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Pareto-Optimality of the Balinski Cut for the Uncapacitated Facility Location Problem

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Abstract—The classical 1962 Benders decomposition scheme is a traditional approach for solving mixed-integer problems such as the uncapacitated facility location problem. Subsequent research has been focused on finding better cutting-plane generation schemes to reduce the solution process time. Pareto-optimal cuts are typically preferred because no other cut can dominate them. However, the Pareto-optimal cut generation process typically requires running a separate linear program at each iteration to determine the appropriate dual variables. It is proven here that the dual variable selection scheme proposed in 1965 by Balinski will generate cuts that are always Pareto-optimal for the uncapacitated facility location problem and can be easily generated without running a time-consuming second linear program at each iteration. Direct comparisons between Benders cuts and Balinski cuts for 25 classical problems from the literature provided an empirical indication of the relative advantage of employing Balinski cuts.

Keywords—Integer programming, Facility location, Decomposition, Cutting planes, Facet generation

1. INTRODUCTION

Many strategic business decisions involve distributing goods from a set of production plants or warehouse facilities to a customer or group of customers. Two aspects of the question often need to be determined simultaneously: the location of the plant and warehouse facilities, and the allocation of customers to those facilities. The Uncapacitated Facility Location Problem (UFLP) is typically used to describe the movement of one product within a one-stage distribution system, such as the movement of goods from a plant to a warehouse, or from a warehouse to a customer. Daskin (1995), Francis et al. (1998), and Drezner and Hamacher (2002) provide comprehensive texts for this area.

The UFLP is a mixed-integer programming model and detailed bibliographies of early solution approaches may be found in Francis and Goldstein (1974), Francis et al. (1983), and Cho et al. (1983). Balinski (1965) proposed using Benders (1962) decomposition to solve the UFLP formulated with disaggregated constraints and presented an adjustment procedure to tighten the usual Benders constraints by considering the extra costs that might be incurred by closing the open plants in the solution at the current iteration. This revised cut procedure was considered a weak improvement over the standard Benders cut, but others (Guignard (1980), Magnanti and Wong (1981), Magnanti and Wong (1990), Magnanti et al. (1986)) have subsequently referred to them as strong cuts.

Magnanti and Wong (1981 and 1990) proposed a technique for accelerating the Benders master problem in which a separate linear programming problem is solved at each iteration for choosing from among the alternate optima solutions to the original Benders subproblem. The dual variables chosen by this second linear program are then used to construct a Pareto-optimal cut. The authors recognized the type of cut as proposed by Balinski as stronger than the regular Benders cut, yet stated that there was no assurance that Balinski cuts were Pareto-optimal. Magnanti et al. (1986) gave computational results for a set of network problems, comparing strong (Balinski) cuts and Pareto-optimal cuts with the regular Benders cuts. Both Balinski cuts and Pareto-optimal cuts dominated the regular Benders cut both in terms of solution time and in terms of the number of cuts required. The method using Pareto-optimal cuts required fewer total cuts, but at the expense of longer computational times for solving the separate linear program at each iteration to generate the alternate dual variables used to construct the cut. No clear preference, however, could be discerned between the strong cuts and the Pareto-optimal cuts.

Much research for efficiently solving with heuristics or to optimality the UFLP continues to appear. Al-Sultan and Al-Fawzan (1999) experimented with a tabu search algorithm and found it to be quite effective in finding solutions efficiently. Shaw (1999) formulated the UFLP as a specially-structured tree and developed a general algorithm that matched the best algorithm dedicated to the

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UFLP. Goldengorin et al. (2003) employed a branch-and-bound modification of data correcting at each branch to find a new instance that will be as close as possible to being solvable in polynomial time. Berman and Krass (2005) considered an improved formulation and found significant improvements in computational time for the UFLP with a special structure for the objective function. Hidaka and Okano (2003) developed several promising heuristics for approximating solutions for a large-scale UFLP in the manufacturing industry and Hoefer (2003) developed an experimental design to compare the performance of five heuristics. Gourdin et al. (2000) considered a variant of the UFLP where at most two clients are allocated to a facility. Similar work has surfaced for the capacitated version of the facility location problem where Wentges (1996) introduced a strong Benders' cut, showed it to be Pareto-optimal, and incorporated it into the cross decomposition algorithm devised by Van Roy (1986).

In this paper, two theorems are presented and proven to indicate that a cut constructed as per the Balinski dual variable selection procedure is always a Pareto-optimal cut for the UFLP. These non-dominated cuts are found without the time expense of solving a separate linear program at each iteration of the algorithm as is necessary for Benders decomposition. The CPU time for solving a UFLP is naturally decreased without solving these additional linear programs and makes the Benders cut procedure quite attractive. Preliminary computational experiments indicate that the use of the proposed Pareto-optimal cuts for the UFLP outperform the Benders decomposition scheme in most cases.

The remainder of this article proceeds with a detailed description of the UFLP formulation, its dual, and the Benders decomposition solution procedure for the UFLP in Section 2. In Section 3 is a description of the Balinski cut for the UFLP and these cuts are proven to be Pareto-optimal cuts in Section 4. In Section 5 are some preliminary computational results and a summary including future research directions are in Section 6.

