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Second-Order Symmetric Duality for Minimax Mixed Integer Programs over Cones

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Abstract—A duality theorem for a pair of Wolfe-type second-order minimax mixed integer symmetric dual programs over cones is proved under separability and η-bonvexity/η-boncavity of the function $k(x, y)$ appearing in the objective, where $k: R^n \times R^m \mapsto R$. Mond-Weir type symmetric duality over cones is also studied under η-pseudobonvexity/η-pseudoboncavity assumptions. Self duality (when the dual problem is identical to the primal problem) theorems are also obtained.

Keywords—Integer programming, Symmetric duality, Minimax, Self duality, η-bonvexity

1. INTRODUCTION

Symmetric duality in mathematical programming, in which dual of the dual is the primal problem, was first introduced by Dorn (1960). He studied symmetric dual quadratic programs. Subsequently, Dantzig et al. (1965) extended the notion of symmetric duality by considering the problems:

Primal (*P*) Min $F = k(x, y) - y^T \nabla_y k(x, y)$

subject to $\nabla_y k(x, y) \leq 0$

$x \geq 0, y \geq 0$

Dual (*D*) Max $G = k(x, y) - x^T \nabla_x k(x, y)$

subject to $\nabla_x k(x, y) \geq 0$

$x \geq 0, y \geq 0$

where $x \in R^n$, $y \in R^m$ and $k: R^n \times R^m \mapsto R$ is a twice differentiable function, called kernel function in the literature. $\nabla_x k(x, y)$ and $\nabla_y k(x, y)$ denote the gradient vectors of k with respect to x and y, respectively.

For the above primal (*P*) and dual (*D*) problems, weak and strong duality results were obtained assuming $k(x, y)$ to be convex in x, for each y, and concave in y, for each x. These problems are general nonlinear programming programs. By substituting

$$k(x, y) = c^T x + b^T y - y^T Ax + \frac{1}{2}(x^T Dx - y^T Ey),$$

where $c \in R^n$, $b \in R^m$, A is an $m \times n$ matrix and D and E are $n \times n$ and $m \times m$ positive semidefinite symmetric matrices respectively, we obtain the following problems studied by Dorn (1960) and Cottle (1963):

Primal (*DP*) Min $F = c^T x + \frac{1}{2}(x^T Dx + y^T Ey)$

subject to $Ax + Ey \geq b$

$x \geq 0, y \geq 0$

Dual (*DD*) Max $G = b^T y - \frac{1}{2}(x^T Dx + y^T Ey)$

subject to $A^T y - Dx \leq c$

$x \geq 0, y \geq 0$

Bazaraa and Goode (1973) extended the work of Dantzig et al. (1965) over arbitrary cones.

Balas (1970) introduced minimax symmetric dual programs by constraining some of the primal and dual variables of problems (*P*) and (*D*) to belong to arbitrary sets, for instance, the set of integers. The problems studied by Balas are

Primal Max$_{x^1}$ Min$_{x^2, y}$ $k(x, y) - (y^2)^T \nabla_{y^2} k(x, y)$

subject to $\nabla_{y^2} k(x, y) \leq 0$

$x^1 \in U,\ y^1 \in V$

$x^2, y^2 \geq 0$

Dual Min$_{y^1}$ Max$_{x, y^2}$ $k(x, y) - (x^2)^T \nabla_{x^2} k(x, y)$

subject to $\nabla_{x^2} k(x, y) \geq 0$

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$$x^1 \in U,\ y^1 \in V$$
$$x^2, y^2 \geq 0$$

where $\nabla_{x^2} k(x, y)$ and $\nabla_{y^2} k(x, y)$ stand for the vectors of partial derivatives of k in the components of $x^2 \in R^{n-n_1}$ and $y^2 \in R^{m-m_1}$, respectively, and U, V are the arbitrary sets of integers in R^{n_1} and R^{m_1}, respectively.

Consider now the following nonlinear programming problems:

$$(P1)\ \text{Min}\ \ f(x)$$
$$\text{subject to}\ \ g(x) \leq 0$$

where the functions $f : R^n \mapsto R$ and $g : R^n \mapsto R^m$ are differentiable on R^n.

