Bräsel and Kleinau (1992b)). They show that $|S^{nm}(O)| \geq \prod_{k=0}^{n-1} \frac{(m+k)!}{k!} = \prod_{k=0}^{m-1} \frac{(n+k)!}{k!}$.

Each subgraph of $K_n \times K_m$ induced by the vertex set $\{o_{i1j1}, o_{i1j2}, \ldots, o_{ik-1jk}, o_{ikjk}, o_{ikj1}\}$ where $k \in \{2, 3, \ldots, \min\{m, n\}\}$, $i_u \neq i_v$ and $j_u \neq j_v$ for all u, v with $u \neq v$, is known as a fundamental cycle $[C_{2k}]$ of length $2k$ (Bräsel and Dhamala (2001a), Dhamala (2002b)). Each $[C_{2k}]$ alternately contains edges of K_m and K_n. There exists 2^{2k} different orientations of $[C_{2k}]$ but only two of them are fundamental dicycles. Following results hold (Bräsel and Dhamala (2001a), Dhamala (2002b)).

Theorem 1. Each non-sequence graph contains at least one fundamental dicycle.

Theorem 2. Let $MO = LS_1$ and $JO = LS_2$ be in $LS[n]$. Then, a necessary and sufficient condition that the pair (MO, JO) is a sequence is that they are identical.

Lemma 1. Let $k \in \{2, 3, ..., \min\{m, n\}\}$ be fixed. Then, the Hamming graph $K_n \times K_m$ contains $\frac{n!m!}{2k(n-k)!(m-k)!}$ different fundamental cycles $[C_{2k}]$.

Therefore, the graph $K_n \times K_m$ contains $\sum_{k=2}^{\min\{n, m\}} \frac{n!m!}{2k(n-k)!(m-k)!}$ different fundamental cycles.

Theorem 3. The number of non-sequences which contain at least one fixed fundamental dicycle C_{2k} is given by $\frac{(n)!^{m+1}(m)!^{n+1}}{k2^{2k}(n-k)!(m-k)!}$, where $k = 2, 3, \ldots, \min\{m, n\}$.

Proof. Consider the set of all (MO, JO), where $G_{MO,JO}$ has at least the fixed fundamental dicycle C_{2k}, where $k = 2, 3, \ldots, \min\{m, n\}$. By definition, exactly one arc in each machine order of all jobs i, $i = 1, \ldots, k$, and one arc in each job order on all machines j, $j = 1, \ldots, k$, is fixed by the arcs of the cycle. All other arcs of MO and JO can be chosen arbitrarily. Therefore, we get $(\frac{m!}{2})^k (m!)^{n-k} = \frac{(m!)^m}{2^k}$ and $(\frac{n!}{2})^k (n!)^{m-k} = \frac{(n!)^m}{2^k}$ possibilities for MO and JO, respectively. Thus, there are overall $\frac{(m!)^n(n!)^m}{2^{2k}}$ possibilities for (MO, JO). By Lemma 1 we give the number of pairs (MO, JO) which contain at least one fixed fundamental dicycle C_{2k} of length $2k$:

$$\frac{2n!m!}{2k(n-k)!(m-k)!} \cdot \frac{(m!)^n(n!)^m}{2^{2k}} = \frac{(n!)^{m+1}(m!)^{n+1}}{k2^{2k}(n-k)!(m-k)!}$$

By means of the fundamental dicycles, a general formula for the number of all sequences can be developed (Dhamala (2002b)). An interesting question would be: how can be a set of pairs (MO, JO) constructed with exactly given number of prescribed dicycles?

Some efforts have been made to compute all sequences. By implementing an enumeration algorithm, Bräsel and Kleinau (1992b) calculate the exact number of sequences for $n = 2 \wedge (2 \leq m \leq 8)$ and for $n = 3 \wedge (3 \leq m \leq 4)$. Further calculations have been made by Bräsel et al. (1999a) for $n = 3 \wedge (5 \leq m \leq 7)$ and for $n = 4 \wedge (4 \leq m \leq 5)$.