2. THE UNCAPACITATED FACILITY LOCATION MODEL AND DECOMPOSITION APPROACHES

In a practical application for a UFLP, a list of candidate sites for plant locations is often given and the problem can be represented as a typical transportation problem. The location aspect of the problem involves selecting the best subset of sources that should be open, where each candidate source has a fixed cost associated with setting up a facility at that location. In addition, an infinite capacity is assumed to be available at each source, or at least enough capacity to handle all the demands that are ultimately assigned to that source according to the problem's solution. For a UFLP with m facility locations and n customers,

Let $x_{ij} = \begin{cases} 1, & \text{if customer } j \text{ is served by location } i \\ 0, & \text{otherwise} \end{cases}$

$$y_i = \begin{cases} 1, & \text{if location } i \text{ is opened} \\ 0, & \text{otherwise} \end{cases}$$

c_{ij} = total cost of shipping goods from i to j, $c_{ij} \geq 0$

f_i = the fixed costs of opening location i, $f_i \geq 0$

The UFLP may then be represented as the following mixed-integer formulation:

Problem UFLP

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{m} f_i y_i \tag{1}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} x_{ij} = 1 \qquad j = 1, 2, ..., n \tag{2}$$

$$x_{ij} \leq y_i \qquad i = 1, 2, ..., m; j = 1, 2, ..., n \tag{3}$$

$$x_{ij} \geq 0 \qquad i = 1, 2, ..., m; j = 1, 2, ..., n \tag{4}$$

$$y_i \in \{0,1\} \qquad i = 1, 2, ..., m \tag{5}$$

Even though the x_{ij} are not explicitly required to be integer, the structure of the constraint matrix is almost unimodular, and the binary requirements on the y_i allow binary x_{ij} to be found in the optimal solution to Problem UFLP. After an appropriate separation of the variables and constraints, a series of smaller subproblems than the original problem may be constructed with the Benders decomposition technique. The solution process then iterates between a master problem and subproblems until convergence to a unique optimal solution is obtained.

Define $x = \{x_{ij}\}$ and $y = \{y_i\}$ as the assignment vector and the facility vector, respectively. Given a feasible facility vector, $\bar{y}$, a corresponding optimal assignment vector, $\bar{x}$, may be found by solving the primal subproblem, Problem PS, or its dual, Problem DS, with corresponding dual variable vectors $v = \{v_j\}$ for Eq. (7) and $w = \{w_{ij}\}$ for Eq. (8).

Problem PS

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \tag{6}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} x_{ij} = 1 \qquad j = 1, 2, ..., n \tag{7}$$

$$x_{ij} \leq \bar{y}_i \qquad i = 1, 2, ..., m; j = 1, 2, ..., n \tag{8}$$

$$x_{ij} \geq 0 \qquad i = 1, 2, ..., m; j = 1, 2, ..., n \tag{9}$$

Problem DS

$$\max \sum_{j=1}^{n} v_j - \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{y}_i w_{ij} \tag{10}$$

$$\text{s.t.} \quad v_j - w_{ij} \leq c_{ij} \qquad i = 1, 2, ..., m; j = 1, 2, ..., n \tag{11}$$

$$w_{ij} \geq 0 \qquad i = 1, 2, ..., m; j = 1, 2, ..., n \tag{12}$$

Define the index sets $I_o = \{i \mid \bar{y}_i = 1\}$ and $I_c = \{i \mid \bar{y}_i =$

0} as the sets of open and closed plants in $\bar{y}$, respectively. The scheme of Benders (1962) for assigning values to the dual variables, which was shown to be an optimal solution for the set of open (closed) plants represented by $\bar{y}$, is:

$$\bar{v}_j = \min \{c_{ij} \mid i \in I_o\} \quad \forall j \tag{13}$$

$$\bar{w}_{ij} = \begin{cases} 0 & \forall i \in I_o \\ \max\ \{(\bar{v}_j - c_{ij}), 0\} & \forall i \in I_c \end{cases} \quad \forall j \tag{14}$$

The dual variables have the following practical interpretation. Given a set of open plants in $\bar{y}$, $\bar{v}_j$ is chosen to equal the smallest cost of getting goods from the open plant(s) to j, thus satisfying Eq. (11). Since $\bar{w}_{ij}$ = 0 for these open plants, nothing is subtracted from the dual objective Eq. (10), even though the corresponding $\bar{y}_i = 1$. If i is closed, $\bar{w}_{ij}$ is chosen to satisfy both Eq. (11) and Eq. (13) by selecting the larger of the two values, but the choice has no affect on Eq. (10), since $\bar{y}_i = 0$ implies $\bar{y}_i\bar{w}_{ij} = 0$. Thus $\bar{w}_{ij}$ is the minimum additional shipping cost to be incurred to customer j if plant i were to become closed.

The optimal solution to Problem DS provides an upper bound on the objective function of Problem UFLP. More important, the dual variables chosen according to Eq. (13) and Eq. (14) are used to construct a constraint, or cut, in the corresponding master problem. Denote the integer solution to the master problem at the rth iteration as $\bar{y}(r)$, and v_j^r and w_{ij}^r as the dual variables chosen when $\bar{y}(r)$ is substituted for $\bar{y}$ in Problem DS. The resulting inequality Eq. (15) is a constraint which must hold for any y feasible to the master problem, and not just the $\bar{y}(r)$ used to generate the cut.

$$z \geq \sum_{j=1}^{n} v_j^r + \sum_{i=1}^{m}\left[f_i - \sum_{j=1}^{n} w_{ij}^r\right] y_i \tag{15}$$

The master problem is constructed with the objective of minimizing z, subject to the r constraints of form Eq. (15) generated from subproblem solutions and the binary requirements on y_i. At least one plant is required to be open for any y_i for the subproblem to have a solution, as shown below in Eq. (18). At the rth iteration, the Master Problem (MP) is:

Problem MP

$$\min\ z \tag{16}$$

$$\text{s.t.}\quad z \geq \sum_{j=1}^{n} v_j^r + \sum_{i=1}^{m}\left[f_i - \sum_{j=1}^{n} w_{ij}^r\right] y_i \qquad \forall r \tag{17}$$

$$y \in Y = \left\{ y \,\middle|\, y_i \in \{0,1\} \text{ and } \sum_{i=1}^{m} y_i \geq 1 \right\} \tag{18}$$

The usual solution procedure is to relax the binary requirements in Y, and solve this relaxed master problem. There is no guarantee, however, that the optimal solution to this relaxed master problem is integer in the y_i. Additional procedures, such as branch-and-bound, are typically used to produce an optimal solution to the Master Problem. The new integer vector obtained, $\bar{y}(r+1)$, is sent to Problem DS and a new cut is generated and appended to the old Problem MP to create the current Problem MP at iteration $r + 1$. The Benders procedure iterates until the solution of the relaxed master problem is integer.