Wolfe (1961) introduced the following dual for (P1):

$$(WD1)\ \text{Max}\ \ f(x) + y^T g(x)$$
$$\text{subject to}\ \ \nabla_x f(x) + y^T \nabla_x g(x) = 0$$
$$y \geq 0$$

and proved duality theorems assuming convexity of the functions f and g. Observing that the duality theorems obtained by Wolfe for the pair (P1) and (WD1) do not hold for generalized convex (pseudoconvex and quasiconvex, defined below) functions, Mond and Weir (1981) introduced the following dual:

$$(MD1)\ \text{Max}\ \ f(x)$$
$$\text{subject to}\ \ \nabla_x f(x) + y^T \nabla_x g(x) = 0$$
$$y^T g(x) \geq 0$$
$$y \geq 0$$

and obtained duality relations under the assumptions that f is pseudoconvex and g is quasiconvex.

A function f is said to be pseudoconvex if for all $x,\ u \in R^n$,

$$(x-u)^T \nabla_x f(x) \geq 0 \ \Rightarrow\ f(x) - f(u) \geq 0,$$

and quasiconvex if for all $x,\ u \in R^n$,

$$f(x) - f(u) \leq 0 \Rightarrow (x-u)^T \nabla_x f(x) \leq 0.$$

The dual problems (WD1) and (MD1) are known as first-order duals and named as Wolfe and Mond-Weir duals, respectively. Kumar et al. (1995) formulated a pair of Mond-Weir type minimax mixed integer symmetric dual programs.

Mangasarian (1975) extended the Wolfe dual (WD1) to second-order dual problem (SD1) as follows:

$$(SD1)\ \text{Max}\ f(x) + y^T g(x) - \frac{1}{2} p_1^T (\nabla_{xx} f(x) + \nabla_{xx} y^T g(x)) p_1$$
$$\text{subject to}\ \ \nabla_x f(x) + y^T \nabla_x g(x)$$
$$+ (\nabla_{xx} f(x) + \nabla_{xx} y^T g(x)) p_1 = 0$$
$$y \geq 0$$

where $p_1 \in R^n$, $\nabla_{xx} f(x)$ denotes the Hessian matrix with respect to x.

For the primal and dual models (P1) and (SD1), the duality results do not hold under convexity/concavity assumptions. Mangasarian established duality results under somewhat involved assumptions (see Bector and Chandra (1987), page 144).

Mond (1974) obtained Mangasarian's duality relations between (P1) and (SD1) under second-order convexity/concavity assumptions. Second-order convex functions were later called bonvex by Bector and Chandra (1986a).

A function $f : R^n \mapsto R$ is said to be bonvex at $u \in R^n$ if for all $x, p_1 \in R^n$,

$$f(x) - f(u) \geq (x-u)^T [\nabla_x f(u) + \nabla_{xx} f(u) p_1]$$
$$- \frac{1}{2} p_1^T \nabla_{xx} f(u) p_1.$$

The definition of convexity follows if the vector $p_1 = 0$.

Bector and Chandra (1986b) formulated second-order Mond-Weir type symmetric dual programs and established duality theorems involving pseudobonvex functions.

A function $f : R^n \mapsto R$ is said to be pseudobonvex at $u \in R^n$ if for all $x,\ p_1 \in R^n$,

$$(x-u)^T [\nabla_x f(u) + \nabla_{xx} f(u) p_1] \geq 0$$
$$\Rightarrow f(x) \geq f(u) - \frac{1}{2} p_1^T \nabla_{xx} f(u) p_1.$$

The definition of pesudoconvexity follows if the vector $p_1 = 0$.

Hanson (1981), replacing the difference vector $(x-u)$ in the definition of a convex function by $\eta(x,u)$ where $\eta : R^n \times R^n \mapsto R^n$, introduced a new class of functions. These functions were named invex by Craven (1981) and η-convex by Kaul and Kaur (1985).

A function f is said to be invex with respect to η (or η-convex) if for all $x,\ u \in R^n$,

$$f(x) - f(u) \geq \eta^T (x,u) \nabla_x f(u).$$

In this paper, we consider Wolfe and Mond-Weir type second-order minimax mixed integer dual programs over arbitrary cones. The symmetric duality theorems are established under separability and η-bonvexity/η-oncavity of the function $k(x, y)$ for the Wolfe type dual and η-pseudobonvexity/η-pseudoboncavity of $k(x, y)$ for the Mond-Weir type dual. Our work subsumes several papers that have appeared in the literature (see Section 6).

