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Duality for Mixed-Integer Linear Programs

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Abstract—The theory of duality for linear programs is well-developed and has been successful in advancing both the theory and practice of linear programming. In principle, much of this broad framework can be extended to mixed-integer linear programs, but this has proven difficult, in part because duality theory does not integrate well with current computational practice. This paper surveys what is known about duality for integer programs and offers some minor extensions, with an eye towards developing a more practical framework.

Keywords—Duality, Mixed-integer linear programming, Value function, Branch and cut

1. INTRODUCTION

Duality has long been a central tool in the development of both optimization theory and methodology. The study of duality has led to efficient procedures for computing bounds, is central to our ability to perform post facto solution analysis, is the basis for procedures such as column generation and reduced cost fixing, and has yielded optimality conditions that can be used as the basis for "warm starting" techniques. Such procedures are useful both in cases where the input data are subject to fluctuation after the solution procedure has been initiated and in applications for which the solution of a series of closely-related instances is required. This is the case for a variety of integer optimization algorithms, including decomposition algorithms, parametric and stochastic programming algorithms, multi-objective optimization algorithms, and algorithms for analyzing infeasible mathematical models.

The theory of duality for linear programs (LPs) is well-developed and has been extremely successful in contributing to both theory and practice. By taking advantage of our knowledge of LP duality, it has been possible to develop not only direct solution algorithms for solving LPs but also sophisticated dynamic methods appropriate for large-scale instances. In theory, much of this broad framework can be extended to mixed-integer linear programs (MILPs), but this has proven largely impractical because a duality theory well-integrated with practice has yet to be developed. Not surprisingly, it is difficult to develop a standard dual problem for MILP with properties similar to those observed in the LP case. Such dual problems are generally either not strong or not computationally tractable. Unlike the LP case, dual information is not easy to extract from the most commonly employed primal solution algorithms. In Section 4.7, we discuss the challenges involved in extracting dual information in the case of branch and cut, which is the most commonly employed solution algorithm for MILPs today.

The study of duality for MILPs can be considered to have two main goals: (1) to develop methods for deriving *a priori* lower bounds on the optimal solution value of a specific MILP instance and (2) to develop methods for determining the effect of modifications to the input data on the optimal solution and/or optimal solution value *post facto*. Methods for producing a priori lower bounds are useful primarily as a means of solving the original problem, usually by embedding the bounding procedure into a branch-and-bound algorithm. Such bounding methods have received a great deal of attention in the literature and are well-studied. Methods for producing dual information post facto, on the other hand, are useful for performing sensitivity analyses and for warm starting solution procedures. Such methods have received relatively little attention in the literature. In both cases, the goal is to produce "dual information." Methods of the second type, however, can take advantage of information produced as a by-product of a primal solution algorithm.

The primary goal of this study is to survey previous work on methods of the second type with an eye towards developing a framework for MILP duality that can be integrated with modern computational practice. Computational methods have evolved significantly since most of the work on integer programming duality was done and a close reexamination of this early work is needed. We have attempted to make the paper as general and self-contained as possible by extending known results from the pure integer to the mixed-integer case whenever possible. We have included proofs for as many results as space would allow, concentrating specifically on results whose proofs were not easily accessible or for which we provide a generalization or alternative approach. The proofs for all results not included here can be found in the

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references cited. This survey draws heavily on the foundational work of a small cadré of authors who contributed greatly to the study of MILP duality, including Gomory, Johnson, Wolsey, Blair, and Jeroslow, and is in many respects an updating and expansion of the excellent papers of Wolsey (1981) and Williams (1996).

1.1 Definitions and notation

Before beginning, we briefly introduce some terminology and notation. A *linear program* is the problem of minimizing a linear objective function over a polyhedral feasible region

$$\mathcal{P} = \{x \in \mathbb{R}^n_+ \mid Ax = b\} \tag{1}$$

defined by rational constraint matrix $A \in \mathbb{Q}^{m\times n}$ and right-hand side vector $b \in \mathbb{R}^m$. A *mixed-integer linear program* is an LP with the additional constraint that a specified subset of the variables must take on integer values. For the remainder of the paper, we address the canonical MILP instance specified by the constraints in (1), with the *integer variables* (those required to take on integer values) indexed by the set $I = \{1, ..., r\} \subseteq N = \{1, ..., n\}$ if $r > 0$ (otherwise, $I = \emptyset$). The remaining variables, indexed by the set $C = N\backslash I$, constitute the *continuous variables*. The feasible region is then given by $\mathcal{S} = \mathcal{P} \cap (\mathbb{Z}^r \times \mathbb{R}^{n-r})$ and the MILP instance can be stated as that of determining

$$z_{IP} = \min_{x\in\mathcal{S}} cx \tag{2}$$

for $c \in \mathbb{R}^n$. The rationality of the constraint matrix A is required to ensure the consistency of (2) and guarantees that any such program is either infeasible, unbounded or has a finite optimal value (Meyer (1974)). In the sequel, we refer to this canonical instance of MILP as the *primal problem*.

For a given MILP, we call any $x \in \mathcal{S}$ a *feasible solution* with *solution value* cx and any x^* such that $cx^* = z_{IP}$ an *optimal solution*. Aside from determining the value z_{IP}, the goal of solving (2) is to find such an optimal solution. The LP obtained from a given MILP by removing the integrality requirements on the variables, i.e., setting $I = \emptyset$, is referred to as the associated *LP relaxation*. The associated *pure integer linear program* (PILP), on the other hand, is obtained by requiring *all* variables to be integer, i.e., setting $r = n$. For any index set $K \subseteq N$, A_K is the submatrix consisting of the corresponding columns of A and similarly, y_K is the vector consisting of just the corresponding components of a vector y.

In what follows, we frequently refer to certain classes of functions, defined below.

Definition 1. Let a function f be defined over domain V. Then f is

- subadditive if $f(v_1) + f(v_2) \geq f(v_1 + v_2)$ $\forall$ $v_1, v_2, v_1 + v_2 \in V$.
- linear if V is closed under addition and scalar multiplication and
  i. $f(v_1) + f(v_2) = f(v_1 + v_2)$ $\forall$ $v_1, v_2 \in V$,
  ii. $f(\lambda v) = \lambda f(v)$ $\forall$ $v \in V$, $\forall$ $\lambda \in \mathbb{R}$.
- convex if V and $epi(f) = \{(v, y): v \in V, y \geq f(v)\}$ are convex sets, and
- polyhedral if $epi(f)$ is a polyhedron.

Definition 2. For a given $k \in \mathbb{N}$, let

- $\Lambda^k = \{f \mid f: \mathbb{R}^k \to \mathbb{R}\}$,
- $\mathcal{L}^k = \{f \in \Lambda^k \mid f \text{ is linear}\}$,
- $\mathcal{C}^k = \{f \in \Lambda^k \mid f \text{ is convex}\}$,
- $\mathcal{F}^k = \{f \in \Lambda^k \mid f \text{ is subadditive}\}$.

The notation $\lceil \lambda \rceil$ for a scalar λ is used to denote the smallest integer greater than or equal to λ. Similarly, we let $\lfloor \lambda \rfloor = -\lceil -\lambda \rceil$. For a function $f \in \Lambda^k$, $\lceil f \rceil$ is the function defined by $\lceil f \rceil(d) = \lceil f(d) \rceil$ $\forall d \in \mathbb{R}^k$. Finally, the l_1 norm of a vector $x = (x_1, ..., x_n)$ is denoted by $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.

1.2 Outline of the paper

The outline of the paper is as follows. In Section 2, we introduce several notions of MILP duality that have appeared in the literature and discuss the relationships between them. In Section 3, we discuss in more detail the well-known *subadditive dual*, which yields a generalization of many duality results from linear programming, but does not integrate easily with current computational practice. In Section 4, we discuss methods for obtaining *dual functions* that provide a lower bound on the objective function value of MILP instances in the neighborhood of a given base instance and can be seen as solutions to certain dual problems we present in Sections 2 and 3. Finally, in Section 5, we discuss future research in this area and indicate how some of the theory presented in this paper may be put into practice.

2. GENERAL FRAMEWORK

A common approach to obtaining an a priori lower bound for a single MILP instance is to construct an optimization problem of the form

$$z_D = \max_{v\in V} f(v), \tag{3}$$

with objective function $f : V \to \mathbb{R}$ and feasible region $V \subseteq \mathbb{R}^k$ for $k \in \mathbb{N}$ such that $z_D \leq z_{IP}$. Such a problem is called a *dual problem* and is a *strong dual problem* if $z_D = z_{IP}$. For any pair (f, V) that comprises a dual problem and any $v \in V$, $f(v)$ is a valid lower bound on z_{IP} and the dual problem is that of finding the best such bound. The usefulness of such a dual problem may be rather limited, since we require only that it provides a valid bound for the single MILP instance being analyzed and since the pair f

and V are not necessarily selected according to any criterion for goodness. A number of methods for producing such dual problems directly from primal input data are already known and include both the dual of the LP relaxation and the Lagrangian dual (see Section 4.5), which are generally easy to construct and solve. The bounds produced by such methods are useful in helping to solve the primal problem, but must necessarily be weak on their own, since computing the exact optimal solution value z_{IP} of the MILP (2) is an *NP*-hard optimization problem in general.

Conceptually, one avenue for improvement in the bound yielded by (3) is to let the dual problem itself vary and try to choose the "best" among the possible alternatives. This leads to the generalized dual

$$z_D = \max_{f,V} \max_{v \in V} f(v), \tag{4}$$

where each pair (f, V) considered in the above maximization is required to comprise a dual problem of the form (3). This dual may yield an improved bound, but it is not clear how to obtain an optimal solution over any reasonable class of dual problems and even less clear how such a solution might help determine the effect of perturbations of the primal problem.

A second avenue for improvement is to focus not just on producing a lower bound valid for a single instance, but on constructing a *dual function* that can produce valid bounds across a range of instances within a neighborhood of a given base instance. Such dual functions may be obtained as a by-product of primal solution algorithms and are needed for effective post facto analysis. In what follows, we generally refer to any function that takes as input an infinite family of MILP instances and returns as output a valid lower bound on each instance as a "dual function." Such a function is considered strong with respect to a given primal problem if the bound returned for that particular instance is z_{IP}.

Because the right-hand side can be thought of as describing the level of resources available within the system being optimized, it is natural to consider the question of how the optimal solution value of a MILP changes as a function of the right-hand side. The *value function* of a MILP is a function that returns the optimal solution value for any given right-hand side, i.e., it is a function $z: \mathbb{R}^m \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$z(d) = \min_{x \in \mathcal{S}(d)} cx, \tag{5}$$

where $\mathcal{S}(d) = \{x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+ \mid Ax = d\}$ for $d \in \mathbb{R}^m$. By convention, we define $z(d) = \infty$ if $d \notin \Omega$, where $\Omega = \{d \in \mathbb{R}^m \mid \mathcal{S}(d) \neq \emptyset\}$. As we discuss below, the value function plays a central role in classical duality theory, but computing it is generally difficult even though it has a closed form. We consider properties of the value function and its structure in more detail in Section 4.1.

In the remainder of the paper, we refer to the following running example.

Example 1. Consider the following MILP instance with right-hand side b:

$$\begin{aligned} z_{IP} = \min \ & \frac{1}{2}x_1 + 2x_3 + x_4 \\ \text{s.t.} \ & x_1 - \frac{3}{2}x_2 + x_3 - x_4 = b \ \text{ and} \\ & x_1,\ x_2 \in \mathbb{Z}_+,\ x_3,\ x_4 \in \mathbb{R}_+. \end{aligned} \tag{6}$$

In this case, the value function (pictured in Figure 1) can be represented explicitly in the form:

$$z(d) = \begin{cases} \vdots & \vdots \\ -d - \frac{3}{2}, & -\frac{5}{2} < d \le -\frac{3}{2} \\ 2d + 3, & -\frac{3}{2} < d \le -1 \\ -d, & -1 < d \le 0 \\ 2d, & 0 < d \le \frac{1}{2} \\ -d + \frac{3}{2}, & \frac{1}{2} < d \le 1 \\ 2d - \frac{3}{2}, & 1 < d \le \frac{3}{2} \\ \vdots & \vdots \end{cases} \tag{7}$$

By considering what optimal solutions to this simple MILP instance look like as the right-hand side is varied, we can get an intuitive feel for why the value function has the shape that it does in this example.

Note that the slope near zero is exactly the objective function coefficients of the continuous variables, since these are the only variables that can have positive value for values of d near zero. Furthermore, the gradient of the function alternates between these two slope values moving away from zero in both directions, as the continuous variables alternate in the role of ensuring that the fractional part of the left-hand side is consistent with that of d. The coefficients of the integer variables, on the other hand, determine the breakpoints between the linear pieces. ■

Although it is generally difficult to construct the value function itself, it is much easier to obtain an approximation that bounds the value function from below, i.e., a function F: $\mathbb{R}^m \to \mathbb{R}$ that satisfies $F(d) \le z(d)$ for all $d \in \mathbb{R}^m$. Given that we can do this, the question arises exactly how to select such a function from among the possible alternatives. A sensible method is to choose one that provides the best bound for the current right-hand side b. This results in the dual

$$z_D = \max\{F(b) : F(d) \le z(d), d \in \mathbb{R}^m, F \in \Upsilon^m\}, \tag{8}$$

where $\Upsilon^m \subseteq \Lambda^m$ and the infinite family of constraints ensures that we only consider dual functions that yield a valid bound for any right-hand side.