To solve the minimization problem, one may be tempted to construct all possible constraints of form Eq. (15). But there are $2^m - 1$ feasible points in Y, and the number of constraints in the Master Problem grows quickly. Furthermore, even if all $2^m - 1$ constraints are added, there is no guarantee that the solution to the corresponding relaxed master problem is integer. In practice, while not all $2^m - 1$ constraints are usually required to determine the optimal solution, a large number are often generated before the optimum z is found. Furthermore, much of the computation time is spent in applying heuristics or branch-and-bound techniques at each iteration to produce an integer y that can then be passed to the subproblem. If better cutting strategies could be employed at each iteration, then perhaps fewer total cuts would be needed, and overall computation time would be shortened.

3. THE THEORY FOR BALINSKI CUTS

Consider the m-dimensional facility vector $\bar{y}$ that is used to construct a cut, where the 1's and 0's represent the open and closed plants, respectively. Define $k = |I_o|$ and $m - k = |I_c|$. The number of open plants, k, is called the level of $\bar{y}$. The notation $\bar{y}^k$ will be used to indicate that k plants are open in the $\bar{y}$ under consideration. A neighbor of $\bar{y}^k$ is defined to be a vector whose components differ in a predictable pattern from the given $\bar{y}^k$. The vector $\bar{y}^{k-1}$ is called a negative neighbor of $\bar{y}^k$ if all closed plants in $\bar{y}^k$ are closed in $\bar{y}^{k-1}$, and exactly one of the open plants in $\bar{y}^k$ is closed as well, while all other open plants in $\bar{y}^k$ remain open in $\bar{y}^{k-1}$. The level of such a negative neighbor is $k - 1$. Any $\bar{y}^k$ has exactly k negative neighbors.

For any negative neighbor of $\bar{y}^k$, if source $q \in I_o$ is the plant currently closed, then J_q is the set of customers that need a new supplier, i.e., for any $i \in I_o$, the index set $J_i = \{j \mid \bar{x}_{ij} = 1\}$ is the set of customers that are being served from any open source in $\bar{y}$. The increased cost of serving $j \in J_q$ will be at least $\Delta_{qj} = \max\{\min_{p \neq q}(c_{pj} - v_j), 0\}$, since j must be served by some other source p. Since any cut that is constructed must hold for all possible y, p must be allowed to range over all remaining plants so that $p \in I_o \cup I_c \backslash \{q\}$. Hence Δ_{qj} is a lower bound on the additional cost of shipping goods to j if q becomes closed.

Using the index sets, Benders constraint Eq. (15) can be rewritten as:

$$z \geq \sum_{j=1}^{n} v_j + \sum_{i \in I_0} f_i + \sum_{i \in I_0} f_i (y_i - 1) + \sum_{i \in I_c} \left[f_i - \sum_{j=1}^{n} w_{ij} \right] y_i \quad (19)$$

Balinski (1965) recognized that alternate optima existed for Problem DS and proposed an adjustment to the dual variable selection rules to effectively tighten Eq. (19) at all the negative neighbors of $\bar{y}^k$ used to construct the cut. Recall that the solution to Problem PS results in a vector of assignments, $\bar{x}$, that are optimal for $\bar{y}$, where $\bar{x}_{ij}$ = 1 for assignments that are used, and $\bar{x}_{ij}$ = 0 for assignments not used.

Balinski (1965) proposed retaining part of the natural Benders scheme and set $v_j = c_{qj}$, where c_{qj} is the lowest shipping cost for getting goods to j for all $q \in I_o$. If in addition, c_{qj} is also the lowest shipping cost to j in $I_o \cup I_c$, then c_{pj} is the second lowest overall cost, and $c_{pj} - c_{qj} \geq 0$. The penalty cost for closing q with respect to j will be at least $\Delta_{qj} = (c_{pj} - c_{qj}) \geq 0$, since it cannot be known for sure that p will be open in any optimal $\hat{y} \in Y$ (where $\hat{y} \neq \bar{y}^k$), and the actual extra cost may even be more.

Alternatively, if $c_{qj} > c_{pj}$ for at least one p, then p is chosen such that c_{pj} is the overall cheapest source for j, and $c_{pj} - c_{qj} < 0$. Consequently $p \in I_c$ (else v_j would have been chosen equal to c_{pj} instead of c_{qj}), and the penalty $\Delta_{qj} = 0$, since one would expect a savings if p were open. Since it is not known if $c_{pj} \geq c_{qj}$ or $c_{pj} < c_{qj}$, the max(·) operation for Δ_{qj} will ensure that the term is nonnegative, and that the smallest additional shipping cost for each $j \in J_q$ is found. The total extra shipping cost from closing q is at least:

$$\sum_{j \in J_q} \Delta_{qj} = \sum_{j \in J_q} max \left\{ \min_{\substack{p \neq q \\ p \in I_o \cup I_c \setminus \{q\}}} (c_{pj} - v_j), 0 \right\} \quad (20)$$