The motivating force for studying such minimax mathematical programming problems has been the fact that they arise frequently in game theory, approximation theory and various situations relating to decision making under uncertainty.

2. NOTATIONS AND PRELIMINARIES

Let R_+^n be the non-negative orthant of R^n. We constrain some of the components of $x \in R^n$ and $y \in R^m$ to belong to arbitrary sets of integers as in Balas (1970). Suppose that the first n_1 $(0 \le n_1 \le n)$ components of x belong to U and the first m_1 $(0 \le m_1 \le m)$ components of y belong to V. Then we write $(x, y) = (x^1, x^2, y^1, y^2)$, where $x^1 = (x_1, x_2, ..., x_{n_1}), y^1 = (y_1, y_2, ..., y_{m_1})$, and x^2 and y^2 belong to R^{n-n_1} and R^{m-m_1}, respectively. Let $\nabla_{x^2x^2} k(\bar{x}, \bar{y})$ and $\nabla_{y^2y^2} k(\bar{x}, \bar{y})$ denote the Hessian matrix with respect to x^2 and y^2 evaluated at $(\bar{x}, \bar{y})$, respectively.

Definition 1. A convex set C of R^n is called a convex cone if for each $x \in C$ and $\lambda \ge 0$, $\lambda x \in C$.

Definition 2. $C^* = \{z \in R^n : x^T z \le 0 \text{ for all } x \in C\}$ is called the polar of the cone C.

Let T_1 and T_2 be closed convex cones in R^n and R^m, respectively, with non-empty interiors. Let $S_1 \subset R^n$ and $S_2 \subset R^m$ be open sets such that $T_1 \times T_2 \subset S_1 \times S_2$.

Definition 3. The function k is said to be η_1-bonvex in the first variable u on S_1 for fixed $v \in S_2$, if there exists a function $\eta_1 : S_1 \times S_1 \mapsto R^n$, such that for any $p_1 \in R^n$,

$$k(x,v) - k(u,v) \ge \eta_1^T(x,u)[\nabla_x k(u,v) + \nabla_{xx} k(u,v) p_1] - \frac{1}{2} p_1^T \nabla_{xx} k(u,v) p_1,$$

for all $x, u \in S_1$ and k is said to be η_2-bonvex in the second variable v on S_2 for fixed $u \in S_1$, if there exists a function $\eta_2 : S_2 \times S_2 \mapsto R^m$ such that for any $r_1 \in R^m$,

$$k(u, y) - k(u,v) \ge \eta_2^T(y,v)[\nabla_y k(u,v) + \nabla_{yy} k(u,v) r_1] - \frac{1}{2} r_1^T \nabla_{yy} k(u,v) r_1,$$

for all $y, v \in S_2$.

Definition 4. The function k is said to be η_1-pseudobonvex in the first variable u on S_1 for fixed $v \in S_2$, if there exists a function $\eta_1 : S_1 \times S_1 \mapsto R^n$ such that for any $p_1 \in R^n$,

$$\eta_1^T(x,u)[\nabla_x k(u,v) + \nabla_{xx} k(u,v) p_1] \ge 0$$
$$\Rightarrow k(x,v) \ge k(u,v) - \frac{1}{2} p_1^T \nabla_{xx} k(u,v) p_1,$$

for all $x, u \in S_1$ and k is said to be η_2-pseudobonvex in the second variable v on S_2 for fixed $u \in S_1$, if there exists a function $\eta_2 : S_2 \times S_2 \mapsto R^m$ such that for any $r_1 \in R^m$,

$$\eta_2^T(y,v)[\nabla_y k(u,v) + \nabla_{yy} k(u,v) r_1] \ge 0$$
$$\Rightarrow k(u, y) \ge k(u,v) - \frac{1}{2} r_1^T \nabla_{yy} k(u,v) r_1,$$

for all $y, v \in S_2$.

The function k is η-boncave or η-pseudoboncave if $-k$ is η-bonvex or η-pseudobonvex, respectively.