Note that if the primal problem has a finite optimal value and $\Upsilon^m \equiv \Lambda^m$, (8) always has a solution F^* that is a strong dual function by setting $F^*(d) = z(d)$ when $d \in \Omega$, and $F^*(d) = 0$ elsewhere. In this case, it also follows that a dual function is optimal to (8) if and only if it bounds the value function from below and agrees with the value function at b. This means that not all optimal solutions to (8) provide the same bound for a given vector d. In fact, there are optimal solutions to this dual that provide arbitrarily poor estimates of the value function for right-hand sides $d \neq b$, even when d is in a local neighborhood of b. It is thus an open question whether (8) in fact provides the best criterion for selecting a dual function or whether it is possible to compute a dual function guaranteed to produce "reasonable" bounds within a specified neighborhood of b.

Consider the value function of the LP relaxation of the MILP (2), given by

$$F_{LP}(d) = \max\{vd : vA \leq c, v \in \mathbb{R}^m\}. \tag{9}$$

Let F be defined by $F(d) = F_{LP}(d)$ $\forall d \in \Omega$ and $F(d) = 0$ elsewhere. Then F is feasible to (8) if $F \in \Upsilon^m$ and the primal problem is bounded, since linear programming duality tells us that $F_{LP}(d) \leq z(d)$ for all $d \in \Omega$. The following example shows the result of computing this dual function for the MILP instance from Example 1.

Example 2. Consider the value function of the LP relaxation of problem (6),

$$\begin{aligned} F_{LP}(d) = \max\ & vd \\ \text{s.t } & 0 \leq v \leq \frac{1}{2}, \text{ and} \\ & v \in \mathbb{R}, \end{aligned} \tag{10}$$

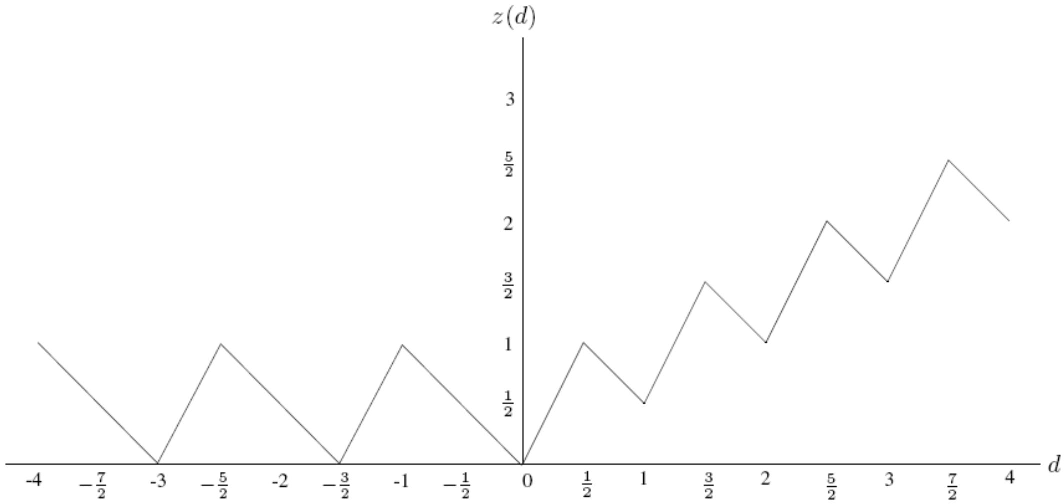

Figure 1. Value function of MILP from Example 1.

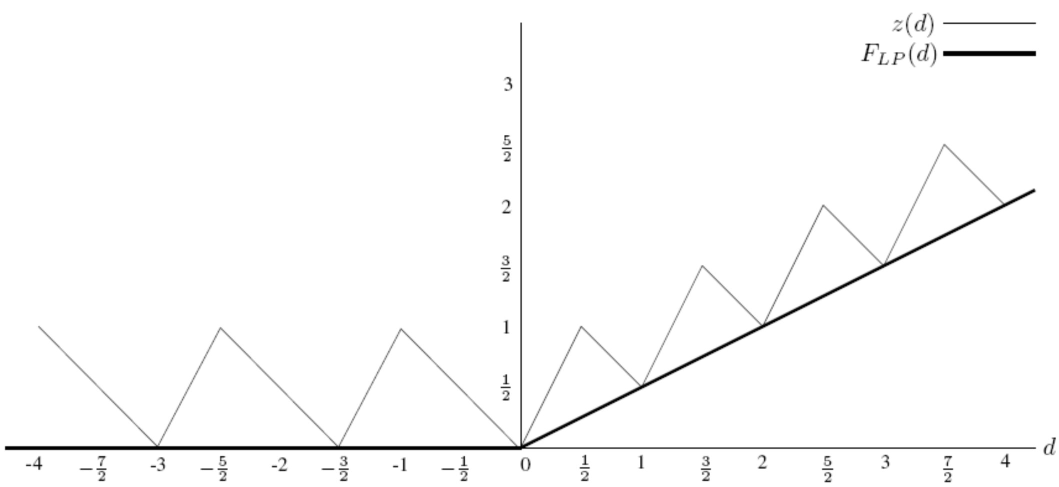

Figure 2. Value function of LP relaxation of problem (6).

which can be written explicitly as

$$F_{LP}(d) = \begin{cases} 0, & d \le 0 \\ \frac{1}{2}d, & d > 0 \end{cases}. \tag{11}$$

This dual function is shown in Figure 2, along with the value function of the original MILP. In this example, F_{LP} can be seen to be the best piecewise-linear, convex function bounding z from below. ■

By considering that

$$\begin{aligned} F(d) \le z(d),\ d \in \mathbb{R}^m &\Leftrightarrow F(d) \le cx,\ x \in \mathcal{S}(d),\ d \in \mathbb{R}^m \\ &\Leftrightarrow F(Ax) \le cx,\ x \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r}, \end{aligned} \tag{12}$$

we see that the dual problem (8) can be rewritten as

$$z_D = \max\ \{F(b) : F(Ax) \le cx, x \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r}, F \in \Upsilon^m\}. \tag{13}$$

In the next section, we will use this equivalence to derive a simpler form of (8) in the case when Υ^m is restricted to a particular class of subadditive functions.

3. THE SUBADDITIVE DUAL

As stated, the dual (8) is rather general and perhaps only of theoretical interest. A natural question is whether it is possible to restrict the class of functions considered in (8) in some reasonable way. Both linear and convex functions are natural candidates for consideration. If we take $\Upsilon^m \equiv \mathcal{L}^m$, then (8) reduces to $z_D = \max\{vb \mid vA \le c, v \in \mathbb{R}^m\}$, which is the dual of the continuous relaxation of the original MILP discussed earlier. Hence, this restriction results in a dual that is no longer guaranteed to produce a strong dual function (see Figure 2). Jeroslow (1979) showed that the optimum z_D obtained by setting $\Upsilon^m \equiv \mathcal{C}^m$ also results in the same optimal solution value obtained in the linear case.

In a series of papers, Johnson (1973, 1974, 1979) and later Jeroslow (1978) developed the idea of restricting Υ^m to a certain subset of $\mathcal{F}^m$ (subadditive functions). The subadditive functions are a superset of the linear functions that retain the intuitively pleasing property of "no increasing returns to scale" associated with linear functions. A strong motivation for considering this class of functions is that the value function itself is subadditive over the domain Ω and can always be extended to a subadditive function on all of $\mathbb{R}^m$ (see Theorem 5). This means that this restriction does not reduce the strength of the dual (8). To see why the value function is subadditive, let $d_1, d_2 \in \Omega$ and suppose $z(d_i) = cx_i$ for some $x_i \in \mathcal{S}(d_i)$, $i = 1, 2$. Then, $x_1 + x_2 \in \mathcal{S}(d_1 + d_2)$ and hence $z(d_1) + z(d_2) = c(x_1 + x_2) \ge z(d_1 + d_2)$.

If $\Upsilon^m \equiv \Gamma^m \equiv \{F \in \mathcal{F}^m \mid F(0) = 0\}$, then we can rewrite (8) as the *subadditive dual*

$$\begin{aligned} z_D = \max\ & F(b) \\ & F(a^j) \le c_j \quad \forall j \in I, \\ & \bar{F}(a^j) \le c_j \quad \forall j \in C, \text{and} \\ & F \in \Gamma^m, \end{aligned} \tag{14}$$

where a^j is the j^{th} column of A and the function $\bar{F}$ is defined by

$$\bar{F}(d) = \limsup_{\delta \to 0+} \frac{F(\delta d)}{\delta} \quad \forall d \in \mathbb{R}^m. \tag{15}$$

Here, $\bar{F}$, first used by Gomory and Johnson (1972) in the context of cutting plane algorithms, is the *upper d-directional derivative* of F at zero. The next result reveals the relation between F and $\bar{F}$.

Theorem 1. (Johnson (1974), Jeroslow (1978), Nemhauser and Wolsey (1988)) If $F \in \Gamma^m$, then $\forall d \in \mathbb{R}^m$ with $\bar{F}(d) < \infty$ and $\lambda \ge 0$, $F(\lambda d) \le \lambda \bar{F}(d)$.

Proof. Let $\lambda > 0$ and $\mu > 0$. Setting $q = \mu - \lfloor \mu \rfloor$, we have

$$\begin{aligned} F(\lambda d) &= F\left(\frac{\mu\lambda d}{\mu}\right) = F\left(\frac{\lfloor \mu \rfloor \lambda d}{\mu} + \frac{q\lambda d}{\mu}\right) \\ &\le \lfloor \mu \rfloor F\left(\frac{\lambda d}{\mu}\right) + F\left(\frac{q\lambda d}{\mu}\right) \\ &= \mu F\left(\frac{\lambda d}{\mu}\right) + F\left(\frac{q\lambda d}{\mu}\right) - qF\left(\frac{\lambda d}{\mu}\right), \end{aligned}$$

where the inequality follows from the fact that $F \in \Gamma^m$. Now, letting $\delta = \frac{1}{\mu}$, we get

$$F(\lambda d) \le \frac{F(\delta\lambda d)}{\delta} + q\delta\left(\frac{F(q\delta\lambda d)}{q\delta} - \frac{F(\delta\lambda d)}{\delta}\right). \tag{16}$$

By taking the limit as $\delta \to 0^+$, we obtain

$$F(\lambda d) \le \bar{F}(\lambda d). \tag{17}$$

Finally, note that

$$\begin{aligned} \bar{F}(\lambda d) &= \limsup_{\delta \to 0+} \frac{F(\delta(\lambda d))}{\delta} \\ &= \limsup_{\delta\lambda \to 0+} \frac{\lambda F(\delta\lambda d)}{\delta\lambda} = \lambda \bar{F}(d). \end{aligned} \tag{18}$$

■

Example 3. Consider the *d*-directional derivative of the value function for the MILP (6), shown in Figure 3:

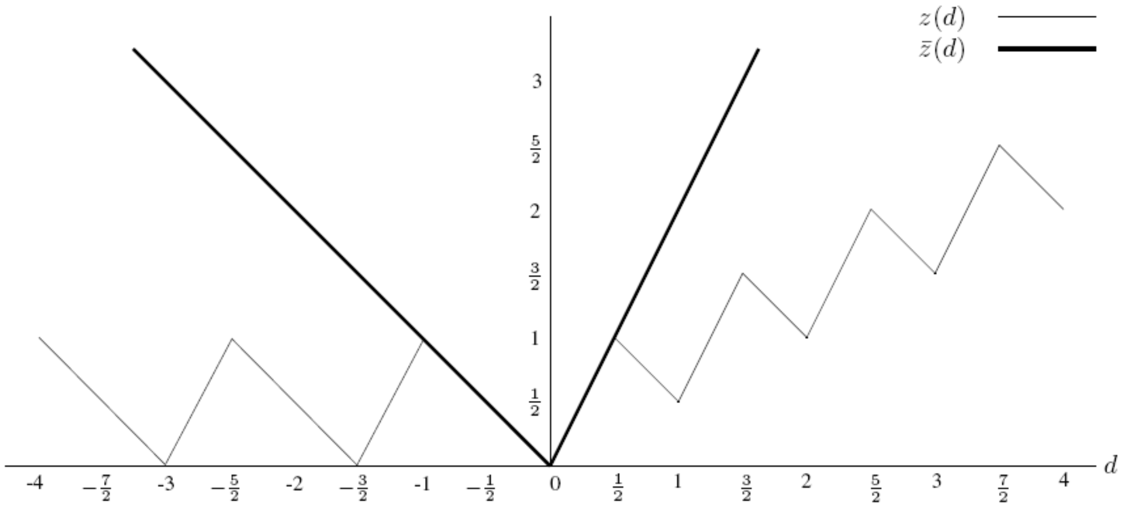

Figure 3. Directional derivative of the value function of problem (6).

$$\bar{z}(d) = \begin{cases} -d, d \le 0 \\ 2d, d > 0 \end{cases}. \tag{19}$$

Note that $\bar{z}$ is a piecewise linear convex function whose directional derivatives near the origin coincide with that of z. As we pointed out in Example 1, these directional derivatives are determined by the coefficients of the continuous variables in (6). ■

The use of $\overline{F}$ is required in (14) due to the presence of the continuous variables and is not needed for pure integer programs. Intuitively, the role of $\overline{F}$ is to ensure that solutions to (14) have gradients that do not exceed those of the value function near zero, since the subadditivity of F alone is not enough to ensure this in the case of MILP. We now show formally that (14) is in fact a valid dual problem.