The contribution of Balinski (1965) was to modify Eq. (19) to include the adjustment term of Eq. (20) to yield the following constraint, hereinafter referred to as a Balinski cut.

$$\begin{aligned} z \geq & \sum_{j=1}^{n} v_j + \sum_{i \in I_0} f_i \\ & + \sum_{i \in I_0} \left[f_i - \sum_{j \in J_i} \Delta_{ij} \right] (y_i - 1) + \sum_{i \in I_c} \left[f_i - \sum_{j=1}^{n} w_{ij} \right] y_i \end{aligned} \quad (21)$$

When the Balinski cut constructed with $\bar{y}^k$ as the point used for generating the cut is evaluated at $\bar{y}^k$, the value of z obtained is the same as for the corresponding Benders cut, because Eq. (20) is not affected by any change in I_c. But when Eq. (21) is evaluated at some $\bar{y}^{k-1}$, for that i which is now closed, $\bar{y}_i^{k-1}$ = 0, the fixed cost for this closed facility is subtracted, and the additional shipping costs are added. The value of z calculated for a negative neighbor with a Balinski cut will then be greater than or equal to (and never less than) the value of z calculated by the corresponding Benders cut. The Balinski cut has been strengthened, and is thus an improvement.

There is no guarantee that the Balinski cut results in the true value of z for any $\bar{y}^{k-1}$. The p that is the least cost source for $j_1 \in J_q$ may not be the same p found for some other $j_2 \in J_q$, and the net effect of that interaction may result in an understatement of the total extra shipping costs to be incurred if $q \in I_o$ is closed. Furthermore, if two or more plants are closed when moving from $\bar{y}^k$ to some other solution vector, the additional shipping costs required may be understated by Eq. (21) because the interactions between plant closings are not considered.

4. PARETO-OPTIMALITY OF THE BALINSKI CUT

Magnanti and Wong (1981 and 1990) developed a technique for accelerating the Benders master problem, in which Problem DS is first solved for a given facility vector $\bar{y}$ to determine initial values for the dual variables. The approach requires that an additional, separate linear programming problem be solved in order to choose from among the alternate optimal solutions to Problem DS. The dual variables resulting from solving this second linear program are then used to construct a cut for Problem MP that is called a Pareto-optimal cut. Here, the Balinski cut is shown to be such a Pareto-optimal cut for Problem UFLP. Another advantage of using the Balinski cut is that the dual variables needed can be readily found without solving a second linear program at each iteration of the master problem. To develop some necessary background, the following definitions from Magnanti and Wong (1990) are provided.

Definition 1. Given two sets of dual variable solutions to a subproblem, u^1 and u^2, the cut $z \geq f(u^1) + yg(u^1)$ is said to dominate or is considered stronger than the cut $z \geq f(u^2) + yg(u^2)$ if $f(u^1) + yg(u^1) \geq f(u^2) + yg(u^2)$ for all y in the polytope P, with strict inequality for at least one $y \in P$. A cut is Pareto-optimal if no cut dominates it. In addition, a dual variable solution u^1 is said to dominate u^2 if the associated cut is stronger, and u^1 is called Pareto-optimal if its corresponding cut is Pareto-optimal.

Let Y^c be the relative interior of the convex hull of the y-space of Problem MP.

Definition 2. A point y^o contained in Y^c is called a core point.

A core point is thus a fractional solution to the relaxed master problem that meets the following additional conditions: each component $y_i^o \in y^o$ must be fractional so that $0 < y_i^o < 1$; the vector y^o must be feasible to the relaxed master problem and in the relative interior of Y^c; and thus $\sum_{i=1}^{m} y_i^o > 1$. Depending upon the choice of y^o, it may be possible to construct many cuts with a given $\overline{y}$ that do not dominate each other. By definition, a facet must be a Pareto-optimal cut since a facet cannot be dominated.

Magnanti and Wong (1981 and 1990) proposed using Pareto-optimal cuts to solve Problem UFLP. The first step was to solve the regular Benders subproblem for a given vector $\overline{y}$ and generate the natural dual variables as per Eq. (13) and Eq. (14). However, because of the network structure of the assignment subproblem, an optimal solution will often have a highly degenerate basis, implying that alternate optimal solutions exist for the dual subproblem. The second step was to choose a Pareto-optimal set of dual variables from among these alternate optima by solving Problem PO for a chosen core point y^o.

Problem PO

$$\max \sum_{j=1}^{n}\left[v_j - \sum_{i=1}^{m} y_i^o w_{ij}\right] \tag{22}$$

$$\text{s.t. } v_j - \sum_{i=1}^{m} \overline{y}_i w_{ij} = \overline{v}_j \qquad j = 1, ..., n \tag{23}$$

$$v_j - w_{ij} \le c_{ij} \qquad i = 1, ..., m; j = 1, ..., n \tag{24}$$

$$v_j \ge 0 \qquad j = 1, ..., n \tag{25}$$

$$w_{ij} \ge 0 \qquad i = 1, ..., m; j = 1, ..., n \tag{26}$$

Note that Problem PO is separable by customer j. Furthermore, Eq. (23) may be substituted into Eq. (22) and Eq. (24) to create the jth Subproblem POj:

Subproblem POj

$$\max \ \overline{v}_j + \sum_{h=1}^{m} (\overline{y}_h - y_h^o) w_{hj} \tag{27}$$

$$\text{s.t. } \overline{v}_j + \sum_{h=1}^{m} \overline{y}_h w_{hj} - w_{ij} \le c_{ij} \qquad i = 1, ..., m \tag{28}$$

$$v_j \ge 0 \qquad j = 1, ..., n \tag{29}$$

$$w_{ij} \ge 0 \qquad i = 1, ..., m \tag{30}$$

After solving each of the n subproblems, POj, a Pareto-optimal set of dual variables, $[v, w]$, is found. Magnanti and Wong (1981 and 1990) noted that varying the core point y^o may generate different Pareto-optimal cuts from the same original vector $\overline{y}$. It will now be shown that for a given $\overline{y}$, there exists a core point, y^o, for which the Balinski cut constructed with $\overline{y}$ is equal to the Pareto-optimal cut generated from y^o. The method of proof requires that the pattern of dual variables for a particular choice of $\overline{y}$ and y^o be examined.