Definition 5. Let $s^1, s^2, ..., s^p$ be elements of an arbitrary vector space. A vector function $G(s^1, s^2, ..., s^p)$ will be called additively separable with respect to s^1 if there exist vector functions $H(s^1)$ (independent of $s^2, ..., s^p$) and $K(s^2, ..., s^p)$ (independent of s^1), such that

$$G(s^1, s^2, ..., s^p) = H(s^1) + K(s^2, ..., s^p).$$

Examples of functions which are η-bonvex (and hence η-pseudobonvex), η-pseudoconvex, or η-quasiconvex but not generalized convex are given in Pandey (1991) and Kaul and Kaur (1985).

In what follows, C_1 and C_2 are closed convex cones in R^{n-n_1} and R^{m-m_1}, respectively.

3. WOLFE TYPE SECOND-ORDER SYMMETRIC DUAL PROGRAMS

We consider the following pair of nonlinear mixed integer symmetric dual programs:

Primal (WP)

$$\text{Max}_{x^1}\text{Min}_{x^2, y} F(x, y, p) = k(x, y) - (y^2)^T \nabla_{y^2} k(x, y) - (y^2)^T \nabla_{y^2y^2} k(x, y) p - \frac{1}{2} p^T \nabla_{y^2y^2} k(x, y) p$$

subject to
$$\nabla_{y^2} k(x, y) + \nabla_{y^2y^2} k(x, y) p \in C_2^* \quad (1)$$
$$x^1 \in U, \; y^1 \in V \quad (2)$$
$$x^2 \in C_1 \quad (3)$$

Dual (WD)

$$\text{Min}_{y^1}\text{Max}_{x,\,y^2} G(x,y,r)=k(x,y)-(x^2)^T\nabla_{x^2}k(x,y)-(x^2)^T\nabla_{x^2x^2}k(x,y)r-\frac{1}{2}r^T\nabla_{x^2x^2}k(x,y)r$$

subject to
$$-\{\nabla_{x^2}k(x,y)+\nabla_{x^2x^2}k(x,y)r\}\in C_1^* \quad (4)$$
$$x^1\in U,\ y^1\in V \quad (5)$$
$$y^2\in C_2 \quad (6)$$

where p and r are $m-m_1$ and $n-n_1$ dimensional vector variables.

Theorem 1. (Symmetric duality). Let $(\bar{x},\bar{y},\bar{p})$ be an optimal solution for (WP). Also let:

(i) $k(x,y)$ be additively separable with respect to x^1 or y^1;
(ii) $k(x,y)$ be η_1-bonvex in x^2 for each (x^1,y) and η_2-boncave in y^2 for each (x^1,y);
(iii) $k(x,y)$ be thrice differentiable in x^2 and y^2;
(iv) $\nabla_{y^2y^2}k(\bar{x},\bar{y})$ be non-singular;
(v) the vector $\bar{p}^T\nabla_{y^2}(\nabla_{y^2y^2}k(\bar{x},\bar{y})\bar{p})=0$ imply that $\bar{p}=0$;
(vi) $\eta_1(x^2,u^2)+u^2\in C_1$ for all $x^2\in C_1$;
(vii) $\eta_2(v^2,y^2)+y^2\in C_2$ for all $v^2\in C_2$.

The $\bar{p}=0$, $F(\bar{x},\bar{y},\bar{p})=G(\bar{x},\bar{y},\bar{r})$, and $(\bar{x},\bar{y},\bar{r}=0)$ is an optimal solution for (WD).

Proof. Let

$$Z=\text{Max}_{x^1}\text{Min}_{x^2,\,y}\{k(x,y)-(y^2)^T\nabla_{y^2}k(x,y)-(y^2)^T\nabla_{y^2y^2}k(x,y)p-\frac{1}{2}p^T\nabla_{y^2y^2}k(x,y)p:(x,y,p)\in S\}$$

and

$$W=\text{Min}_{y^1}\text{Max}_{x,\,y^2}\{k(x,y)-(x^2)^T\nabla_{x^2}k(x,y)-(x^2)^T\nabla_{x^2x^2}k(x,y)r-\frac{1}{2}r^T\nabla_{x^2x^2}k(x,y)r:(x,y,r)\in T\}$$

where S and T are feasible regions for (WP) and (WD), respectively.