4. SETS OF IRREDUCIBLE SEQUENCES

Gonzalez and Sahni (1976) solve the problem $O2||C_{\max}$ in $O(n)$ time by constructing a certain schedule with $C_{\max} = \max\{\max_j \sum_i p_{ij}, \max_i \sum_j p_{ij}\}$. Bräsel and Kleinau (1996) describe in detail the set of all irreducible sequences and present new algorithm of the same complexity to this problem by means of block-matrices model. This approach obtains the optimal sequence in the set of irreducible sequences. In the case of $n = 2$, for every $A \in S_I^{2m}(O)$ there exists a $k \in \{2, 3, \ldots, m\}$ such that A can be obtained by a permutation of the columns of the following open shop irreducible sequence

$$B_k = \begin{pmatrix} 1 & \ldots & k-1 & k & k+1 & \ldots & m \\ m-k+2 & \ldots & m & 1 & 2 & \ldots & m-k+1 \end{pmatrix}$$

The main idea of their proof is to show that any sequence in this class is irreducible and any sequence not belonging to this class reduces to a sequence belonging to this class. Clearly, the total number of os-irreducible sequences for $O|n = 2|C_{\max}$ is $m!(m-1)$. As the total number of sequences for $O|n = 2|C_{\max}$ is $m!(m! + \sum_{k=1}^{m} \binom{m}{k} \frac{m!}{k!})$, the asymptotic relation $\lim_{m\to\infty} \frac{|S_I^{2m}(O)|}{|S^{2m}(O)|} = 0$ holds.

Further studies deal the problem $O||C_{\max}$ (see Harborth (1999)). Using the definitions of these terms, the decision problem whether a given sequence is reducible, similar or strongly reducible to another given sequence takes exponential time. However, it is solved in $O(n^2m^2)$ time by Bräsel et al. (1999b). They use the algorithms on the transitive closures for graphs which needs $O(n^2m^2)$ time to determine the transitive closure of a sequence graph and to check if $\left[G_B^{tr}\right]$ is a subgraph of $\left[G_A^{tr}\right]$.

Theorem 4 Let $A, B \in S^{nm}(O)$. Then A is similar, reducible or strongly reducible to B for $O||C_{\max}$ if and only if $\left[G_B^{tr}\right] = \left[G_A^{tr}\right]$, $\left[G_B^{tr}\right] \subseteq \left[G_A^{tr}\right]$ or $\left[G_B^{tr}\right] \subset \left[G_A^{tr}\right]$, respectively.

Let E_d be the set of all diagonal arcs in the transitive closure $G_A^{tr} = (SIJ, E_A^{tr})$, and let $E_A = E_A^{tr} \setminus E_d$ be the set difference. Then the graph $G_A = (SIJ, E_A)$ is such that $[G_A] \cong K_n \times K_m$ for any $n \times m$ sequence A. Bräsel et al. (1999b) prove that a sequence A is open shop irreducible if and only if there exists no comparability graph $G_C = (SIJ, E_C)$ such that $[G_A] \subseteq G_C \subset \left[G_A^{tr}\right]$. Assuming that a given graph G contains no comparability graph G_C with $[G_A] \subseteq G_C \subset G$, they present an algorithm of complexity $O(n^2m^2)$ to test whether there is a sequence A with $G = \left[G_A^{tr}\right]$.