For a given $\overline{y}$, the corresponding index sets I_o and I_c are used to rewrite the objective function and constraints in Subproblem POj. The jth subproblem now consists of Eq. (31), Eq. (32), and the non-negativity requirements for the dual variables.

$$\max \ \overline{v}_j + \sum_{q \in I_o} (\overline{y}_q - y_q^o) w_{qj} + \sum_{p \in I_c} (\overline{y}_p - y_p^o) w_{pj} \tag{31}$$

$$\sum_{q \in I_o} \overline{y}_q w_{qj} + \sum_{p \in I_c} \overline{y}_p w_{pj} - w_{ij} \le c_{ij} - \overline{v}_j \qquad i = 1, ..., m \tag{32}$$

Since $\overline{y}_q = 1$ for all $q \in I_o$, the coefficients in the second term of Eq. (31) will all be positive. Since $\overline{y}_p = 0$ for all $p \in I_c$, the coefficients in the third term of Eq. (31) all reduce to $-y_p^o$. But these coefficients are all negative, since each fractional component of a core point is strictly greater than zero. To maximize the value of the objective function, an optimal choice for the dual variables is needed. If feasible to constraints Eq. (32), $w_{pj} = 0$ for all $p \in I_c$ would be optimal. Also, all the w_{qj} for $q \in I_o$ should be set as large as the constraints will allow. Since $\overline{y}_p = 0$ for all $p \in I_c$, the second summation of Eq. (32) is zero. But since $\overline{y}_q = 1$ for all $q \in I_o$, Eq. (32) reduces to:

$$\sum_{q \in I_o} w_{qj} - w_{ij} \le c_{ij} - \overline{v}_j \qquad i = 1, ..., m. \tag{33}$$

The pattern of the w_{ij} will now be examined by expanding Eq. (33). For the first group of k constraints associated with I_o, $k = |I_o|$, the $-w_{ij}$ term can be combined with its corresponding term in the expansion of the summation to produce the following pattern of constraints:

$$w_{2j} + w_{3j} + w_{4j} + \ldots + w_{(k-1)j} + w_{kj} \le c_{1j} - \overline{v}_j \tag{34}$$

$$w_{1j} + w_{3j} + w_{4j} + \ldots + w_{(k-1)j} + w_{kj} \le c_{2j} - \overline{v}_j \tag{35}$$

$$w_{1j} + w_{2j} + w_{4j} + \ldots + w_{(k-1)j} + w_{kj} \le c_{3j} - \overline{v}_j \tag{36}$$

$$w_{1j} + w_{2j} + w_{3j} + \ldots + w_{(k-1)j} + w_{kj} \le c_{4j} - \overline{v}_j \tag{37}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$w_{1j} + w_{2j} + w_{3j} + w_{4j} + \ldots + w_{(k-1)j} \le c_{kj} - \overline{v}_j \tag{38}$$

In the second group of $m - k$ constraints associated with I_c, $m - k = |I_c|$, the $-w_{ij}$ term has no corresponding term from the expansion of the summation with which to combine:

$$w_{1j} + w_{2j} + w_{3j} + w_{4j} + \ldots + w_{(k-1)j} + w_{kj} - w_{(k+1)j} \le c_{(k+1)j} - \overline{v}_j \tag{39}$$

$$w_{1j} + w_{2j} + w_{3j} + w_{4j} + \ldots + w_{(k-1)j} + w_{kj} - w_{(k+2)j} \le c_{(k+2)j} - \overline{v}_j \tag{40}$$

$$w_{1j} + w_{2j} + w_{3j} + w_{4j} + \ldots + w_{(k-1)j} + w_{kj} - w_{(k+3)j} \le c_{(k+3)j} - \overline{v}_j \quad (41)$$

$$\vdots$$

$$w_{1j} + w_{2j} + w_{3j} + w_{4j} + \ldots + w_{(k-1)j} + w_{kj} - w_{mj} \le c_{mj} - \overline{v}_j \quad (42)$$

Assume without loss of generality that the first open plant has the cheapest shipping cost for delivering goods to plant j, so that $c_{1j} = \overline{v}_j = \min\{c_{ij} | i \in I_o\}$. But then the right-hand side of Eq. (34) becomes zero, whereupon the non-negativity conditions imply that the dual variables w_{2j} through w_{kj} must all be set equal to zero. Constraints Eq. (35)–(38) are thus simplified, and can be rewritten as Eq. (43) to impose an upper bound condition on w_{1j}:

$$0 \le w_{1j} \le \min_{p=2,\ldots,k} (c_{pj} - \overline{v}_j) \quad (43)$$

For Eq. (43), note that each $(c_{pj} - \overline{v}_j) \ge 0$, since c_{pj} was not the cheapest shipping cost to customer j from the open plants. Furthermore, once w_{2j} through w_{kj} are set to zero, Eq. (39)–(42) may also be simplified, and combined with the non-negativity conditions to produce the constraints:

$$0 \le w_{1j} \le (c_{(k+1)j} - \overline{v}_j) + w_{(k+1)j} \quad (44)$$

$$0 \le w_{1j} \le (c_{(k+2)j} - \overline{v}_j) + w_{(k+2)j} \quad (45)$$

$$0 \le w_{1j} \le (c_{(k+3)j} - \overline{v}_j) + w_{(k+3)j} \quad (46)$$

$$\vdots \qquad \vdots$$

$$0 \le w_{1j} \le (c_{mj} - \overline{v}_j) + w_{mj} \quad (47)$$

To maximize Eq. (31), the value of w_{1j} should be chosen as large as possible. It would also be desirable to set all of the dual variables $w_{(k+1)j}$ through w_{mj} equal to zero. There are two cases to consider: 1) If any $(c_{hj} - \overline{v}_j) \ge 0$ for $h = k + 1$ through m in constraints Eq. (44) – (47), then there is a feasible, nonnegative solution for w_{1j}, and the corresponding w_{hj} may now be set equal to zero. The resulting condition, $0 \le w_{1j} \le (c_{hj} - \overline{v}_j)$, must also be imposed on w_{1j} and is of the same form as Eq. (43). 2) If any $(c_{hj} - \overline{v}_j) < 0$, then w_{hj} must be set at least equal to $|c_{hj} - \overline{v}_j|$ so that $w_{1j} \ge 0$ as required by non-negativity.

We now develop two theorems. Theorem 1 is a sufficient condition for the w_{hj} to result in a Pareto-optimal cut. Theorem 2 is a sufficient condition for the existence of a core point y^o, such that the Balinski cut constructed with $\overline{y}$ will be equal to the Pareto-optimal cut generated from using y^o.

Theorem 1. Given an m-dimensional facility vector $\overline{y} = [1, 1, 1, ..., 1; 0, 0, 0, ..., 0]$, with level $= k$ and a core point $y^o = [\eta_1, \eta_2, \eta_3, ..., \eta_k, \quad \beta_{k+1}, \beta_{k+2}, \beta_{k+3}, ..., \beta_m]$ that meets the following additional condition: $1 - \eta_q < \beta_p$, for all $q = 1, \ldots, k$ and $p = k + 1, \ldots, m$. With $\overline{v}_j - c_{hj} > 0$, an optimal solution to Subproblem POj includes setting $w^*_{hj} = (\overline{v}_j - c_{hj})$ so that $w^*_{1j} = 0$.

Proof. Since y^o is a core point, the following conditions must also hold:

a) The feasibility requirement for Problem MP: $m > \sum_{q \in I_o} \eta_q + \sum_{p \in I_c} \beta_p > 1$.

b) $0 < \eta_q < 1$ and $0 < \beta_p < 1$, for all $q \in I_o$ and all $p \in I_c$.

For this choice of core point, Subproblem POj becomes Eq. (48), subject to Eq. (32), Eq. (29), and Eq. (30).

$$max\ \overline{v}_j + \sum_{q \in I_o} \left(1 - \eta_q\right) w_{qj} - \sum_{p \in I_c} \beta_p w_{pj} \quad (48)$$

However, for a Pareto-optimal cut, Eq. (32) may be replaced by Eq. (43) and Eq. (44)–(47). Assume without loss of generality that $c_{1j} = \min\{c_{ij} | i \in I_o\}$, and $\overline{v}_j$ is set equal to c_{1j}. Then Eq. (43) is satisfied. If any $(c_{hj} - \overline{v}_j) \ge 0$ for $h = k + 1$ through m, there is a feasible, nonnegative solution for w_{1j} from Eq. (44)–(47).

Suppose there exists some $(c_{hj} - \overline{v}_j) < 0$ for $h = k + 1$ through m. Define $w^*_{hj} = (\overline{v}_j - c_{hj})$ and suppose that w_{hj} is set strictly greater than $(\overline{v}_j - c_{hj})$ in a feasible solution to Subproblem POj. Let s represent the amount of surplus, so that the following constraint may be derived for w_{1j}:

$$0 \le w_{1j} \le (c_{hj} - \overline{v}_j) + w_{hj} = (c_{hj} - \overline{v}_j) + s + w^*_{hj} = s \quad (49)$$

Then w_{1j} could be made as large as s. But $1 - \eta_1 < \beta_h$ by definition. Furthermore, since $(c_{hj} - \overline{v}_j) < 0$, then $w^*_{hj} = (\overline{v}_j - c_{hj}) > 0$, which further implies:

$$(1 - \eta_1) w_{1j} \le (1 - \eta_1) s < \beta_h s < \beta_h s + \beta_h w^*_{hj} = \beta_h w_{hj} \quad (50)$$

When Eq. (48) is evaluated at the claimed optimal choice of dual variables $w^*_{1j} = 0$ and $w^*_{hj} = (\overline{v}_j - c_{hj})$, the objective value z_1 is produced for Subproblem POj.

$$z_1 = \overline{v}_j + (1 - \eta_1) w^*_{1j} - \beta_h w^*_{hj} = \overline{v}_j - \beta_h w^*_{hj} \quad (51)$$

But when Eq. (48) is evaluated at any other feasible choice of w_{hj}, such as $s + w^*_{hj}$, the objective function value z_2 is produced:

$$z_2 = \overline{v}_j + (1 - \eta_1) w_{1j} - \beta_h (s + w^*_{hj}) \quad (52)$$

Since $w_{1j} \le s$ from Eq. (49), then $z_2 \le z_S$, where

$$z_S = \bar{v}_j + (1-\eta_1)s - \beta_h(s + w_{hj}^*)$$
$$= \bar{v}_j - \beta_h w_{hj}^* + (1-\eta_1-\beta_h)s \tag{53}$$

The last term in this relation is strictly negative, thus $z_2 \le z_S < z_1$, and the claim is proven. Therefore, $w_{1j}^* = 0$ and $w_{hj}^* = (\bar{v}_j - c_{hj})$.