As $k(x,y)$ is taken to be additively separable with respect to x^1 or y^1 (say with respect to x^1), it follows that

$$k(x,y)=k^1(x^1)+k^2(x^2,y). \quad (7)$$

Therefore, $\nabla_{y^2}k(x,y)=\nabla_{y^2}k^2(x^2,y)$ and Z can be written as

$$Z=\text{Max}_{x^1}\text{Min}_{x^2,\,y}\left\{\begin{array}{l}k^1(x^1)+k^2(x^2,y)\\ \quad-(y^2)^T\nabla_{y^2}k^2(x^2,y)\\ \quad-(y^2)^T\nabla_{y^2y^2}k^2(x^2,y)p\\ \quad-\frac{1}{2}p^T\nabla_{y^2y^2}k^2(x^2,y)p:\\ \nabla_{y^2}k^2(x^2,y)+\nabla_{y^2y^2}k^2(x^2,y)p\\ \quad\in C_2^*,\ x^2\in C_1,\ x^1\in U,\ y^1\in V\end{array}\right\}$$

$$=\text{Max}_{x^1}\text{Min}_{y^1}\text{Min}_{x^2,\,y^2}\left\{\begin{array}{l}k^1(x^1)+k^2(x^2,y)\\ \quad-(y^2)^T\nabla_{y^2}k^2(x^2,y)\\ \quad-(y^2)^T\nabla_{y^2y^2}k^2(x^2,y)p\\ \quad-\frac{1}{2}p^T\nabla_{y^2y^2}k^2(x^2,y)p:\\ \nabla_{y^2}k^2(x^2,y)+\nabla_{y^2y^2}k^2(x^2,y)p\\ \quad\in C_2^*,\ x^2\in C_1, x^1\in U,\ y^1\in V\end{array}\right\}$$

Or,

$$Z=\text{Max}_{x^1}\text{Min}_{y^1}\{k^1(x^1)+\phi(y^1):x^1\in U,\ y^1\in V\}, \quad (8)$$

where

$$\phi(y^1)=\text{Min}_{x^2,\,y^2}\left\{\begin{array}{l}k^2(x^2,y)-(y^2)^T\nabla_{y^2}k^2(x^2,y)\\ \quad-(y^2)^T\nabla_{y^2y^2}k^2(x^2,y)p\\ \quad-\frac{1}{2}p^T\nabla_{y^2y^2}k^2(x^2,y)p:\\ \nabla_{y^2}k^2(x^2,y)+\nabla_{y^2y^2}k^2(x^2,y)p\\ \quad\in C_2^*,\ x^2\in C_1\end{array}\right\} \quad (9)$$

Similary,

$$W=\text{Min}_{y^1}\text{Max}_{x^1}\{k^1(x^1)+\psi(y^1):x^1\in U,\ y^1\in V\}, \quad (10)$$

where

$$\psi(y^1)=\text{Max}_{x^2,\,y^2}\left\{\begin{array}{l}k^2(x^2,y)-(x^2)^T\nabla_{x^2}k^2(x^2,y)\\ \quad-(x^2)^T\nabla_{x^2x^2}k^2(x^2,y)r\\ \quad-\frac{1}{2}r^T\nabla_{x^2x^2}k^2(x^2,y)r:\\ -\{\nabla_{x^2}k^2(x^2,y)+\nabla_{x^2x^2}k^2(x^2,y)r\}\\ \quad\in C_1^*,\ y^2\in C_2\end{array}\right\}$$
(11)

For any given y^1, (9) and (11) are a pair of Wolfe type second-order symmetric dual nonlinear programs and in view of assumptions (ii)-(vii), Theorem 3.2 in Gulati et al. (2007) becomes applicable. Therefore, for $y^1 = \bar{y}^1$ we obtain

$$\bar{p} = 0 \text{ and } \phi(\bar{y}^1) = \psi(\bar{y}^1)$$

where the functions ϕ and ψ are given by (9) and (11), respectively.