Theorem 2. (Weak Duality by Jeroslow (1978, 1979)) Let x be a feasible solution to the MILP (2) and let F be a feasible solution to the subadditive dual (14). Then, $F(b) \le cx$.

Proof. Let x be a given feasible solution to the MILP (2). Note that

$$\begin{aligned} F(b) &= F(Ax) \\ &\le F(\sum_{j\in I} a^j x_j) + F(\sum_{j\in C} a^j x_j) \\ &\le \sum_{j\in I} F(a^j)x_j + \sum_{j\in C} \overline{F}(a^j)x_j \\ &\le cx. \end{aligned}$$

The first inequality follows from the subadditivity of F. Next, $F(\sum_{j\in I} a^j x_j) \le \sum_{j\in I} F(a^j)x_j$, since F is subadditive, F(0) = 0 and $x_j \in \mathbb{Z}_+$, $j \in I$. Similarly, $F(\sum_{j\in C} a^j x_j) \le \sum_{j\in C} F(a^j x_j) \le \sum_{j\in C} \overline{F}(a^j)x_j$, since $\overline{F}(0) = 0$ and $F(a^j x_j) \le \overline{F}(a^j)x_j$, $x_j \in \mathbb{R}_+$, $j \in C$ by Theorem 1. Therefore, the second inequality holds. For the last inequality, $F(a^j) \le c_j$, $j \in I$ and $\overline{F}(a^j) \le c_j, j \in C$ by dual feasibility and x_j is nonnegative for all $j \in N$ by primal feasibility. ■

Example 4. For the MILP (6), the subadditive dual problem is

$$\begin{aligned} \max \quad & F(b) \\ & F(1) \le \frac{1}{2} \\ & F(-\tfrac{3}{2}) \le 0 \\ & \overline{F}(1) \le 2 \\ & \overline{F}(-1) \le 1 \\ & F \in \Gamma^1. \end{aligned} \tag{20}$$

As described above, the last two constraints require that the slope of F going away from the origin (the d-directional derivative) be less than or equal to that of the value function, whereas the first two constraints require that $F(1)$ and $F(-\frac{3}{2})$ not exceed $z(1)$ and $z(-\frac{3}{2})$, respectively. Note that in this example, the constraint $\overline{F}(-1) \le 1$ is actually equivalent to the constraint $F(-1) \le 1$, but replacing $\overline{F}(1) \le 2$ with $F(1) \le 2$ results in the admission of invalid dual functions.

If we require integrality of all variables in (6), then the value function becomes that shown in Figure 4, defined only at discrete values of the right-hand side d. In this case, $\overline{F}$ is replaced by F in (20) and the third constraint becomes redundant. This can be seen by the fact that x_3 cannot take on a positive value in any optimal solution to the pure integer restriction of (6). ■

Although the value function itself yields an optimal solution to the subadditive dual of any given MILP, irrespective of the value of the original right-hand side b, the set of all dual functions that are optimal to (14) can be affected dramatically by the initial value of b considered. This is because F is required to agree with the value function only at b and nowhere else. In the following example, we consider optimal solutions to (20) for different values of b.

Example 5. Consider optimal solutions to (14) for the MILP (6) for different values of b.

1. $F_1(d) = \frac{d}{2}$ is an optimal dual function for $b \in \{0, 1, 2, ...\}$ (see Figure 2),
2. $F_2(d) = 0$ is an optimal dual function for $b \in \{ ..., -3, -\frac{3}{2}, 0\}$ (see Figure 2).
3. $F_3(d) = \max\left\{\frac{1}{2}\left\lceil d - \frac{\lceil\lceil d\rceil - d\rceil}{4}\right\rceil,\ 2d - \frac{3}{2}\left\lceil d - \frac{\lceil\lceil d\rceil - d\rceil}{4}\right\rceil\right\}$ is an optimal dual function for $b \in \{[0, \frac{1}{4}] \cup [1, \frac{5}{4}] \cup [2, \frac{9}{4}] \cup ...\}$ (see Figure 5).
4. $F_4(d) = \max\left\{\frac{3}{2}\left\lceil \frac{2d}{3} - \frac{2}{3}\left\lceil\left\lceil \frac{2d}{3}\right\rceil - \frac{2d}{3}\right\rceil\right\rceil - d,\ -\frac{3}{4}\left\lceil \frac{2d}{3} - \frac{2}{3}\left\lceil\left\lceil \frac{2d}{3}\right\rceil - \frac{2d}{3}\right\rceil\right\rceil + \frac{d}{2}\right\}$ is an optimal dual function for $b \in \{ \ldots \cup [-\frac{7}{2}, -3] \cup [-2, -\frac{3}{2}] \cup [-\frac{1}{2}, 0]\}$ (see Figure 5). ■

As in LP duality, weak duality yields results concerning the relationship between primal and dual when no finite optimum exists. Before proving the main corollary, we need the following important lemma.

Lemma 3. For the MILP (2) and its subadditive dual (14), the following statements hold:

1. The primal problem is unbounded if and only if $b \in \Omega$ and $z(0) < 0$.
2. The dual problem is infeasible if and only if $z(0) < 0$.

Proof. First, note that $0 \in \Omega$ and $z(0) \leq 0$, since $x = 0$ is a feasible solution to the MILP (2) with right-hand side 0.

1. If $b \in \Omega$ and $z(0) < 0$, then there exist $\bar{x} \in \mathcal{S}$ and $\hat{x} \in \mathcal{S}(0)$ with $c\hat{x} < 0$. Then $\bar{x} + \lambda\hat{x} \in \mathcal{S}$ for all $\lambda \in \mathbb{R}_+$ and it follows that λ can be chosen to make $z(b)$ arbitrarily small. Conversely, if $b \in \Omega$ and $z(0) = 0$, then we must also have that $\min\{cx \mid Ax = 0, x \in \mathbb{R}^n_+\} = 0$. Otherwise, there must exist an $\hat{x} \in \mathbb{Q}^n_+$ for which $A\hat{x} = 0$ and $c\hat{x} < 0$, which can be scaled to yield an integer solution to (2) with right-hand side 0, contradicting the assumption that $z(0) = 0$. Since no such vector exists, the LP relaxation of (2), and hence the MILP itself, must be bounded.
2. If $z(0) = 0$, then $\min\{cx \mid Ax = 0, x \in \mathbb{R}^n_+\} = \max\{v0: vA \leq c, v \in \mathbb{R}^m\} = 0$ (see the proof of part 1 above) and therefore, (14) is feasible by setting $F(d) = v^*d \ \forall d \in \mathbb{R}^m$, where v^* is the optimal dual solution. This implies that if the dual is infeasible, then $z(0) < 0$. If $z(0) < 0$, on the other hand, the dual cannot be feasible since any feasible solution F has to satisfy $F(0) = 0$ and this would contradict weak duality. ■

Corollary 4. For the MILP (2) and its subadditive dual (14),

1. If the primal problem (resp., the dual) is unbounded then the dual problem (resp., the primal) is infeasible.
2. If the primal problem (resp., the dual) is infeasible, then the dual problem (resp., the primal) is infeasible or unbounded.

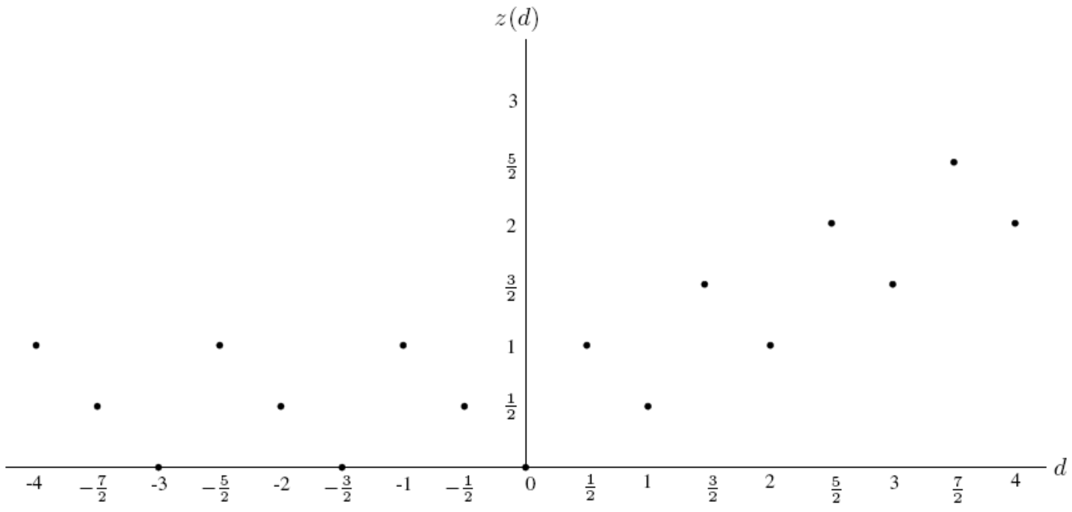

Figure 4. Value function of problem (6) with all variables integer.

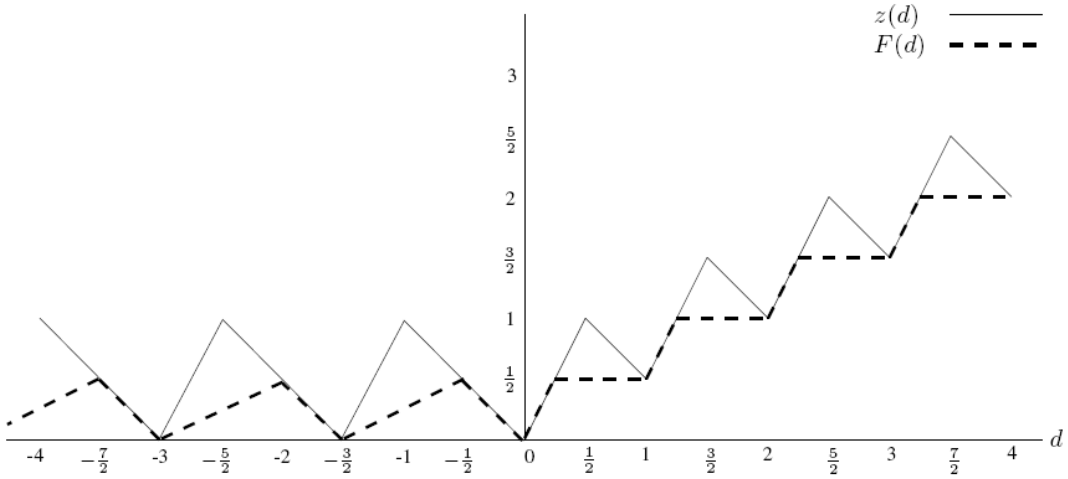

Figure 5. Observe that $F(d) = \max\{F_3(d), F_4(d)\}$ is an optimal dual function for (20) for some values of b and only feasible otherwise.

Proof.

1. This part follows directly from Theorem 2.
2. Assume that the primal problem is infeasible. Then there are two cases. If $z(0) < 0$, the dual is infeasible by Lemma 3. Otherwise, by LP duality, $\exists\ v \in \mathbb{R}^m$ with $vA \leq c$. Let $F_1 \in \Gamma^m$ be defined by $F_1(d) = vd$, $\forall d \in \mathbb{R}^m$. Note that $\overline{F}_1 = F_1$. Next, consider the value function $F_2(d) = \min\{x_0\text{: } Ax + x_0 d = d,\ x \in \mathbb{Z}^r_+ \times \mathbb{R}^{n-r}_+,\ x_0 \in \mathbb{Z}_+\}$. F_2 is defined and finite for all $d \in \mathbb{R}^m$ since $x = 0$ with $x_0 = 1$ is a feasible solution for any right-hand side. Therefore, $F_2 \in \Gamma^m$. Furthermore, for any $j \in I$, $F_2(a^j) \leq 0$, since e^j (the j^{th} unit vector) together with $x_0 = 0$ is a feasible solution to the corresponding problem. On the other hand, for any $j \in C$ and $\delta > 0$, $F_2(\delta a^j) \leq 0$ due to the fact that $x = \delta e^j$ and $x_0 = 0$ is feasible. Thus, $\overline{F}_2(a^j) \leq 0$, $\forall\ j \in C$. In addition, $F_2(b) = 1$ since there cannot be an optimal solution with $x_0 = 0$ as a consequence of $\mathcal{S} = \emptyset$. Then, observe that for any scalar $\lambda \in \mathbb{R}_+$, $F_1 + \lambda F_2$ is dual feasible to (14), which means that the dual is unbounded as λ can be chosen arbitrarily large.
   If the dual problem is infeasible, then, by Lemma 3, $z(0) < 0$ and the primal problem is unbounded if $b \in \Omega$ and infeasible otherwise. ■

Before moving on to prove strong duality, we need the following theorem that states that any given MILP can be "extended" to one that is feasible for all right-hand sides and whose value function agrees with that of the original MILP for all right-hand sides $d \in \Omega$.