Bräsel et al. (1999a) and Harborth (1999) present several sufficient conditions for sequence reducibility which can be tested without computing the transitive closures of the associated sequence graphs. For example, an $n \times m$ sequence $[a_{ij}]$, where $\min\{n, m\} \geq 3$, having an operation o_{ij} with $a_{ij} \geq nm - 2$ is strongly open shop reducible. Likewise, any sequence with o_{ij} such that o_{ij} has at least one successor but none of its successors in row i or column j has a direct predecessor outside row i and column j, respectively, is strongly reducible to some sequence for $O||C_{\max}$. They prove

Theorem 5. Let A be a sequence such that each job $i \in \{1, 2, \ldots, n\}$ is first processed on the same machine $j \in \{1, 2, \ldots, m\}$. Then there exists a sequence $B \in S^{nm}(O)$ such that $B \prec A$ for $O||C_{\max}$.

If we wish to test whether a given sequence can be strongly reduced to another sequence by deleting an operation and reinserting it as a sink or a source, we have to ensure that no new path is created in the latter and at least one path is destroyed in the former sequence. This test can be performed in $O(n^2m^2)$ time and in $O(n^2m^2)$ space, given an $n \times m$ sequence (Bräsel et al., 1999a).

Bräsel et al. (1999a) prove that a sequence with only one sequence implication class is irreducible with respect to the open shop. As an application of this result, Willenius (2000) shows Theorem 6. Thus if $n = m$, each rank minimal sequence is os-irreducible. But in general, there exist rank minimal sequences which are reducible in the open shop (see Section 2 for an example).

Theorem 6. Any complete latin square is open shop irreducible for the makespan objective.

On the other hand, to check the reducibility of a sequence by the reversion of a certain set of arcs in it having more than a single implication class, one has to consider one implication class totally. In this reducibility test, for each subset of the set of sequence implication classes in given sequence, we have to construct a possibly reduced sequence from it by reversing the orientations of all arcs belonging to these sequence implication classes. This test depends on the number of sequence implication classes which may cause exponential cost. In the worst case, if all operations belong to a single path, one has to check all $O(2^{nm^2+mn^2})$ subsets of sequence implication classes (Bräsel et al. (1999a)). The lower ranks do not necessarily imply a less number of implication classes (Bräsel et al. (1999a)).

Experiments show that the set of all latin squares does not guarantee the existence of an optimal solution for $O||C_{\max}$ (Tautenhahn and Willenius (2000)). There exist irreducible sequences that are not rank minimal (Bräsel et al. (1999a)). It still seems interesting to investigate how higher is the density of optimal sequences in the set of rank minimal sequences.

Bräsel et al. (1999b) propose an enumeration algorithm which computes all os-irreducible sequences constructing inclusion minimal comparability graphs by successively inserting diagonal arcs into $[G_A]$. Each sequence in such a set to minimize $C_{\max}$ is similar to exactly one sequence in this class. This algorithm constructs graphs G such that $G = \left[G_A^{tr}\right]$ for some sequence A. Number of diagonal arcs play crucial role in this algorithm. For min$\{n, m\} \geq 2$, a lower bound on the number of diagonal arcs of an $n \times m$ sequence on the complete operation set is $\frac{1}{4}n(n-2)\binom{m}{2}$ (Bräsel et al. (1999b)). Such enumeration strategy can also be used to enumerate js-irreducible sequences considering partially directed comparability graphs.

The complexity of a job shop problem depends on the given machine order matrix. Bräsel et al. (1999b) study the relation between the hardness of a job shop problem in terms of the number of js-irreducible sequences. For example, an MO is of latin rectangle type if no two jobs have the same machine at the same position in their machine orders and it is of near latin rectangle type if it is obtained from a latin rectangle by interchanging two machines in the machine order of the same job. The numerical computations show that these classes of machine order matrices contain comparably a small number of sequences but, on the contrary, contain a large number of irreducible sequences because of which these classes are thought as easier in comparison to the other ones.