To summarize Theorem 1, note that given $\bar{v}_j = c_{1j} = \min\{c_{ij} | i \in I_o\}$, an optimal choice for the w_{hj} for the Pareto-optimal cut is as follows:

$$w_{1j} = \max\left[0,\ \min(c_{pj} - \bar{v}_j)\right] \qquad \text{for } p = 2, \ldots, m \tag{54}$$

$$w_{2j} = w_{3j} = \ldots = w_{kj} = 0 \tag{55}$$

$$w_{hj} = \begin{Bmatrix} 0 & \forall(c_{hj} - \bar{v}_j) \ge 0 \\ \bar{v}_j - c_{hj} & \forall(c_{hj} - \bar{v}_j) < 0 \end{Bmatrix} \tag{56}$$

for $h = k + 1, \ldots, m$

These dual variables have the following economic interpretation. Variable w_{1j} from Eq. (54) is for determining the potential extra cost of serving customer j if plant 1 is closed, and may be interpreted as a penalty term to be imposed for closing plant 1. Variables w_{2j} through w_{kj} are zero because there is no extra shipping cost to customer j, and thus no penalty to be imposed, if one of these open plants are closed. Finally, variables w_{hj} are the shipping cost savings that may be obtained by opening the closed plant h. Note that this interpretation parallels the interpretation of the w_{ij} for the Balinski cut discussed earlier.

Theorem 2. Given an m-dimensional facility vector $\bar{y}$ = [1, 1, 1, ..., 1; 0, 0, 0, ..., 0], with level = k and core point y^o = [η_1, η_2, η_3, ..., η_k, β_{k+1}, β_{k+2}, β_{k+3}, ..., β_m] that meets the following additional condition: $1 - \eta_q < \beta_p$, for all $q \in I_o$ and all $p \in I_c$. Then a Pareto-optimal cut constructed with $\bar{y}$ and this core point y^o, according to Eq. (54) – (56), is the same constraint as the Balinski cut constructed with $\bar{y}$.

Proof. Assume without loss of generality that $\bar{v}_j = c_{1j} = \min\{c_{ij} | i \in I_o\}$. Since the core point y^o meets the conditions of Theorem 1, the Pareto-optimal conditions Eq. (54)–(56) for the w_{ij} are satisfied. The Pareto-optimal value for v_j is then computed from Eq. (23), rearranged as:

$$v_j = \bar{v}_j + \sum_{i=1}^{m} \bar{y}_i w_{ij} \tag{57}$$

But the second term of Eq. (57) may be rewritten over I_o and I_c.

$$v_j = \bar{v}_j + \sum_{q \in I_o} \bar{y}_q w_{qj} + \sum_{p \in I_c} \bar{y}_p w_{pj} \tag{58}$$

Since $\bar{y}_p$ = 0 for all $p \in I_c$, the third term of the right-hand side of Eq. (58) is zero. Since $\bar{y}_q$ = 1 for all $q \in I_o$, and the dual variables w_{2j} through w_{kj} are all zero, the second term of Eq. (58) is simplified to yield:

$$v_j = \bar{v}_j + w_{1j} \tag{59}$$

Given the assumption that plant 1 was the cheapest source for customer j, the Pareto-optimal solution is to construct the Benders cut based on the v_j from Eq. (59) and the w_{ij} from Eq. (54)–(56). But this produces the same cut coefficients as the set of dual variable solutions for the Balinski cut, where the penalty term $w_{1j} = \Delta_{1j}$, and the savings terms $w_{hj} = w_{pj}$ for all $p \in I_c$. The initial assumption may be relaxed by requiring only that some plant $q \in I_o$ be the current cheapest source for customer j. Thus there exists a core point such that the Balinski cut is Pareto-optimal.

As a result of Theorem 2, the Balinski choices for the dual variables for the jth subproblem given in Eq. (60)–(62) are thus provably Pareto-optimal:

$$v_j = \min\ \{c_{ij} \mid i \in I_o\} \tag{60}$$

$$w_{ij} = \begin{Bmatrix} 0 & \forall i \in I_o \\ \max\ \{(\bar{v}_j - c_{ij}), 0\} & \forall i \in I_c \end{Bmatrix} \tag{61}$$

$$\Delta_{qj} = \max\ \{\min\ (c_{pj} - v_j),\ 0\} \qquad \text{for } p \in I_o \cup I_c \setminus \{q\} \tag{62}$$

The condition $1 - \eta_q < \beta_p$, for all pairs of open and closed plants, imposed upon the core point in Theorem 2 is not very restrictive. For example, one may choose a core point where all η_q = .8 and all β_p = .9. This satisfies the additional condition imposed upon the core point in the theorem, as well as the feasibility and fractionality conditions of a core point. If all components of y^o are increased, such that each η_q = .998 and each β_p = .999, the new core point still satisfies all the conditions of the theorem, and y^o approaches the vector $\bar{y}^m$, representing the case of all plants open. Alternatively, one may choose to set all η_q = .01 and all β_p = .99, and still satisfy all the conditions of Theorem 2. The resulting core point approaches the facility vector $\bar{y}^c$ =[0, 0, 0, ..., 0; 1, 1, 1, ..., 1], with level $c = m - k$. This $\bar{y}^c$ may be considered a complement of the original $\bar{y}$. Thus, it is relatively easy to construct a core point that meets the conditions of the theorem.