Theorem 5. (Blair and Jeroslow (1977)) For the MILP (2), there exists an extension $z_e(d) = \min\ \{c_e x\text{: } A_e x = d,\ x \in \mathbb{Z}^l_+ \times \mathbb{R}^{k-l}_+\}$, with c_e and A_e obtained by adding new coefficients and columns to c and A, such that $z_e(d) = z(d)$ $\forall\ d \in \Omega$ and $z_e(d) < \infty\ \forall\ d \in \mathbb{R}^m$.

We will not give the proof here, but note that the idea depends on iteratively adding columns orthogonal to the span of the columns of A with objective function coefficients chosen so that $z_e(d) = z(d)$ whenever $z(d) < \infty$. The following result then shows formally that the dual (14) is strong.

Theorem 6. (Strong duality by Jeroslow (1978, 1979), Wolsey (1981)) If the primal problem (2) (resp., the dual) has a finite optimum, then so does the dual problem (14) (resp., the primal) and they are equal.

Poof. Note that if the primal or the dual has a finite optimum, then Corollary 4 requires the other also to have a finite optimum. Now, we prove the claim by verifying that the value function z (or an extension to z) is a feasible dual function whenever the primal has a finite optimum.

i. $\Omega \equiv \mathbb{R}^m$: In this case, $z \in \Gamma^m$, and with a similar argument in the second part of the proof of Corollary 4, z is feasible to the dual problem.
ii. $\Omega \subset \mathbb{R}^m$: By Theorem 5, $\exists\ z_e \in \Gamma^m$ with $z_e(d) = z(d)$ $\forall d \in \Omega$ and $z_e(d) < \infty\ \forall d \in \mathbb{R}^m$. By construction, z_e must satisfy the constraints of the dual of the original MILP (2), since the dual of the extended MILP from Theorem 5 includes the constraints of (14) ($I_e \supseteq I$ and $N_e \supseteq N$). Therefore, z_e is feasible to the dual of the original MILP and hence, this dual has an optimal solution value of $z_e(b) = z(b)$. ■

One can further use the strong duality property of (14) to derive a generalization of Farkas' Lemma. This result is stated more formally in the following corollary.

Corollary 7. (Blair and Jeroslow (1982)) For the MILP (2), exactly one of the following holds:

1. $\mathcal{S} \neq \emptyset$.
2. There is an $F \in \Gamma^m$ with $F(a^j) \leq 0\ \forall j \in I$, $\overline{F}(a^j) \leq 0\ \forall j \in C$ and $F(b) > 0$.

Proof. The proof follows directly from applying Corollary 4 and Theorem 6 to the MILP (2) with $c = 0$. ■

The subadditive dual (14) can also be used to extend familiar concepts such as *reduced costs* and the *complementary slackness* conditions to MILPs. For a given optimal solution F^* to (14), the reduced costs can be defined as $c_j - F^*(a^j)$ for $j \in I$ and $c_j - \bar{F}^*(a^j)$ for $j \in C$. These reduced costs have an interpretation similar to that in the LP case, except that we do not have the same concept of "sensitivity ranges" within which the computed bounds are exact. Complementary slackness conditions can be stated as follows.

Theorem 8. (Jeroslow (1978), Johnson (1979), Bachem and Schrader (1980), Wolsey (1981)) For a given right-hand side b, let x^* and F^* be feasible solutions to the primal problem (2) and the subadditive dual problem (14). Then, x^* and F^* are optimal if and only if

$$
\begin{aligned}
&x_j^*(c_j - F^*(a^j)) = 0,\ \forall j \in I,\\
&x_j^*(c_j - \bar{F}^*(a^j)) = 0,\ \forall j \in C, \text{ and}\\
&F^*(b) = \sum_{j\in I} F^*(a^j)x_j^* + \sum_{j\in C} \bar{F}^*(a^j)x_j^*
\end{aligned}
\tag{21}
$$

Proof. If x^* and F^* are optimal, then, from the properties of F^* and strong duality,

$$F^*(b) = F^*(Ax^*) = \sum_{j\in I} F^*(a^j)x_j^* + \sum_{j\in C} \bar{F}^*(a^j)x_j^* = cx^*. \tag{22}$$

Then, we have

$$\sum_{j\in I} x_j^*(c_j - F^*(a^j)) + \sum_{j\in C} x_j^*(c_j - \bar{F}^*(a^j)) = 0.$$

By primal and dual feasibility, $x_j^* \geq 0\ \forall\ j \in N$, $c_j - F^*(a^j) \geq 0\ \forall\ j \in I$ and $c_j - \bar{F}^*(a^j) \geq 0\ \forall\ j \in C$. Therefore, (21) holds.

On the other hand, if the conditions (21) are satisfied, then (22) holds, which in turn, yields $F^*(b) = cx^*$. ■

These conditions, if satisfied, yield a certificate of optimality for a given primal-dual pair of feasible solutions. They can further be used to develop an integer programming analog of the well-known primal-dual algorithm for linear programming. Llewellyn and Ryan (1993) give the details of one such algorithm.

The subadditive duality framework also allows the use of subadditive functions to obtain inequalities valid for the convex hull of $\mathcal{S}$. In fact, subadditive functions alone can, in theory, yield a complete description of conv($\mathcal{S}$). It is easy to see that for any $d \in \Omega$ and $F \in \Gamma^m$ with $\bar{F}^*(a^j) < \infty\ \forall j \in C$, the inequality

$$\sum_{j\in I} F(a^j)x_j + \sum_{j\in C} \bar{F}(a^j)x_j \geq F(d) \tag{23}$$

is satisfied for all $x \in \mathcal{S}(d)$. The proof follows the same steps as that of weak duality, with x restricted to be in $\mathcal{S}(d)$. Furthermore, the following result shows that any valid inequality is either equivalent to or dominated by an inequality in the form of (23).

Theorem 9. (Johnson (1973), Jeroslow (1978)) For the MILP (2) and $\pi \in \mathbb{R}^n$, $\pi_0 \in \mathbb{R}$, we have that $\pi x \geq \pi_0$ $\forall x \in \mathcal{S}$ if and only if there is an $F_\pi \in \Gamma^m$ such that

$$
\begin{aligned}
&F_\pi(a^j) \leq \pi_j\ \ \forall j \in I,\\
&\bar{F}_\pi(a^j) \leq \pi_j\ \ \forall j \in C, \text{ and}\\
&F_\pi(b) \geq \pi_0.
\end{aligned}
\tag{24}
$$

Proof. First assume that $\pi \in \mathbb{R}^n$ and $\pi_0 \in \mathbb{R}$ are given such that $\pi x \geq \pi_0\ \forall x \in \mathcal{S}$. Consider the MILP

$$z_\pi = \min\{\pi x \mid x \in \mathcal{S}\}. \tag{25}$$

Clearly, $z_\pi \geq \pi_0$ because otherwise, there exists an $\bar{x} \in \mathcal{S}$ with $\pi\bar{x} < \pi_0$. Applying Theorem 6 to (25), we find that there must be a dual feasible function F_π satisfying (24).

Conversely, assume that there exists an $F_\pi \in \Gamma^m$ satisfying (24) for a given $\pi \in \mathbb{R}^n$ and $\pi_0 \in \mathbb{R}$. Then F_π is also feasible to the subadditive dual of (25) and from weak duality, $\pi x \geq F_\pi(b) \geq \pi_0$ for all $x \in \mathcal{S}$. ■

Example 6. The subadditive dual function $F_3(d)$ in Example 5 is feasible to (20). Since $F_3(1) = \frac{1}{2}$, $F_3(-\frac{3}{2}) = -\frac{1}{2}$, $\bar{F}_3(1) = 2$, $\bar{F}_3(-1) = 1$, then

$$\frac{x_1}{2} - \frac{x_2}{2} + 2x_3 + x_4 \geq F_3(b) \tag{26}$$

is a valid inequality for (6). ■

As an extension to this theorem, Bachem and Schrader (1980) showed that the convex hull of $\mathcal{S}$ can be represented using only subadditive functions and that rationality of A is enough to ensure the existence of such a representation, even if the convex hull is unbounded.

Theorem 10. (Jeroslow (1978), Blair (1978), Bachem and Schrader (1980)) For any $d \in \Omega$,

$$
\begin{aligned}
conv(\mathcal{S}(d)) = \{x : &\sum_{j\in I} F(a^j)x_j\\
&+ \sum_{j\in C} \bar{F}(a^j)x_j \geq F(d),\ F \in \Gamma^m, x \geq 0\}.
\end{aligned}
\tag{27}
$$

For a fixed right-hand side, it is clear that only finitely many subadditive functions are needed to obtain a complete description, since every rational polyhedron has finitely many facets. In fact, Wolsey (1979) showed that for PILPs, there exists a finite representation that is valid for all right-hand sides.

Theorem 11. (Wolsey (1979)) For a PILP in the form (2), there exist finitely many subadditive functions F_i, $i = 1, ..., k$, such that

$$conv(\mathcal{S}(d)) = \{x : \sum_{j=1}^{n} F_i(a^j)x_j \geq F_i(d),\quad i = 1, ..., k, \ x \geq 0\} \tag{28}$$

for any $d \in \Omega$.

Proof. Wolsey (1979) showed that when $A \in \mathbb{Z}^{m \times n}$, there exists finitely many subadditive functions F_i, $i = 1, ..., k$, such that

$$conv(\mathcal{S}(d)) = \{x : Ax = d, \ \sum_{j=1}^{n} F_i(a^j)x_j \geq F_i(d),\quad i = 1, ..., k, \ x \geq 0\} \ \forall d \in \mathbb{Z}^m. \tag{29}$$

However, the assumption that $A \in \mathbb{Z}^{m \times n}$ is without loss of generality, since A can be scaled appropriately. After scaling, we must have $\Omega \subseteq \mathbb{Z}^m$ and the result follows. ∎

Finally, it is possible to show not only that any facet can be generated by a subadditive function, but that this is true of any *minimal inequality*. Recall that $\pi \in \mathbb{R}^m$ and $\pi_0 \in \mathbb{R}$ define a minimal valid inequality if there is no other valid inequality defined by $\hat{\pi} \in \mathbb{R}^m$ and $\hat{\pi}_0 \in \mathbb{R}$ such that $\hat{\pi}_j \leq \pi_j$ for all $j \in N$ and $\hat{\pi}_0 \geq \pi_0$. Although the next theorem was originally stated for either rational constraint matrices (Johnson (1974), Blair (1978)) or bounded feasible regions (Jeroslow (1979)), Bachem and Schrader (1980) showed that the same result holds without any restriction on $\mathcal{S}$.

Theorem 12. (Bachem and Schrader (1980)) If $\pi \in \mathbb{R}^m$ and $\pi_0 \in \mathbb{R}$ define a minimal valid inequality for the MILP (2), then there is an $F \in \Gamma^m$ such that

$$\begin{aligned} &F(a^j) = \pi_j = F(b) - F(b - a^j) \qquad \forall j \in I, \\ &\bar{F}(a^j) = \pi_j = \lim_{\delta \to 0^+} \tfrac{F(b) - F(b - \delta a^j)}{\delta} \ \ \forall j \in C \text{ and,} \\ &F(b) = \pi_0. \end{aligned} \tag{30}$$

The converse of Theorem 12 holds for any subadditive function that is the value function for the MILP (2) with objective function π, where $\pi_0 = \min\{\pi x \mid x \in \mathcal{S}\}$ (as in (25)).

4. CONSTRUCTING DUAL FUNCTIONS

It is reasonable to conclude that neither the general dual problem (8) nor the subadditive dual problem (14) can be formulated as manageable mathematical program solvable directly using current technology.

However, there are a number of methods we can use to obtain feasible (and in some cases optimal) dual functions indirectly. We focus here on dual functions that provide bounds for instances of a given MILP after modifying the right-hand side, since these are the ones about which we know the most. Such dual functions are intuitive because they allow us to extend traditional notions of duality from the realm of linear programming to that of integer programming. However, we emphasize that they are not the only dual functions of potential interest in practice. Dual functions that accommodate changes to the objective function are also of interest in many applications, particularly decomposition algorithms. Dual functions that allow changes to the constraint matrix are closely related to those for studying the right-hand side, but may also yield further insight. Dual functions of these latter two types have not been well-studied. We discuss future research on these dual functions in Section 5.