Now, we need only to show that $(\bar{x}, \bar{y}, \bar{r} = 0)$ is optimal for (*WD*). If this is not the case, there exist $y^{*1} \in V$ such that $\psi(y^{*1}) < \psi(\bar{y}^1)$. But then, in view of assumptions (iv) and (v), we have

$$\phi(\bar{y}^1) = \psi(\bar{y}^1) > \psi(y^{*1}) = \phi(y^{*1}),$$

contradicting the optimality of $(\bar{x}, \bar{y}, \bar{p} = 0)$ for (*WP*). Hence $(\bar{x}, \bar{y}, \bar{r} = 0)$ is an optimal solution for (*WD*).

4. MOND-WEIR TYPE SECOND-ORDER SYMMETRIC DUAL PROGRAMS

We now consider the following pair of nonlinear mixed integer symmetric dual programs:

Primal (*MP*)

$$\text{Max}_{x^1} \text{Min}_{x^2, y} M(x, y, p) = k(x, y) - \frac{1}{2} p^T \nabla_{y^2 y^2} k(x, y) p$$

$$\text{subject to } \nabla_{y^2} k(x, y) + \nabla_{y^2 y^2} k(x, y) p \in C_2^* \quad (12)$$

$$(y^2)^T \left[\nabla_{y^2} k(x, y) + \nabla_{y^2 y^2} k(x, y) p \right] \geq 0 \quad (13)$$

$$x^1 \in U, y^1 \in V \quad (14)$$

$$x^2 \in C_1 \quad (15)$$

Dual (*MD*)

$$\text{Min}_{y^1} \text{Max}_{x, y^2} N(x, y, r) = k(x, y) - \frac{1}{2} r^T \nabla_{x^2 x^2} k(x, y) r$$

$$\text{subject to } -\{\nabla_{x^2} k(x, y) + \nabla_{x^2 x^2} k(x, y) r\} \in C_1^* \quad (16)$$

$$(x^2)^T \left[\nabla_{x^2} k(x, y) + \nabla_{x^2 x^2} k(x, y) r \right] \leq 0 \quad (17)$$

$$x^1 \in U, y^1 \in V \quad (18)$$

$$y^2 \in C_2 \quad (19)$$

Theorem 2. (Symmetric duality). Let $(\bar{x}, \bar{y}, \bar{p})$ be an optimal solution for (*MP*). Also, let:

(i) $k(x, y)$ be additively separable with respect to x^1 or y^1;
(ii) $k(x, y)$ be η_1-pseudobonvex in x^2 for each (x^1, y) and η_2-pseudoboncave in y^2 for each (x^1, y);
(iii) $k(x, y)$ be thrice differentiable in x^2 and y^2;
(iv) either $\nabla_{y^2 y^2} k(\bar{x}, \bar{y})$ be positive definite and $\bar{p}^T \nabla_{y^2} k(\bar{x}, \bar{y}) \geq 0$ or $\nabla_{y^2 y^2} k(\bar{x}, \bar{y})$ be negative definite and $\bar{p}^T \nabla_{y^2} k(\bar{x}, \bar{y}) \leq 0$;
(v) $\nabla_{y^2} k(\bar{x}, \bar{y}) + \nabla_{y^2 y^2} k(\bar{x}, \bar{y}) \bar{p} \neq 0$;
(vi) $\eta_1(x^2, u^2) + u^2 \in C_1$ for all $x^2 \in C_1$;
(vii) $\eta_2(v^2, y^2) + y^2 \in C_2$ for all $v^2 \in C_2$.

Then $\bar{p} = 0$, $M(\bar{x}, \bar{y}, \bar{p}) = N(\bar{x}, \bar{y}, \bar{r})$, and $(\bar{x}, \bar{y}, \bar{r} = 0)$ is an optimal solution for (*MD*).

Proof. Let

$$Z = \text{Max}_{x^1} \text{Min}_{x^2, y} \left\{ \begin{array}{l} k(x, y) - \frac{1}{2} p^T \nabla_{y^2 y^2} k(x, y) p : \\ (x, y, p) \in P \end{array} \right\}$$

and

$$W = \text{Max}_{y^1} \text{Min}_{x, y^2} \left\{ \begin{array}{l} k(x, y) - \frac{1}{2} r^T \nabla_{x^2 x^2} k(x, y) r : \\ (x, y, r) \in Q \end{array} \right\}$$

where P and Q are feasible regions for (*MP*) and (*MD*), respectively.