Bräsel and Kleinau (1992b) present an insertion method for the enumeration of all sequences. Bräsel et al. (1999a) employ a modified version of this method and present an enumeration algorithm for the set of all os-irreducible sequences for a given format. In their algorithm, a set of non-isomorphic sequences is computed and thereafter tested for irreducibility. One sequence per isomorphic class is sufficient for this purpose as open shop irreducibility with respect to the makespan is invariant within each isomorphic class. For this they make the use of lexicographic minimality and the concepts of isomorphisms and automorphisms. They give the number of all sequences, irreducible sequences and isomorphism classes. Their computational results show that the ratio between the number of irreducible sequences and all sequences decreases with growing n and m. As there exist nonisomorphic irreducible sequences (Bräsel et al. (1999a)), a potentially optimal solution must have the cardinality smaller than the cardinality of all irreducible sequences.

Tautenhahn and Willenius (2000) examine a dominance relation between sets of sequences. They present several necessary and sufficient conditions for a sequence to be dominated by a set of other sequences. They formulate the dominance relation as a mixed integer programming problem. Furthermore, they give sets of unavoidable sequences for small formats. Among seven classes of all 3 × 3 open shop irreducible sequences only three of them are unavoidable in the sense that these together with their reverses are the unique optimal solutions for certain matrices of processing times. This set is the minimal one ensuring of at least one optimal solution for the open shop problem. A sequence of biggest rank five among all $O3|n = 3|C_{\max}$ irreducible sequences belongs to the class of unavoidable sequences. For the problem $O3|n = 2|C_{\max}$, the minimal cardinality of a potentially optimal solution is 3. There are two disjoint potentially optimal solutions of this cardinality.

Willenius (2002) generalizes the concept of irreducibility with respect to some other regular objective functions and arbitrary numerical input data. A sequence $A \in S^{nm}(\alpha)$ is called general-reducible to $B \in S^{nm}(\alpha)$, written as $B \prec_g A$, if $C_i(B) \leq C_i(A)$ for all jobs i and all possible instances of numerical data. If $C_i(B) \leq C_i(A)$ for all jobs and all numerical data, then $\gamma(B) \leq \gamma(A)$ holds for all regular γ. A sequence $A \in S^{nm}(\alpha)$ is called r-reducible to $B \in S^{nm}(\alpha)$, denoted by $B \preceq_r A$, if $C_{\max}(B) \leq C_{\max}(A)$ for all instances of processing times P and release dates $r = [r_i]$. The definitions of strong reducibility, similarity and irreducibility have been extended similarly. Along with a number of results in terms of comparability graphs and precedence relations between operations, Willenius (2002) presents interesting relations between the general- and r-reducibility. For instance, for any A, $B \in S^{nm}(\alpha)$, it holds $B \preceq_r A$ if and only if $B_{-1} \preceq_g A_{-1}$.

5. A DECOMPOSITION OF SEQUENCES

A generalized decomposition on irreducibility is studied by Dhamala (2002b). This gives us a possibility to generate sequences of higher sizes with prescribed characterization on irreducibility of lower sizes. For this, we consider an underlying 2 × 2 open shop by the assignment of an operation to each part. In 2 × 2 open shop, we have 2 irreducible (1 unavoidable) and 12 reducible sequences. We partition an $n \times m$ sequence A called the $A(i, j)$ decomposition with i rows and j columns as $A(i, j) = \left(\begin{array}{c|c} S_{i,j} & S_{i,m-j} \\ \hline S_{n-j,j} & S_{n-j,m-j} \end{array}\right)$. Here, $A(i, j)$ decomposition is a block decomposition where at least one part in $\{S_{i,j}, S_{i,m-j}, S_{n-i,j}, S_{n-i,m-j}\}$ is a block of size no less than 2 × 2. In the cases when any one of the sequences $S_{k,l}$ in $A(i, j)$ represents an 1 × 1 matrix, we represent it by $A(1, 1)$ and we call an operation decomposition. In the $A(1, 1)$ decomposition one of the blocks $S_{k,l}$ contains only one but arbitrary o_{uv}, without loss of generality, assume that the block $S_{1,1} = (o_{11})$. Furthermore, we consider the following three class representatives: $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, and $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ which are the lexicographically minimal in their structure isomorphic classes of the 2 × 2 open shop and build the corresponding $A(i, j)$ decompositions. These decompositions are denoted by type$_1$, type$_2$ and type$_3$, respectively. The following sequences, respectively,