Dual functions of the right-hand side can be grouped into three categories: (1) those constructed explicitly in closed form using a finite procedure, (2) those obtained from known families of relaxations, and (3) those obtained as a by-product of a primal solution algorithm, such as branch and cut. In Sections 4.1 and 4.2 below, we discuss two different methods of explicitly constructing the value function of a PILP and give an idea of how those might be extended to the MILP case. In Sections 4.3 through 4.5, we discuss methods for obtaining dual functions from relaxations. In Section 4.6, we discuss a method by which the subadditive dual of a bounded PILP can be formulated as a linear program. Finally, in Section 4.7, we discuss how to obtain a dual function as a by-product of the branch-and-cut algorithm, the method used most commonly in practice for solving MILPs.

4.1 The value function

The value function itself is the most useful dual function we can obtain for studying the effect of perturbations of the right-hand side vector, since it provides an exact solution value for any right-hand side vector. Unfortunately, it is unlikely that there exist effective methods for producing the value function for general MILPs. For PILPs, Blair and Jeroslow (1982) showed that a procedure similar to Gomory's cutting plane procedure can be used to construct the value function in a finite number of steps. Unfortunately, the representation so produced may have exponential size. From this procedure, however, they were able to characterize the class of functions to which value functions belong, namely, *Gomory functions*, a subset of a more general class called *Chvátal functions*.

Definition 3. Chvátal functions are the smallest set of functions $\mathscr{C}^m$ such that

1. If $h \in \mathcal{L}^m$, then $h \in \mathscr{C}^m$.
2. If $h_1, h_2 \in \mathscr{C}^m$ and $\alpha, \beta \in \mathbb{Q}_+$, then $\alpha h_1 + \beta h_2 \in \mathscr{C}^m$.
3. If $h \in \mathscr{C}^m$, then $\lceil h \rceil \in \mathscr{C}^m$.

Gomory functions are the smallest set of functions $\mathscr{G}^m \subseteq \mathscr{C}^m$ with the additional property that

4. If $h_1, h_2 \in \mathscr{G}^m$, then $\max\{h_1, h_2\} \in \mathscr{G}^m$.

The relationship between $\mathscr{C}^m$ and $\mathscr{G}^m$ is evident from the following theorem.

Theorem 13. (Blair and Jeroslow (1982)) Every Gomory function can be written as the maximum of finitely many Chvátal functions, that is, if $g \in \mathscr{G}^m$, then there exist $h_i \in \mathscr{C}^m$ for $i = 1, ..., k$ such that

$$g = \max\{h_1, ..., h_k\}. \tag{31}$$

This theorem also makes evident the relationship between $\mathscr{G}^m$ and the property of subadditivity. Note that if h_1, h_2 are subadditive and $\alpha, \beta \in \mathbb{Q}_+$, then it is easy to show that the functions $\alpha h_1 + \beta h_2$ and $\lceil h_1 \rceil$ are both subadditive. Consequently, one can show that Chvátal functions are subadditive by induction on the rank of functions (i.e., the number of operations of the type specified in Definition 3 needed to derive a given Chvátal function from the base class $\mathcal{L}^m$). Since $\max\{h_1, h_2\}$ is subadditive, Gomory functions are also subadditive. As a result of subadditivity, both Chvátal and Gomory functions can be used for generating valid inequalities. The following lemma, needed for the proof of Theorem 15 shows that for PILPs, Chvátal functions can be used to obtain a description of the convex hull of solutions.

Lemma 14. (Schrijver (1980)) The subadditive functions in Theorem 11 can be taken to be Chvátal functions.

The above lemma then allows us to characterize the value function of a PILP for which $z(0) = 0$.

Theorem 15. (Blair and Jeroslow (1982)) For a PILP in the form (2), *if* $z(0) = 0$, then there is a $g \in \mathscr{G}^m$ such that $g(d) = z(d)$ for all $d \in \Omega$.

Proof. Consider the parameterized family of PILPs $\min\{cx \mid x \in conv(\mathcal{S}(d))\}$ $\forall\ d \in \Omega$, where $\mathrm{conv}(\mathcal{S}(d))$ is represented by the finite set of Chvátal functions whose existence is guaranteed by Lemma 14. Applying LP duality, we get $g(d) = z(d) = \max\{\sum_{i=1}^{k} v_i F_i(d) \mid v \in V\}$ where V is the finite set of dual basic feasible solutions. Then the proof is complete by Theorem 13. ■

Example 7. The value function of problem (6) with all variables assumed to be integer can be written as $z(d) = \frac{3}{2}\max\{\lceil \frac{2d}{3} \rceil, \lceil d \rceil\} - d\ \forall\ d \in \Omega$, which is a Gomory function (see Figure 4). ■

For PILPs, it is also worth mentioning that there always exists an optimal solution to the subadditive dual problem (14) that is a Chvátal function.

Theorem 16. (Blair and Jeroslow (1982)) For a PILP in the form (2), if $b \in \Omega$ *and* $z(b) > -\infty$, then there exists $h \in \mathscr{C}^m$ that is optimal to the subadditive dual (14).

Proof. Note from the proof of Theorem 6 that either the value function itself, or an extension of the value function is a feasible solution to the subadditive dual. Denote this function as z_e. From Theorem 15, we know that there is a $g \in \mathscr{G}^m$ with $g(d) = z_e(d)$ for all $d \in \Omega$ and hence, feasible to the subadditive dual (14). By Theorem 13, g is the maximum of finitely many Chvátal functions, $h_1, ..., h_k$. For right-hand side b, since $z_e(b) = \max\{h_1(b), ..., h_k(b)\}$, there exists $l \in \{1, ..., k\}$ with $z(b) = z_e(b) = h_l(b)$. Then h_l is an optimal solution to the subadditive dual (14) since it is subadditive, and $h_l(a^j) \leq g(a^j) \leq c_j$ for all $j \in I$. ■

Using a result similar to Corollary 7 above, Blair and Jeroslow (1982) introduced the more general concept of a *consistency tester* to detect the infeasibility of the problem for any right-hand side. They showed that for a given PILP, there is a $g \in \mathscr{G}^m$ such that for every $d \in \mathbb{R}^m$, $g(d) \leq 0$ if and only if $d \in \Omega$. Using the consistency tester concept, we can state a converse of Theorem 15. That is, for Gomory functions g_1, g_2, there exist $\bar{A}$, $\bar{c}$ such that $g_1(d) = \min\{\bar{c}x \mid \bar{A}x = d, x \geq 0 \text{ and integral}\}$ for all d with $g_2(d) \leq 0$. In this sense, there is one-to-one correspondence between PILP instances and Gomory functions.

For MILPs, neither Theorem 15 nor its converse holds. However, Blair and Jeroslow (1984) argue that the value function z can still be represented by a Gomory function if $c_j = 0\ \forall\ j \in C$ or can be written as a minimum of finitely many Gomory functions. A deeper result is contained in the subsequent work of Blair (1995), who showed that the value function of a MILP can be written as a *Jeroslow formula*, consisting of a Gomory function and a correction term. Here, rather than the formula itself (see Blair and Jeroslow (1984), Blair (1995) for details), we present a simplified version to illustrate its structure.

For a given $d \in \Omega$, let the set $\mathscr{E}$ consist of the index sets of dual feasible bases of the linear program

$$\min\{c_C x_C : A_C x_C = d,\ x \geq 0\}. \tag{32}$$

By the rationality of A, we can choose $M \in \mathbb{Z}_+$ such that for any $E \in \mathscr{E}$, $MA_E^{-1}a^j \in \mathbb{Z}^m$ for all $j \in I$. For $E \in \mathscr{E}$, let v_E be the corresponding basic feasible solution to the dual of

$$\min\left\{\frac{1}{M}c_C x_C : \frac{1}{M}A_C x_C = d,\ x \geq 0\right\}, \tag{33}$$

which is a scaled version of (32). Finally, for a right-hand side d and $E \in \mathcal{E}$, let $\lfloor d \rfloor_E = A_E \lfloor A_E^{-1} d \rfloor$.

Theorem 17. (Blair (1995)) For the MILP (2), there is a $g \in \mathscr{G}^m$ such that

$$z(d) = \min_{E\in\mathcal{E}} g(\lfloor d \rfloor_E) + v_E(d - \lfloor d \rfloor_E) \tag{34}$$

for any $d \in \Omega$.

Proof. Assume that c_C and A_C are scaled as in (33) and consider the PILP instance

$$\begin{aligned} z_{JF}(\phi) &= \min\ cx + z(\varphi) y \\ &\quad \text{s.t } Ax + \varphi y = \phi \\ &\quad\quad x \in \mathbb{Z}^n_+,\ y \in \mathbb{Z}_+, \end{aligned} \tag{35}$$

where $\varphi = -\sum_{j\in C} a^j$. Then we have the following:

1. For any $E \in \mathcal{E}$ and $d \in \mathbb{R}^m$, (35) is feasible for $\phi = \lfloor d \rfloor_E$. To see this, observe that if $\lfloor A_E d \rfloor \geq 0$, then $x_E = \lfloor A_E^{-1} d \rfloor$, $x_{N\setminus E} = 0$, $y = 0$ is a feasible solution. Otherwise, there exists $\Delta \in \mathbb{Z}_+$ such that $x_E = (\lfloor A_E^{-1} d \rfloor + \Delta A_E^{-1} \sum_{j\in E} a^j) \in \mathbb{Z}^m_+$, since $A_E^{-1} \sum_{j\in E} a^j = \mathbf{1}^m$. Therefore, together with x_E, $x_I = 0$, $x_j = \Delta$ for $j \in C\setminus E$ and $y = \Delta$ is a feasible solution.
2. For any $E \in \mathcal{E}$ and $d \in \mathbb{R}^m$, $z_{JF}(\lfloor d \rfloor_E) \geq z(\lfloor d \rfloor_E)$. To see this, assume that $z(\varphi) = cx_1$ and $z_{JF}(\lfloor d \rfloor_E) = cx_2 + z(\varphi)\hat{y} = c(x_2 + x_1\hat{y})$. But then, clearly, $z_{JF}(\lfloor d \rfloor_E) \geq z(\lfloor d \rfloor_E)$ since $(x_2 + x_1\hat{y}) \in \mathcal{S}(\lfloor d \rfloor_E)$.

Now, we know from Theorem 15 that there is a $g \in \mathscr{G}^m$ with $g(\lfloor d \rfloor_E) = z_{JF}(\lfloor d \rfloor_E)$ for all $d \in \mathbb{R}^m$, $E \in \mathcal{E}$. Let $d \in \Omega$ be given and $\bar{x} \in \mathcal{S}(d)$. By LP duality, there is an $E \in \mathcal{E}$ with $c_E A_E^{-1} A_C \bar{x}_C \leq c_C \bar{x}_C$. Noting that $\lfloor d \rfloor_E = \lfloor A\bar{x} \rfloor_E = A_I \bar{x}_I + \lfloor A_C \bar{x}_C \rfloor_E$, we have

$$\begin{aligned} &g(\lfloor d \rfloor_E) + v_E(d - \lfloor d \rfloor_E) \\ &= z_{JF}(A_I\bar{x}_I + \lfloor A_C\bar{x}_C \rfloor_E) + v_E(d - \lfloor d \rfloor_E) \\ &\leq c_I\bar{x}_I + c_E A_E^{-1}\lfloor A_C\bar{x}_C \rfloor_E + v_E(d - \lfloor d \rfloor_E) \\ &= c_I\bar{x}_I + c_E A_E^{-1} A_C\bar{x}_C \\ &\leq c\bar{x}, \end{aligned}$$

where the first inequality follows from the fact that $x_I = \bar{x}_I$, $x_j = 0$ for $j \in C\setminus E$, $x_E = A_E^{-1}\lfloor A_C\bar{x}_C \rfloor_E$, and $y = 0$ is a feasible solution to (35) with right-hand side $A_I\bar{x}_I + \lfloor A_C\bar{x}_C \rfloor_E$ and the last equality follows from the fact that

$$\begin{aligned} v_E(d - \lfloor d \rfloor_E) &= v_E(A_C\bar{x}_C - \lfloor A_C\bar{x}_C \rfloor_E) \\ &= c_E A_E^{-1}(A_C\bar{x}_C - \lfloor A_C\bar{x}_C \rfloor_E). \end{aligned}$$