5. PRELIMINARY COMPUTATIONAL RESULTS

A study was performed to compare the effectiveness of the traditional Benders cut method with the Balinski cut method. For each algorithm, the total number of cuts employed to find the solution, the total number of pivots employed to find the solution, the solution times in seconds needed for the algorithm, and whether the optimal solution was found or not was recorded. Solutions were

Table 1. Comparisons of Benders cuts and Balinski cuts for 25 test problems

	Results with Benders Cuts				Results with Balinski Cuts			
	Total Solution				Total Solution			
Problem	Cuts Used	Total Pivots	Time (Seconds)	Optimal Found?	Cuts Used	Total Pivots	Time (Seconds)	Optimal Found?
KHS1	33	586,252	1,890.31	No	11	804	5.05	Yes
KHS2	68	511,792	2,328.24	Yes	12	919	5.10	Yes
KHS3	33	70,146	234.59	No	16	2,452	12.91	Yes
KHS4	19	6,152	21.36	Yes	12	442	3.24	Yes
KHS5	79	1,813,794	10,705.76	Yes	11	344	3.07	Yes
KHS6	57	577,096	2,479.77	Yes	11	488	3.40	Yes
KHS7	47	219,736	844.53	Yes	10	780	3.96	Yes
KHS8	50	123,905	520.65	Yes	16	4,430	14.29	Yes
KHS9	60	1,109,337	5,172.56	Yes	83	1,572	8.18	Yes
KHS10	25	51,694	150.33	Yes	14	446	4.40	Yes
KHS11	17	3,956	17.36	Yes	10	195	2.09	Yes
KHS12	20	3,125	15.05	Yes	9	65	1.15	Yes
KHS13	72	1,668,777	9,039.92	Yes	10	302	2.52	Yes
KHS14	78	1,027,194	5,644.09	Yes	10	288	2.20	Yes
KHS15	82	618,010	3,494.74	Yes	20	9,733	38.40	Yes
KHS16	45	196,917	735.02	Yes	17	3,847	13.18	Yes
SE1	12	123,130	329.06	No	10	194	1.75	Yes
SE2	44	*	>57,600	No	37	4,295,215	15,965.08	Yes
SE3	94	**	>86,400	No	48	2,068,154	8,607.43	Yes
SE4	103	9,154,431	67,912.68	No	54	1,923,023	8,393.38	Yes
SE5	103	3,908,558	28,265.32	No	74	1,993,787	11,362.54	Yes
SE6	102	2,664,779	19,837.94	Yes	62	901,347	4,507.13	Yes
SE7	70	1,165,609	6,277.49	Yes	43	374,700	1,472.18	Yes
SE8	25	128,000	385.31	Yes	19	45,289	122.82	Yes
SE9	21	56,743	164.95	Yes	15	19,528	58.22	Yes

KHS1–KHS16 are problems from Kuehn and Hamburger (1963) and Sá (1969).

SE1–SE9 are problems from Schrage (1975) and Erlenkotter (1978).

* – Terminated because of ≥ 10 million pivots after 16 hours.

** – Terminated because of ≥ 10 million pivots after 24 hours.

obtained with a user-written interface for LINDO and solved on a personal computer. Problem solution techniques were specifically designed to indicate the relative performances of Benders cuts vs. Balinski cuts in a controlled and common setting and not designed to be competitive with state-of-the-art solution procedures with respect to number of cuts, number of pivots, and solution times but were. First, five small problems were solved to verify the programming and check for initial differences. These problems included an $m = n = 4$ problem from Manne (1964), an $m = n = 10$ problem from Spielberg (1969), an $m = 5$ and $n = 8$ problem from Khumawala (1972 and 1973), an $m = 5$ and $n = 4$ problem from Bilde and Krarup (1977), and an $m = 3$ and $n = 5$ problem from Martin (1998). There were no discernible differences between Benders and Balinski cut methods for these problems with respect to number of cuts, number of pivots, or solution times.

The methods were then compared with two larger sets of problems. Kuehn and Hamburger (1963) and Sá (1969) together developed 16 problems, each with $m = 25$ and $n = 50$, for four different levels of fixed costs and three different configurations. Schrage (1975) and Erlenkotter (1978) both considered an $m = n = 33$ problem of Karg and Thompson (1964) for various levels of fixed costs. In Table 1 are the results of the differences obtained with the two cutting procedures for these problems. Note that for this initial comparison, different solution times were employed as the stopping criteria for the problems.

All of the problems converged to an optimal solution when Balinski cuts were employed while seven of 25 problems failed to converge to an optimal solution with Benders cuts. Employing Balinski cuts resulted in the number of cuts, number of pivots, and solution times being clearly dominant for every case except for the one outlier, problem KHS9, where the total number of cuts employed was greater for the Balinski procedure, but the total number of pivots and overall solution time were dominated by the Balinski procedure.

6. SUMMARY AND FUTURE RESEARCH DIRECTIONS

In this paper a Balinski cut constructed by employing any feasible facility vector for the master problem of a UFLP was shown to be a Pareto-optimal cut. Furthermore, these Balinski cuts can be easily generated, without running a time consuming second linear program at each iteration to determine the appropriate dual variables used to construct such a Pareto-optimal cut. Initial computational results were provided to indicate the relative efficiency of Balinski cuts over Benders cuts.

Pareto-optimality of Balinski cuts for the UFLP, and thus the natural preference of Balinski cuts over Benders cuts, are nice properties. An indication of where more promising approaches to solving the UFLP might be found is indicated. Now the challenge is to explore incorporating Balinski cuts and/or variants of it to produce a competitive algorithm(s) for efficiently solving the UFLP. Comparisons to current state-of-the-art methodologies for solving the UFLP, such as branch-and-bound methods based on Lagrangian relaxation incorporating dual ascent and subgradient optimization, are also worthy research endeavors.

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