$$\left(\begin{array}{c|ccc} 1 & 4 & 5 & 6 \\ \hline 5 & 1 & 2 & 3 \\ 6 & 2 & 3 & 1 \\ 4 & 3 & 1 & 2 \end{array}\right), \left(\begin{array}{c|ccc} 1 & 2 & 4 & 3 \\ \hline 3 & 4 & 6 & 7 \\ 4 & 6 & 7 & 5 \\ 2 & 7 & 5 & 6 \end{array}\right) \text{ and } \left(\begin{array}{c|ccc} 1 & 2 & 3 & 4 \\ \hline 10 & 3 & 5 & 7 \\ 9 & 6 & 7 & 5 \\ 8 & 7 & 4 & 6 \end{array}\right)$$

represent $A(1, 1)$ decompositions corresponding to the 2×2 sequences of type$_1$, type$_2$ and type$_3$. An $n \times n$ os-irreducible sequence A is said to satisfy the irr-decomposition property if it contains exactly n sinks and there exists no similar sequence $S \neq A_{-1}$ to it. We conjecture that the following result holds even if the second condition of this property is violated. For a proof, we recall that open shop irreducibility with respect to the makespan is invariant within each isomorphic class.

Theorem 7. Consider the $A(1, 1)$ decomposition of type$_1$ in an $n \times n$ sequence A where $S_A = S_{n-1,n-1}$ is irreducible for $O||C_{\max}$. Then, A is irreducible for $O||C_{\max}$ if S_A satisfies the irr-decomposition property.

Proof. It is sufficient to consider $S_{1,1}$, $S_{1,n-1}$ and $S_{n-1,n-1}$ since there are no edges between operations of any two diagonal blocks. Because S^A is already os-irreducible, the sequence A cannot be reduced to any other sequence B of type$_1$ for which $S^A_{n-1,\,n-1} \neq S^B_{n-1,\,n-1}$. Therefore, we consider the cases $S^A_{n-1,\,n-1} \simeq S^B_{n-1,\,n-1}$. If the sequence S^B contains less than $n-1$ sinks, there must be a row u and a column v such that no common sink is contained in them. Then there exists a sink vertex o_{kl} in both sequences A and B such that $\{o_{uv}, o_{kl}\} \in \left[G^{tr}_B\right] \backslash \left[G^{tr}_A\right]$ where o_{uv} is a sink in A.

For any other sequence B different than A with $S^A_{n-1,\,n-1} \simeq S^B_{n-1,\,n-1}$, there exists an edge $\{o_{1v}, o_{1l}\}$ such that $\{o_{1v}, o_{1l}\} \in \left[G^{tr}_B\right] \backslash \left[G^{tr}_A\right]$. If $S^A_{1,\,n-1} \neq S^B_{1,\,n-1}$ but we have $S^A_{n-1,\,n-1} = S^B_{n-1,\,n-1}$, there exists at least one reversion in these permutations that produces at least one path in B that was not in A. To show this, let o_{1v} and o_{1l} be two operations with different orientations in $S^A_{1,\,n-1}$ and $S^B_{1,\,n-1}$, say $(o_{1v}, o_{1l}) \in G^{tr}_A$ and $(o_{1l}, o_{1v}) \in G^{tr}_B$. As there are $n-1$ sinks in $S^A_{n-1,\,n-1}$ (and in $S^B_{n-1,\,n-1}$), there exists a unique operation o_{ul} which is the sink in row u and column l restricted to the corresponding lower right corner block so that $\{o_{ul}, o_{1v}\} \in \left[G^{tr}_B\right] \backslash \left[G^{tr}_A\right]$. If $S^B_{n-1,\,n-1} \neq S^A_{n-1,\,n-1}$, then an existence of $\{o_{ul}, o_{1v}\} \in \left[G^{tr}_B\right] \backslash \left[G^{tr}_A\right]$ follows by the fact that each sink in $S^A_{n-1,\,n-1}$ is a source in $S^B_{n-1,\,n-1}$.