On the other hand, for any $E \in \mathcal{E}$,

$$\begin{aligned} &g(\lfloor d \rfloor_E) + v_E(d - \lfloor d \rfloor_E) = z_{JF}(\lfloor d \rfloor_E) + v_E(d - \lfloor d \rfloor_E) \\ &\geq z(\lfloor d \rfloor_E) + z(d - \lfloor d \rfloor_E) \geq z(d). \end{aligned}$$

by the subadditivity of z. ■

Example 8. Consider the MILP (6). With $M = 2$, the set of index sets of dual feasible bases of $\min\{x_3 + \frac{1}{2}x_4 \mid \frac{1}{2}x_3 - \frac{1}{2}x_4 = d,\ x_3, x_4 \geq 0\}$ is $\mathcal{E} = \{\{3\}, \{4\}\}$. Furthermore, $v_{\{3\}} = 2$ and $v_{\{4\}} = -1$. Since $\varphi = \frac{1}{2} - \frac{1}{2} = 0$ and $z(0) = 0$, (35) reduces to $z_{JF}(\phi) = \{\frac{1}{2}x_1 + x_3 + \frac{1}{2}x_4 \mid x_1 - \frac{3}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 = \phi,\ x_i \in \mathbb{Z}_+,\ i = 1, \ldots, 4\}$. The value function of this problem is the same as z in Example 7. Thus, $g(d) = \frac{3}{2}\max\{\lceil \frac{2d}{3} \rceil, \lceil d \rceil\} - d$ solves this problem. Then the value function (see Figure 1) of (6) is

$$\min\left\{\frac{3}{2}\max\left\{\left\lceil\frac{\lfloor 2d \rfloor}{3}\right\rceil, \left\lceil\frac{\lfloor 2d \rfloor}{2}\right\rceil\right\} + \frac{3\lceil 2d \rceil}{2} + 2d,\right.$$
$$\left.\frac{3}{2}\max\left\{\left\lceil\frac{\lceil 2d \rceil}{3}\right\rceil, \left\lceil\frac{\lceil 2d \rceil}{2}\right\rceil\right\} - d\right\}.$$

■

4.2 Generating functions

Lasserre (2004a, b) recently introduced a different method for constructing the value function of PILPs that utilizes generating functions. This methodology does not fit well into a traditional duality framework, but nevertheless gives some perspective about the role of basic feasible solutions of the LP relaxation in determining the optimal solution of a PILP.

Theorem 18. (Lasserre (2003)) For a PILP in the form (2) with $A \in \mathbb{Z}^{m\times n}$, define

$$z(d, c) = \min_{x\in\mathcal{S}(d)} cx, \tag{36}$$

and let the corresponding summation function be

$$\hat{z}(d, c) = \sum_{x\in\mathcal{S}(d)} e^{cx}\ \ \forall d \in \mathbb{Z}^m. \tag{37}$$

Then the relationship between z and $\hat{z}$ is

$$e^{z(d,c)} = \lim_{q\to-\infty} \hat{z}(d,qc)^{1/q} \text{ or equivalently,}$$
$$z(d,c) = \lim_{q\to-\infty} \frac{1}{q} \ln \hat{z}(d,qc). \tag{38}$$

In order to get a closed form representation of $\hat{z}$, we can solve the two sided $\mathbb{Z}$- transform $\hat{F}: \mathbb{C}^m \to \mathbb{C}$ defined by

$$\hat{F}(s,c) = \sum_{d\in\mathbb{Z}^m} s^{-d} \hat{z}(d,\ c) \tag{39}$$

with $s^d = s_1^{d_1} \ldots s_m^{d_m}$ for $d \in \mathbb{Z}^m$. Substituting $\hat{z}$ in this formula, we get

$$\begin{aligned}\hat{F}(s,c) &= \sum_{d\in\mathbb{Z}^m} \sum_{x\in\mathcal{S}(d)} s^{-d} e^{cx} \\ &= \sum_{x\in\mathbb{Z}^n_+} e^{cx} \sum_{d=Ax} s^{-d} \\ &= \sum_{x\in\mathbb{Z}^n_+} e^{cx} s^{-Ax} \\ &= \prod_{j=1}^{n} \frac{1}{1-e^{c_j} s^{-a^j}},\end{aligned} \tag{40}$$

where the last equality is obtained by applying Barvinok (1993)'s short form equation for summation problems over a domain of all non-negative integral points. The formula (40) is well-defined if $\left|s^{a^j}\right| > e^{c_j}$, j = 1, ..., n and the function $\hat{z}$ is then obtained by solving the inverse problem

$$\begin{aligned}\hat{z}(d,c) &= \frac{1}{(2i\pi)^m} \int_{|s|=\gamma} \hat{F}(s,c) s^{d-1^m} ds \\ &= \frac{1}{(2i\pi)^m} \int_{|s_1|=\gamma_1} \cdots \int_{|s_m|=\gamma_m} \hat{F}(s,c) s^{d-1^m} ds,\end{aligned} \tag{41}$$

where γ is a vector satisfying $\gamma^{a^j} > e^{c_j}$ j = 1, ..., n and $\mathbf{1}^m$ = (1, ..., 1) $\in \mathbb{R}^m$.

Although it is possible to solve (41) directly by Cauchy residue techniques, the complex poles make it difficult. One alternative is to apply Brion and Vergne's (see Brion and Vergne (1997), Lasserre (2003) for details) lattice points counting formula in a polyhedron to get the reduced form, which, for each $d \in \mathbb{R}^m$, is composed of the optimal solution value of the LP relaxation and a correction term. The correction term is the minimum of the sum of the reduced costs of certain nonbasic variables over all basic feasible solutions, obtained by the degree sum of certain real-valued univariate polynomials. Another approach using generating functions is to apply Barvinok (1994)'s algorithm for counting lattice points in a polyhedron of fixed dimension to a specially constructed polyhedron that includes for any right-hand side the corresponding minimal test set (see Loera et al. (2004a, b) for details).

4.3 Cutting plane method

Cutting plane algorithms are a broad class of methods for obtaining lower bounds on the optimal solution value of a given MILP by iteratively generating inequalities valid for the convex hull of $\mathcal{S}$ (called *cutting planes* or *cuts*). The procedure works by constructing progressively tighter polyhedral approximations of *conv*($\mathcal{S}$), over which a linear program is then solved to obtain a bound. To be more precise, in iteration k, the algorithm solves the following linear program:

$$\begin{aligned}\min\ & cx \\ \text{s.t. } & Ax = b \\ & \Pi x \ge \Pi_0 \\ & x \ge 0,\end{aligned} \tag{42}$$

where $\Pi \in \mathbb{R}^{k\times n}$ and $\Pi_0 \in \mathbb{R}^k$ represents the cutting planes generated so far. At the time of generation, each of the valid inequalities is constructed so as to eliminate a portion of the feasible region of the current relaxation that contains the current solution to (42), but does not contain any feasible solutions to the original MILP.

As noted earlier, the LP dual of (42), i.e.,

$$\begin{aligned}\max\quad & vb + w\Pi_0 \\ & vA + w\Pi \le c \\ & v \in \mathbb{R}^m,\ w \in \mathbb{R}^k_+,\end{aligned} \tag{43}$$

is also a dual problem for the original MILP, but does not yield a dual function directly because the cutting planes generated may only be valid for the convex hull of solutions to the original MILP and not for instances with a modified right-hand side. However, one can extract such a dual function if it is possible to systematically modify each cut to ensure validity after replacing the original right-hand side b with a new right-hand side d. Assuming that a subadditive representation (23) of each cut is known, the i^{th} cut can be expressed parametrically as a function of the right-hand side $d \in \mathbb{R}^m$ in the form

$$\sum_{j\in I} F_i(\sigma_i(a^j))x_j + \sum_{j\in C} \bar{F}_i(\bar{\sigma}_i(a^j))x_j \ge F_i(\sigma_i(d)), \tag{44}$$

where F_i is the subadditive function representing the cut, and the functions $\sigma_i, \bar{\sigma}_i : \mathbb{R}^m \to \mathbb{R}^{m+i-1}$ are defined by

- $\sigma_1(d) = \bar{\sigma}_1(d) = d$,
- $\sigma_i(d) = [\ d\ F_1(\sigma_1(d)) \ldots F_{i-1}(\sigma_{i-1}(d))\ ]$ for $i \ge 2$, and
- $\bar{\sigma}_i(d) = [\ d\ \bar{F}_1(\bar{\sigma}_1(d)) \ldots \bar{F}_{i-1}(\bar{\sigma}_{i-1}(d))\ ]$ for $i \ge 2$.

Furthermore, if (v^k, w^k) is a feasible solution to (43) in the k^{th} iteration, then the function

$$F_{CP}(d) = v^k d + \sum_{i=1}^{k} w_i^k F_i(\sigma_i(d)) \tag{45}$$

is a feasible solution to the subadditive dual problem (14).

As noted earlier, Wolsey (1981) showed how to construct a dual function optimal to the subadditive dual for a given PILP using the Gomory fractional cutting plane algorithm under the assumption that cuts are generated using a method guaranteed to yield a sequence of LPs with lexicographically increasing solution vectors (this method is needed to guarantee termination of the algorithm in a finite number of steps with either an optimal solution or a proof that original problem is infeasible). In Gomory's procedure, the subadditive function F_i, generated for iteration i, has the following form

$$F_i(d) = \left\lceil \sum_{k=1}^{m} \lambda_k^{i-1} d_k + \sum_{k=1}^{i-1} \lambda_{m+k}^{i-1} F_k(d) \right\rceil \tag{46}$$
where $\lambda^{i-1} = (\lambda_1^{i-1}, \ldots, \lambda_{m+i-1}^{i-1}) \geq 0$.

Assuming that $b \in \Omega$, $z(b) > -\infty$, and that the algorithm terminates after k iterations, the function F_G defined by

$$F_G(d) = v^k d + \sum_{i=1}^{k} w_i^k F_i(d) \tag{47}$$

is optimal to the subadditive dual problem (14). Note that F_G is a Chvátal function and hence, this can be seen as an alternative proof for Theorem 16.

In practice, it is generally not computationally feasible to determine a subadditive representation for each cut added to the LP relaxation. However, since our goal is simply to ensure the validity of each cut after modification of the right-hand side, an alternative approach that is feasible for some classes of valid inequalities is simply to track the dependency of each cut on the original right-hand side in some other way. If this information can be functionally encoded, as it is with the subadditive representation, the right-hand side of each cut can be modified to make it valid for new instances and these functions can be used to obtain a dual function similar in form to (45). As an example of this, Schrage andWolsey (1985) showed how to construct a function tracking dependency on the right-hand side for cover inequalities by expressing the right-hand side of a cut of this type as an explicit function of the right-hand side of the original knapsack constraint. To illustrate, suppose that $\pi \in \mathbb{R}^n$ and $\pi_0 \in \mathbb{R}$ is such that $\pi \geq 0$ and $\pi_0 \geq 0$. We define $U \subseteq \{1, \ldots, n\}$ to be a *cover* if $\sum_{j\in U}\pi_j > \pi_0$. It is then well-known that $\sum_{j\in U}x_j \leq |U| - 1$ for all $x \in \{0, 1\}^n$ satisfying $\pi x \leq \pi_0$. The following proposition shows how to modify the given inequality so that it remains valid if π_0 is changed to $\bar{\pi} \in \mathbb{R}$.

Theorem 19. (Schrage and Wolsey (1985)) Let $\pi_v = \max\{\pi_j \mid j \in U\}$ for a given knapsack constraint with nonnegative parameters and a cover U. Then,

$$\sum_{j\in U} x_j \leq \left\lfloor |U| - \frac{\sum_{j\in U}\pi_j - \bar{\pi}_0}{\pi_v} \right\rfloor \tag{48}$$

for all $x \in \{0, 1\}^n$ satisfying $\pi x \leq \bar{\pi}_0$, where $\bar{\pi}_0$ is the modified right-hand side.

In the same paper, it is further discussed that a similar construction can also be obtained for lifted cover inequalities where some of the coefficients of the left side of the cover inequality are increased to strengthen the inequality.

4.4 Corrected linear dual functions

A natural way in which to account for the fact that linear functions are not sufficient to yield strong dual functions in the case of MILPs is to consider dual functions that consist of a linear term (as in the LP case) and a correction term accounting for the duality gap. One way to construct such a function is to consider the well-known *group relaxation*. Let B be the index set of the columns of a dual feasible basis for the LP relaxation of a PILP and denote by $N\backslash B$ the index set of the remaining columns. Consider the function F_B defined as

$$\begin{aligned} F_B(d) = \min\ & c_B x_B + c_{N\backslash B} x_{N\backslash B} \\ \text{s.t. } & A_B x_B + A_{N\backslash B} x_{N\backslash B} = d \\ & x_B \in \mathbb{Z}^m, x_{N\backslash B} \in \mathbb{Z}_+^{n-m}. \end{aligned} \tag{49}$$

Substituting $x_B = A_B^{-1} d - A_B^{-1} A_{N\backslash B} x_{N\backslash B}$ in the objective function, we obtain the group relaxation (Gomory (1969))

$$\begin{aligned} F_B(d) = c_B A_B^{-1} d - \max\ & \bar{c}_{N\backslash B} x_{N\backslash B} \\ & A_B x_B + A_{N\backslash B} x_{N\backslash B} = d, \\ & x_B \in \mathbb{Z}^m, x_{N\backslash B} \in \mathbb{Z}_+^{n-m} \end{aligned} \tag{50}$$

where $\bar{c}_{N/B} = (c_B A_B^{-1} A_{N/B} - c_{N/B})$. Here, dual feasibility of the basis A_B is required to ensure that $\bar{c}_{N/B} \leq 0$.