Thus, there exists no sequence $B \in S^{nm}$ of type$_1$ to which A can be reduced. Neither A reduced to any other sequence B not of type$_1$ as there is no edge between the blocks $S_{1,1}$ and $S_{n-1,n-1}$ in A but there must be at least one edge between these corresponding blocks in B.

As a consequence of Theorems 6 and 7, any $A(1, 1)$ decomposition of type$_1$ in an $(n+1) \times (n+1)$ sequence A, where $S_{n,n}$ is a latin square sequence of order n, is os-irreducible for $C_{\max}$. Observe that a_{i1} and a_{1j} in this sequence are the permutations of $\{n+1, \ldots, 2n\}$ for all $i, j \in \{2, 3, \ldots, n+1\}$. Therefore, there exists an os-irreducible sequence of format $(n+1) \times (n+1)$ with maximal rank $2n$ for $O \,||\, C_{\max}$.

Remark that the irr-decomposition property is not a necessary condition for the irreducibility of a sequence for $O \,||\, C_{\max}$. For the following sequences

$$\left(\begin{array}{c|ccc} 1 & 4 & 5 & 6 \\ \hline 7 & 1 & 2 & 5 \\ 5 & 2 & 4 & 3 \\ 6 & 3 & 1 & 4 \end{array}\right) \text{ and } \left(\begin{array}{c|ccc} 1 & 4 & 5 & 6 \\ \hline 6 & 1 & 2 & 5 \\ 5 & 2 & 4 & 3 \\ 7 & 3 & 1 & 4 \end{array}\right)$$

the former is os-irreducible whereas the latter is a reducible one; the latter reduces to the former one. Here, the partition on the lower right corner contains only two sinks shown by bold faced entries. In fact in $S_{3,3}$, operation o_{42} is the sink in column 2 but o_{44} is the sink in row 4 and the irr-decomposition property fails.

Corollary 1. Any $A(i, i)$ decomposition of type$_1$, where all of its 4 partitions are latin squares of order at least 2 is an os-reducible sequence for $C_{\max}$.

Corollary 1 follows from Theorem 6. Any $A(1, 1)$ decompositions of type$_2$ and type$_3$ in an $n \times m$ sequence A is a strongly os-reducible sequence for $C_{\max}$. However, a natural interest arises: do there exist irreducible sequences of type$_2$ and type$_3$ in these block-decompositions? In general, if we consider an $A(i, j)$ decomposition for $i \geq 2$ and $j \geq 2$ with type$_1$, then an example illustrates that there exists a reducible sequence even if all 4 partitions contain the same irreducible sequence. But, it is also possible to construct an os-irreducible sequence by considering the same irreducible sequence as its all 4 partitions. In the following, both considered sequences A and B are os-irreducible for the makespan:

$$A = \begin{pmatrix} 2 & 6 & 7 & 8 \\ 1 & 5 & 6 & 7 \\ 3 & 4 & 5 & 1 \\ 4 & 7 & 1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 6 & 7 & 8 \\ 2 & 5 & 6 & 7 \\ 3 & 4 & 5 & 1 \\ 4 & 7 & 1 & 2 \end{pmatrix}$$

But $\left(\begin{array}{c|c} A & A \\ \hline A & A \end{array}\right)$ and $\left(\begin{array}{c|c} B & B \\ \hline B & B \end{array}\right)$ are, respectively, os-irreducible and reducible for $C_{\max}$.