F_B is feasible to the subadditive dual (14). To see this, note that F_B is subadditive since it is the sum of a linear function and the value function of a PILP. Also, we have $F_B(a^j) \leq c_B A_B^{-1} a^j - (c_B A_B^{-1} a^j - c_j) = c_j$, $j \in N\backslash B$ and $F_B(a^j) = c_B A_B^{-1} a^j = c_j$, $j \in B$. Therefore, for the PILP (2), $F_B(b) \leq z(b)$. Gomory (1969) further discusses sufficient conditions for FB to be strong. Observe that $F_B(b) = z(b)$ when there exists an optimal solution to (50) with $x_B \geq 0$.

Another way to construct an optimal solution to the subadditive dual using a linear function with a correction term is given by Klabjan (2002).

Theorem 20. (Klabjan (2002)) For a PILP in the form (2), and a given vector $v \in \mathbb{R}^m$, define the function F_v as

$$F_v(d) = vd - \max\left\{(vA_{D_v} - c_{D_v})x \mid A_{D_v}x \leq d,\ x \in \mathbb{Z}_+^{|D_v|}\right\},$$

where $D_v = \{i \in I : va^i > c_i\}$. Then, F_v is a feasible solution to the subadditive dual problem (14) and furthermore, if $b \in \Omega$ and $z(b) > -\infty$, there exists a $v \in \mathbb{R}^m$ such that $F_v(b) = z(b)$.

Proof. For a given v, F_v is subadditive using an argument similar to that made above for group relaxation problems. Now, consider the problem $\max\{(vA_{D_v} - c_{D_v})x \mid A_{D_v}x \le a^i,\ x \in \mathbb{Z}_+^{|D_v|}\}$ for a given i. If $i \in I \backslash D_v$, $x = 0$ is feasible. Otherwise the i^{th} unit vector is a feasible solution. Thus, for any $i \in I$, $F_v(a^i) \le c_i$. Therefore, F_v is a feasible solution to the subadditive dual (14) and $F_v(b) \le z(b)$.

Next, suppose that the original PILP is solved with Gomory's procedure (42) after k iterations. Let the set of generated Chvátal inequalities be represented by (π^j, π_0^j) for $j \in J = \{1, \ldots, k\}$. Let v^k and w^k be the corresponding components of the optimal dual solution with respect to the set of original constraints and the set of valid inequalities. With $x \in \{x \in \mathbb{Z}_+^{|D_{v^k}|} \mid A_{D_{v^k}}x = b\}$,

$$\begin{aligned}(v^k A_{D_{v^k}} - c_{D_{v^k}})x &\le \sum_{i \in D_{v^k}} \sum_{j \in J} \pi_i^j w_j^k x_i \\ &= -\sum_{j \in J} w_j^k \sum_{i \in D_{v^k}} \pi_i^j x_i \\ &\le -\sum_{j \in J} w_j^k \pi_0^j \\ &= v^k b - z(b),\end{aligned}$$

where the first inequality follows from the dual feasibility of v^k and w^k, i.e., $v^k a^i + \sum_{j \in J} \pi_i^j w_j^k \le c_i$, $i \in D_{v^k}$, and the last inequality follows from the fact that $\pi^j x \ge \pi_0^j$, $j \in J$, are valid inequalities for $\{A_{D_{v^k}}x = b, x \in \mathbb{Z}_+^{|D_{v^k}|}\}$ and $w^k \ge 0$. Rearranging, we have

$$z(b) \le v^k b - (v^k A_{D_{v^k}} - c_{D_{v^k}})x \le F_{v^k}(b). \tag{51}$$

Combining this result with weak duality, we get $z(b) = F_{v^k}(b)$. ■

Klabjan (2002) also introduced an algorithm that finds the optimal dual function utilizing a subadditive approach from (Burdet and Johnson (1977)) together with a row generation approach that requires the enumeration of feasible solutions. Unfortunately, even for the set partitioning problems that the author reports on, this algorithm seems not to be practical.

4.5 Lagrangian relaxation

Another widely used framework for generating dual problems is that of *Lagrangian duality* (Fisher (1981)). A mathematical program obtained by relaxing and subsequently penalizing the violation of a subset of the original constraints, called the *complicating* constraints, is a *Lagrangian relaxation*. Generally, this relaxation is constructed so that it is much easier to solve than the original MILP, in which case a dual problem can be constructed as follows. Suppose for a given $d \in \mathbb{R}^m$ that the inequalities defined by matrix A and right-hand side d are partitioned into two subsets defined by matrices A^1 and A^2 and right-hand sides d^1 and d^2. Furthermore, let $\mathcal{S}_{LD}(d^2) = \{x \in \mathbb{Z}_+^r \times \mathbb{R}_+^{n-r} : A^2x = d^2\}$. Then, for a given penalty multiplier $v \in \mathbb{R}^{m-l}$, the corresponding Lagrangian relaxation can be formulated as

$$L(d, v) = \min_{x \in \mathcal{S}_{LD}(d^2)} cx + v(d^1 - A^1x) \tag{52}$$

Assuming $z(0) = 0$ and that $x^*(d)$ is an optimal solution to the original MILP with right-hand side d, we have $L(d, v) \le cx^*(d) + v(d^1 - A^1x^*(d)) = cx^*(d) = z(d)\ \forall\ v \in \mathbb{R}^{m-l}$. Thus, the Lagrangian function defined by

$$L_D(d) = \max\{L(d, v) : v \in V\}, \tag{53}$$

with $V \equiv \mathbb{R}^{m-l}$, is a feasible dual function in the sense that $L_D(d) \le z(d)\ \forall\ d \in \Omega$.

Note that for a given $d \in \Omega$, $L(d, v)$ is a concave, piecewise-polyhedral function. Therefore, the set V_d of extreme points of $epi(L(d, v))$ is finite. Setting $V_\Omega = \cup_{d \in \Omega} V_d$, we can rewrite $L_D(d) = \max\{L(d, v) : v \in V_\Omega\}$. It follows that if V_Ω is finite, then L_D reduces to the maximization of finitely many subadditive functions and therefore, is subadditive and feasible to the subadditive dual problem (14). Furthermore, in the PILP case, L_D corresponds to a Gomory function, since for a fixed v, (52) can be represented by a Gomory function and the maximum of finitely many Gomory functions is also a Gomory function.

L_D above is a weak dual function in general, but Blair and Jeroslow (1979) showed that it can be made strong for PILP problems by introducing a quadratic term. To show this, we first need the following proximity relation.

Theorem 21. (Blair and Jeroslow (1977)) For a given PILP with $z(0) = 0$, there is a constant $\varepsilon > 0$ such that

$$|z(d_1) - z(d_2)| \le \varepsilon \, \|d_1 - d_2\|_1. \tag{54}$$

for all $d_1, d_2 \in \Omega$.

Let the quadratic Lagrangian relaxation be defined as

$$L(d, v, \rho) = \min_{x \in \mathbb{Z}_+^n} \left\{ (c - vA)x + \rho \sum_{i=1}^{m} (A_i x - d_i)^2 + vd \right\}, \tag{55}$$

where $v \in \mathbb{R}^m$, $\rho \in \mathbb{R}_+$ and A_i is the i^{th} row of A.

Theorem 22. (Blair and Jeroslow (1979)) For a PILP in the form (2), denote the quadratic Lagrangian dual function as

$$L_D(d,v) = \max_{\rho \in \mathbb{R}_+} L(d,v,\rho). \tag{56}$$

Then for a given $v \in \mathbb{R}^m$, $L_D(d, v) \le z(d)$ $\forall$ $d \in \Omega$ and furthermore, if $b \in \Omega$ and $z(b) > -\infty$, then for any $v \in \mathbb{R}^m$, $L_D(b, v) = z(b)$.

Proof. The first part follows from the fact that for any $d \in \Omega$ and $\rho \in \mathbb{R}_+$,

$$\begin{aligned} L(d,v,\rho) &\le \min_{x \in \mathcal{S}(d)} \left\{ (c - vA)x + \rho \sum_{i=1}^{m} (A_i x - d_i)^2 + vd \right\} \\ &= \min_{x \in \mathcal{S}(d)} cx = z(d). \end{aligned} \tag{57}$$

For the second part, we show that for right-hand side $b \in \Omega$ with $z(b) > -\infty$ and a given $v \in \mathbb{R}^m$, there exists $\rho(v) \in \mathbb{R}_+$ such that, $L(b, v, \rho(v)) = z(b)$. Let $\rho(v) = 1 + \varepsilon + \|v\|_1$, with ε defined as in (54), assume that $\bar{x} \in \mathbb{Z}^n_+$ is an optimal solution to yield $L(b, v, \rho(v))$ and let $\bar{b} = A\bar{x}$. Then,

$$\begin{aligned} &(c - vA)\bar{x} + \rho(v)\sum_{i=1}^{m}(A_i\bar{x} - b_i)^2 + vb \\ &= c\bar{x} + v(b - A\bar{x}) + \rho(v)\sum_{i=1}^{m}(\bar{b}_i - b_i)^2 \\ &\ge z(\bar{b}) + v(b - \bar{b}) + \rho(v)\left\|b - \bar{b}\right\|_1 \\ &\ge z(b) - \varepsilon\left\|b - \bar{b}\right\|_1 - \|v\|_1\left\|b - \bar{b}\right\|_1 + \rho(v)\left\|b - \bar{b}\right\|_1 \\ &= z(b) + \left\|b - \bar{b}\right\|_1 \\ &\ge z(b) \end{aligned} \tag{58}$$

by Theorem 21 and the fact that $\left\|b - \bar{b}\right\|_1 \le \sum_{i=1}^{m}(b_i - \bar{b}_i)^2$. Therefore, $L(b, v, \rho(v)) \ge z(b)$ and due first part, $L_D(b, v) = L(b, v, \rho(v)) = z(b)$. ∎

Note that one can verify that (56) attains its maximum at a point x^* that is also optimal to the PILP. This is because in order to get equality in (58), the conditions $b = \bar{b}$ and $cx^* = z(b)$ have to be satisfied at the same time. Otherwise, $L_D(b, v) > z(b)$. In addition, it is clear that $\rho(v)$ can be replaced by any $\bar{\rho}$ such that $\bar{\rho} \ge \rho(v)$ for a given v in (58). In fact, if we let $\bar{v}$ be the optimal solution to the dual of the LP relaxation of PILP, then choosing $\bar{\rho} > z(b) - \bar{v}b$ is adequate, since

$$(c - \bar{v}A)\bar{x} + \bar{\rho}\sum_{i=1}^{m}(\bar{b}_i - b_i)^2 + \bar{v}b \ge \bar{\rho} + \bar{v}b > z(b). \tag{59}$$

Due to dual feasibility, $L(b, \bar{v}, \bar{\rho})$ is forced to have its infimum at an x^* that is also optimal to the PILP, since equality in (59) is attained only in that case.

4.6 Linear representation of the subadditive dual

For bounded PILPs with $A \in \mathbb{Q}^{m \times n}_+$, the subadditive dual can be reformulated as an equivalent LP

$$\begin{aligned} \max\ & \eta(b) \\ \text{s.t. } & \eta(\lambda) + \eta(\mu) \ge \eta(\lambda + \mu), \\ & 0 \le \lambda \le b,\ 0 \le \mu \le b,\ 0 \le \lambda + \mu \le b, \\ & \eta(a^j) \le c_j,\ j = 1, \ldots, n \\ & \eta(0) = 0, \end{aligned} \tag{60}$$

after scaling A and b to be integer. This follows from the fact that the subadditive dual function in this case can be represented by the values it takes over the finite domain $\{\lambda \in \mathbb{Z}^m_+ \mid \lambda \le b\}$ (Gomory (1969), Johnson (1979)). The variables in the above LP represent the values of the subadditive function to be constructed at each point in this domain and the constraints ensure that the function η: $\{\alpha \mid \alpha \le b\} \rightarrow$ R is actually subadditive.

Lasserre (2004c, b) further decreases the row dimension of this LP using a discrete version of Farkas' lemma. Let $\mathbb{R}[s_1, \ldots, s_m]$ be the ring of real-valued polynomials in the variables s_i, $i = 1, \ldots, m$. Then, a polynomial $Q \in \mathbb{R}[s_1, \ldots, s_m]$ can be written as

$$Q(s) = \sum_{\alpha \in \zeta} \lambda^\alpha s^\alpha = \sum_{\alpha \in \zeta} \lambda^\alpha s_1^{\alpha_1} \ldots s_m^{\alpha_m},$$

where $\zeta \subset \mathbb{Z}^m_+$ and $\lambda^\alpha \in \mathbb{R}$ $\forall \alpha \in \zeta$.