6. CONCLUDING REMARKS

The problem is interesting and important from both theoretical and practical point of view. We studied the structural properties of potentially optimal solutions that are important for shop problems with uncertain processing times. The set of all irreducible sequences is such a solution. The similarity class representatives of irreducible sequences yield a solution of still smaller cardinality. The cardinality of minimal solutions is 3 for $O3|n = 2|C_{\max}$. There exists unique minimal solution for $O3|n = 3|C_{\max}$. But, the existence of unique minimal solution is unlikely in general.

The status of the set of irreducible sequences with respect to the maximum completion time have been studied. No polynomial time algorithm is known for the decision whether a sequence is irreducible though a number of tests have been proposed. Investigations about irreducible sequences are believed to provide a powerful tool to improve exact and heuristic algorithms. Only a very small fraction of all sequences is irreducible. Algorithms on the very small percentage of irreducible sequences among all sequences can have better performances than conventional ones which will affect the computational complexity.

We are interested in developing new constructive and iterative neighborhood structures in this smaller set of irreducible sequences. We presented a decomposition approach on the class of irreducible sequences. This decomposition approach hints a construction of irreducible super sequences based on given irreducible sequences of smaller size (Theorem 7).

In this paper, we proposed an alternate approach of counting sequences by counting infeasible solutions from the whole set. This is an interesting unsolved counting problem. Our approach provides an exact formula for the number of non-sequences which contain at least one fixed fundamental dicycle (Theorem 3). The bounds on the number of sequences will be improved if exact formulas for the number of non-sequences which contain given numbers of fixed fundamental dicycles are proved. This interesting problem still remains open.

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REFERENCES

1. Akers, S.B. and Friedman, J. (1955). A non-numerical approach to production scheduling problems. *Operations Research*, 3: 429-442.
2. Ashour, S. (1972). *Sequencing Theory*, Springer-Verlag, Berlin, NY.
3. Bräsel, H. and Dhamala, T.N. (2002). On some structures of decomposition approaches in shop problems. *Proceedings of the 15-th Workshop on Discrete Optimization*, Wittenberg, Germany.
4. Bräsel, H. and Dhamala, T.N. (2001a). On algebraic properties in shop scheduling problems. Preprint, University of Magdeburg, Germany.
5. Bräsel, H., Harborth, M., and Willenius, P. (2001). Isomorphism for digraphs and sequences of shop scheduling problems. *Journal of Ccombinatorial Mathematics and Combinatorial Computin*, 37: 115-128.
6. Bräsel, H., Harborth, M., Tautenhahn, T., and Willenius, P. (1999a). On the set of solutions of the open shop problem. *Annals of Operations Research*, 92: 241-263.
7. Bräsel, H., Harborth, M., Tautenhahn, T., and Willenius, P. (1999b). On the hardness of the classical job shop problem. *Annals of Operations Research*, 92: 265-279.
8. Bräsel, H., Sotskov, Yu.N. and Werner, F. (1996). Stability of a schedule minimizing mean flow time. *Mathematical and Computer Modelling*, 24: 39-53.
9. Bräsel, H. and Kleinau, M. (1996). New steps in the amazing world of sequences and schedules. *Mathematical Methods of Operations Research*, 43: 195-214.
10. Bräsel, H. and Kleinau, M. (1992a). On number problems for the open shop problems. *System Modeling and Optimization, Proceedings of the 15th IFIP Conference*, Zurich, Switzerland, pp. 145-155.
11. Bräsel, H. and Kleinau, M. (1992b). On the number of feasible solutions of the open shop problem-an application of special latin rectangles. *Optimization*, 23: 251-260.
12. Bräsel, H. (1990). *Latin Rectangle in Scheduling Theory*, Dissertation B, University of Magdeburg, German.
13. Conway, R.W., Maxwell, W.L., and Miller, L.W. (1967). *Theory of Scheduling*, Addison-Wesley, London.
14. Dhamala, T.N. (2002a). Recognition and isomorphism algorithms of shop graphs. *The Nepali Mathematical Sciences Report*, 20: 17-31.
15. Dhamala, T.N. (2002b). *Shop Scheduling Solution-Spaces with Algebraic Characterizations*, PhD Thesis, University of Magdeburg, Shaker Verlag, Germany.
16. Golumbic, M.C. (1980). *Algorithmic Graph Theory and Perfect Graphs*, Academic, NY.
17. Gonzalez, T. and Sahni, S. (1976). Open shop scheduling to minimize finish times. *Journal of the Association of Computing Machinery*, 23: 665-679.
18. Graham, R.E., Lawler, E.L., Lenstra, J.K., and Rinnooy Kan, A.H.G. (1979). Optimization and approximation in deterministic sequencing and scheduling-A survey. *Annals of Discrete Mathematics*, 5: 287-326.
19. Harborth, M. (1999). *Structural Investigation of Shop Scheduling Problems: Number Problems, Potential Optimality and New Enumeration Algorithms*, PhD Thesis, University of Magdeburg, German.
20. Johnson, S.M. (1954). Optimal two- and three-stage production schedules with setup times included. *Naval Research Logistic Quarterly*, 1: 61-68.
21. Kravchenko, S.A., Sotskov, Yu.N. and Werner, F.(1995). Optimal schedules with infinitely large stability radius. *Optimization*, 33: 271-280.
22. Kleinau, M. (1993). *On the Structure of Shop Scheduling Problems: Number Problems, Reducibility and Complexity*. PhD Thesis, University of Magdeburg, German.
23. Lai, T.-C. and Sotskov, Y.N. (1999). Sequencing with uncertain numerical data for makespan minimization. *Journal of the Operations Research Society*, 50: 230-243.
24. Lai, T.-C., Sotskov, YN., Sotskova, N.Yu., and Werner, F. (1997). Optimal makespan scheduling with given bounds of processing times. *Mathematical and Computer Modelling*, 26: 67-86.
25. Lenstra, J.K. and Rinnooy Kan, A.H.G. (1979).

Computational complexity of discrete optimization problems. *Annals of Discrete Mathematics*, 4: 121-140.

26. Lenstra, J.K., Rinnooy Kan, A.H.G., and Brucker, P. (1977). Complexity of machine scheduling problems. *Annals of Discrete Mathematics*, 1: 343-362.
27. Prins, C. (1994). An overview of scheduling problems arising in satellite communications. *Journal of the Operational Research Society*, 40: 611-623.
28. Sotskov, Yu.N., Tanaev, V.S., and Werner, F. (1998). Stability Radius of an Optimal Schedule: A Survey and Recent Developments. In: Gang, Yu.(Ed), *Industrial Applications of Combinatorial Optimization*, Kluwer Academic Publishers, Boston, pp 72-108
29. Sotskov, Y., Sotskova, NY., and Werner, F. (1997). Stability of an optimal schedule in a job shop. *Omega, International Journal of Management Science*, 25: 397-414.
30. Sotskov, Y.N. (1991). Stability of an optimal schedule. *European Journal of Operations Research*, 55: 91-102.
31. Sotskova, N. (2001). *Optimal Scheduling with Uncertainty in the Numerical Data on the Basis of a Stability Analysis*, PhD Thesis, University of Magdeburg, Germany.
32. Sussman, B. (1972). Scheduling problems with interval disjunctions. *Zeitschrift fur Operations Research*, 16: 165-178.
33. Tautenhahn, T. and Willenius, P. (2000). How many sequences solve an open shop problem? Preprint, University of Magdeburg, Germany.
34. Tautenhahn, T. (2002). Irreducible sequences on tree-like operation sets. Preprint, University of Magdeburg, Germany.
35. Willenius, P. (2002). *Irreducibility Theory in Scheduling Theory*, PhD Thesis, University of Magdeburg, Shaker Verlag, German.