Theorem 23. (Lasserre (2003)) The following two properties are equivalent:

1. $Ax = b$ has a solution $x \in \mathbb{Z}^n_+$.
2. The real valued polynomial $s^b - 1$ can be written as

$$s^b - 1 = \sum_{j=1}^{n} Q_j(s)(s^{a^j} - 1) \tag{61}$$

for some real-valued polynomials $Q_j \subset \mathbb{R}[s_1, \ldots, s_m]$, $j = 1, \ldots, n$, all with nonnegative coefficients.

Proof. (1) → (2). Let $x \in \mathcal{S}$. Writing

$$s^b - 1 = s^{a^1 x_1} - 1 + s^{a^1 x_1}(s^{a^2 x_2} - 1) + \ldots + s^{\sum_{j=1}^{n-1} a^j x_j}(s^{a^n x_n} - 1)$$

with

$$s^{a^j x_j} - 1 = (s^{a^j} - 1)\left[1 + s^{a^j} + \ldots + s^{a^j(x_j - 1)}\right],\ j = 1, \ldots, n,$$

we obtain

$$Q_j(s) = s^{\sum_{k=1}^{j-1} a^k x_k}\left[1 + s^{a^j} + \ldots + s^{a^j(x_j - 1)}\right],\ j = 1, \ldots, n. \tag{62}$$

(2) → (1). Let $q \in \mathbb{R}^k_+$ be the vector of nonnegative coefficients of all polynomials Q_j, $j = 1, ..., n$, and $M \in \mathbb{R}^{p\times k}$ be such that the set of constraints defining the polyhedron $\Theta = \{q \mid Mq = \tau,\ q \geq 0\} \neq \emptyset$ equalizes the respective coefficients of the polynomials $s^b - 1$ and $\sum_{j=1}^n Q_j(s)(s^{a^j} - 1)$. It is easy to show that each Q_j, $j = 1, ..., n$, may be restricted to contain only monomials $\{s^\alpha: \alpha \leq b - a^j,\ \alpha \in \mathbb{Z}^m_+\}$ and therefore

$$p = \prod_{i=1}^m (b_i + 1)$$

$$k = \sum_{j=1}^n k_j \text{ with } k_j = \prod_{i=1}^m (b_i - a_i^j + 1),\ j = 1, ..., n.$$

In other words, p is the number of monomials y^α with $\alpha \leq b$ and k_j is the number of monomials y^α with $\alpha - a^j \leq b$. With this construction, it is not hard to see that M is totally unimodular and each extreme point of Θ, if it exists, is integral, since τ is also integral.

Next, recall that $\mathbf{1}^{k_j} = (1, ..., 1) \in \mathbb{R}^{k_j}$, $j = 1, ..., n$, and let $K \in \mathbb{Z}^{n\times k}_+$ be the n-block diagonal matrix, whose each diagonal block is a row vector 1^{k_j}, that is,

$$K = \begin{bmatrix} 1^{k_1} & 0 & \dots & 0 \\ 0 & 1^{k_2} & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1^{k_n} \end{bmatrix}.$$

Now, let Q_j, $j = 1, ..., n$, be the set of polynomials satisfying (61). Then, $\Theta \neq \emptyset$ and there exists an integral $\overline{q} \in \Theta$. If we denote by $\overline{Q}_j$, $j = 1, ..., n$, the corresponding monomials $\overline{q}$ represents and take the derivative of both sides with respect to s_i, $i = 1, ..., m$, at $(1, ..., 1)$, we get

$$b_i = \sum_{j=1}^n \overline{Q}_j(1, ..., 1) a_i^j = \sum_{j=1}^n a_i^j (K\overline{q})_j,\ i = 1, ..., m.$$

Observe that setting $x = K\overline{q}$ completes the proof. ■

The converse of the last part of the proof is also valid, i.e., for any $x \in \mathcal{S}$, $x = Kq$ for some $q \in \Theta$. As a consequence, we have the following corollary.

Corollary 24. (Lasserre (2004c)) For a PILP in the form (2) with $A \in \mathbb{Z}^{m\times n}_+$, let K, M, τ be defined as before. Then, $z(b) = \min\{cKq \mid Mq = \tau,\ q \geq 0\}$. Moreover, if q^* is an optimal solution, then $x^* = Kq^*$ is an optimal solution to the PILP.

Lasserre further shows that the LP dual of the problem in the first part of Corollary 24 can be reduced to a subadditive formulation that is also dual to PILP. Compared to (60), the number of variables is the same, however, this one has $\mathcal{O}(np)$ constraints, whereas (60) has $\mathcal{O}(p^2)$ constraints.

4.7 Branch and cut

The most common technique for solving MILPs in practice today is the branch-and-cut algorithm. Developing a procedure for obtaining a dual function as a by-product of this procedure is of great importance if duality is to be made computationally useful. Here we discuss "vanilla" branch and cut, in which branching is done only by restricting variable bounds and no standard computational enhancements, such as preprocessing, are used. Such an algorithm works by attempting to solve the subproblem of each branch-and-cut tree node utilizing a cutting plane method, as described in Section 4.3. If the subadditive characterization or a functional encoding of the right-hand side dependency is available for each cut, then we can obtain a dual function for the corresponding subproblem. Below, we show how this dual information can be gathered together to yield a feasible dual function for the original problem.

Assume that the MILP (2) has a finite optimum and has been solved to optimality with a branch-and-cut algorithm. Let T be the set of leaf nodes of the tree and let $\nu(t)$ be the number of cuts generated so far on the path from the root node to node $t \in T$ (including the ones generated at t). To obtain a bound for this node, we solve the LP relaxation of the following problem

$$\begin{aligned} z^t(b) = \min\ & cx \\ \text{s.t. } & x \in \mathcal{S}_t(b), \end{aligned} \tag{63}$$

where the feasible region $\mathcal{S}_t(b) = \{x \in \mathbb{Z}^r \times \mathbb{R}^{n-r} \mid Ax = b,\ x \geq l^t,\ -x \geq -u^t,\ \Pi^t x \geq \Pi_0^t\}$ and u^t, $l^t \in \mathbb{Z}^n_+$ are the branching bounds applied to the integer variables, $\Pi^t \in \mathbb{R}^{\nu(t)\times n}$ and $\Pi_0^t \in \mathbb{R}^{\nu(t)}$.

For each cut k, $k = 1, ..., \nu(t)$, suppose that the subadditive representation F_k^t is known and let the function σ_k^t be defined for (63) as in Section 4.3, considering also the branching bounds. For each feasibly pruned node $t \in T$, let $(v^t, \underline{v}^t, \overline{v}^t, w^t)$ be the corresponding dual feasible solution used to obtain the bound that allowed the pruning of node t. Note that such a solution is always available if the LP relaxations are solved using a dual simplex algorithm. For each infeasibly pruned node $t \in T$, let $(v^t, \underline{v}^t, \overline{v}^t, w^t)$ be a corresponding dual feasible solution that can be obtained from the parent of node t.

Theorem 25. If $b \in \Omega$ and $z(b) > -\infty$, then the function

$$F_{BC}(d) = \min_{t\in T}\{v^t d + \underline{v}^t l^t - \overline{v}^t u^t + \sum_{k=1}^{\nu(t)} w_k^t F_k^t(\sigma_k^t(d))\} \qquad (64)$$

is an optimal solution to the dual (8).

Proof. The proof follows the outline of Wolsey (1981)'s proof for validating an optimal dual function for the branch-and-bound algorithm. Because of the way branch and cut partitions $\mathcal{S}$, we are guaranteed that for any $d \in \Omega$ and $\hat{x} \in \mathcal{S}(d)$, there must exist a leaf node $t \in T$ such that $\hat{x} \in \mathcal{S}_t(d)$. Then, from LP duality,

$$c_j\hat{x}_j \geq v^t a^j \hat{x}_j + \underline{v}_j^t \hat{x}_j - \overline{v}_j^t \hat{x}_j + w^t \Pi_j^t \hat{x}_j \quad j = 1,\ldots,n, \qquad (65)$$

where Π_j^t is the j^{th} column of Π^t. Adding the above inequalities over all columns, we get

$$\begin{aligned} c\hat{x} &\geq v^t A\hat{x} + \underline{v}^t \hat{x} - \overline{v}^t \hat{x} + w^t \Pi^t \hat{x} \\ &\geq v^t d + \underline{v}^t l^t - \overline{v}^t u^t + \sum_{k=1}^{\nu(t)} w_k^t F_k^t(\sigma_k^t(d)) \qquad (66) \\ &\geq F_{BC}(d) \end{aligned}$$

Now assume that x^* is an optimal solution to MILP with right-hand side b. In this case, we know that for some node t^*, $z(b) = cx^* = z^{t^*}(b)$ and we also have that $z(b) \geq z^{t^*}(b)$ for all $t \in T$. Therefore, $F_{BC}(b) = z(b)$. ■

Unfortunately, (64) is not subadditive due to the the constant term resulting from the bounds imposed by branching and hence is not feasible for the subadditive dual (14). One can, however, obtain a subadditive dual function in the case where the original MILP has explicit upper and lower bounds on all variables by including these bounds as part of the right-hand side. Suppose that

$$\overline{z}(\overline{b}) = \min\{cx \mid \overline{A}x \geq \overline{b},\ x \in \mathbb{Z}^r \times \mathbb{R}^{n-r}\} \qquad (67)$$

with $\overline{A} = [A\ I - I]'$ and $\overline{b} = [\,b\ l - u\,]$ where l and u are the lower and upper bounds pre-defined on the variables. With this construction, at each node $t \in T$, we solve the LP relaxation of the following subproblem

$$\begin{aligned} \overline{z}_t(\overline{b}^t) = \min\ & cx \\ \text{s.t. } & \overline{A}x \geq \overline{b}^t \\ & \Pi^t x \geq \Pi_0^t \\ & x \in \mathbb{Z}^r \times \mathbb{R}^{n-r} \end{aligned} \qquad (68)$$

with $\overline{b}^t = [\,b\ l^t - u^t\,]$.

Theorem 26. If $\mathcal{S}(\overline{b}) \neq \emptyset$ and $\overline{z}(\overline{b}) > -\infty$, then the function

$$F_{BCS}(d) = \max_{t\in T}\{\overline{v}^t d + \sum_{k=1}^{\nu(t)} \overline{w}_k^t F_k^t(\sigma_k^t(d))\},\ d \in \mathbb{R}^{m+2n} \qquad (69)$$

is feasible to the subadditive dual problem (14) of the MILP (67).

Proof. For any $t \in T$, LP duality yields

$$c_j \geq \overline{v}^t \overline{a}^j + \sum_{k=1}^{\nu(t)} \overline{w}_k^t \Pi_j^t \quad j = 1,\ldots,n.$$

Therefore, it is clear that $c_j \geq F_{BCS}(\overline{a}^j)$ if $j \in I$ and likewise, $c_j \geq \overline{F}_{BCS}(\overline{a}^j)$ when $j \in C$. In addition, since $F_k^t \in \Gamma^{m+2n+k-1}$, $k = 1, \ldots, \nu(t)$, $F_{BCS} \in \Gamma^{m+2n}$. ■

Note that in this case, the dual function may not be strong. As in Theorem 19, it is not strictly necessary to have a subadditive representation of each cut in order to apply the results of this section. They remain valid as long as a functional dependency of each cut on the right-hand-side is known (see Section 4.3).

5. CONCLUSION

In this paper, we presented a survey of existing theory and methodology for constructing dual functions. Many of the ideas presented here were developed more than three decades ago and it would seem that little progress has been made towards a practical framework. From the standpoint of computational practice, the importance of these methodologies is that they may allow us to extend to the realm of MILP some of the useful techniques already well-developed for linear programming, such as the ability to perform post facto sensitivity analyses and the ability to warm start solution processes. The development of such techniques is the underlying motivation for our work. Constructing a strong dual function for a given MILP is at least as difficult as solving the primal problem, so there is little hope of or use for constructing such functions independent of primal solution algorithms. This leaves two possible options—constructing dual functions that are not necessarily strong, but may still give us some ability to analyze the effect of changes in the input data, or constructing dual functions as a by-product of existing primal solution algorithms, namely branch and cut.

Currently, the techniques discussed in Section 4.7 represent the clearest path to bringing a practical notion of duality to fruition, since these techniques work in tandem with algorithms that are already effective in solving the primal problem. Execution of the branch-and-cut algorithm produces a tremendous amount of dual information, most of which is normally discarded. By retaining this information and using it effectively, one may be able to develop procedures for both sensitivity analysis and warm starting. Ralphs and Guzelsoy (2005) describe an implementation of these techniques, using the

SYMPHONY MILP solver framework, that supports both warm starting and basic sensitivity analysis for branch and bound. They also describe recent results using warm starting to accelerate the solution process for algorithms that involve solution of a sequence of related MILPs (Ralphs and Guzelsoy (2006)). These results are still preliminary, but demonstrate the potential for further development.

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