As $k(x, y)$ is taken to be additively separable with respect to x^1 or y^1 (say with respect to x^1), it follows that

$$k(x, y) = k^1(x^1) + k^2(x^2, y). \quad (20)$$

Therefore, $\nabla_{y^2} k(x, y) = \nabla_{y^2} k^2(x^2, y)$ and Z can be written as

$$Z = \text{Max}_{x^1} \text{Min}_{x^2, y} \left\{ \begin{array}{l} k^1(x^1) + k^2(x^2, y) \\ \quad - \frac{1}{2} p^T \nabla_{y^2 y^2} k^2(x^2, y) p : \\ \nabla_{y^2} k^2(x^2, y) \\ \quad + \nabla_{y^2 y^2} k^2(x^2, y) p \in C_2^*, \\ (y^2)^T \{\nabla_{y^2} k^2(x^2, y) \\ \quad + \nabla_{y^2 y^2} k^2(x^2, y) p\} \geq 0, \\ x^2 \in C_1, x^1 \in U, \ y^1 \in V \end{array} \right\} \quad (21)$$

Or,

$$Z = \text{Max}_{x^1} \text{Min}_{y^1} [k^1(x^1) + \phi(y^1) : x^1 \in U, y^1 \in V],$$

where

$$\phi(y^1) = \text{Min}_{x^2, y^2} \left\{ \begin{array}{l} k^2(x^2, y) - \frac{1}{2} p^T \nabla_{y^2 y^2} k^2(x^2, y) p : \\ \nabla_{y^2} k^2(x^2, y) + \nabla_{y^2 y^2} k^2(x^2, y) p \in C_2^*, \\ (y^2)^T \{\nabla_{y^2} k^2(x^2, y) \\ \quad + \nabla_{y^2 y^2} k^2(x^2, y) p\} \geq 0, \\ x^2 \in C_1 \end{array} \right\} \quad (22)$$

Similarly,

$$W = \text{Min}_{y^1} \text{Max}_{x^1} [k^1(x^1) + \psi(y^1) : x^1 \in U, y^1 \in V],$$

where

$$\psi(y^1) = \text{Max}_{x^2, y^2} \left\{ \begin{array}{l} k^2(x^2, y) - \frac{1}{2} r^T \nabla_{x^2 x^2} k^2(x^2, y) r : \\ -\{\nabla_{x^2} k^2(x^2, y) \\ \quad + \nabla_{x^2 x^2} k^2(x^2, y) r\} \in C_1^*, \\ (x^2)^T \{\nabla_{x^2} k^2(x^2, y) \\ \quad + \nabla_{x^2 x^2} k^2(x^2, y) r\} \leq 0, \; y^2 \in C_2 \end{array} \right\} \quad (23)$$

For any given y^1, programs (22) and (23) are a pair of Mond-Weir type second-order symmetric dual nonlinear programs and in view of assumptions (ii)-(vii), Theorem 4.2 in Gulati et al. (2007) becomes applicable. Therefore, for $y^1 = \bar{y}^1$ we obtain

$$\bar{p} = 0 \quad \text{and} \quad \phi(\bar{y}^1) = \psi(\bar{y}^1)$$

where the functions ϕ and ψ are given by (22) and (23), respectively.

Now, we need only to show that $(\bar{x}, \bar{y}, \bar{r} = 0)$ is optimal for (*MD*). If this is not the case, there exist $y^{*1} \in V$ such that $\psi(y^{*1}) < \psi(\bar{y}^1)$. But then, in view of assumptions (iv) and (v), we have

$$\phi(\bar{y}^1) = \psi(\bar{y}^1) > \psi(y^{*1}) = \phi(y^{*1}),$$

contradicting the optimality of $(\bar{x}, \bar{y}, \bar{p} = 0)$ for (*MP*). Hence $(\bar{x}, \bar{y}, \bar{r} = 0)$ is an optimal solution for (*MD*).

5. SELF DUALITY

A mathematical problem is said to be self dual if it is formally identical with its dual, that is, if the dual is recast in the form of the primal, the new problem so obtained is the same as the primal. In general, (*WP*) and (*WD*) are not self dual without an added restriction on k. The vector function $k : R^n \times R^n \mapsto R$ is said to be skew symmetric if for all $x, y \in R^n$,

$$k(y, x) = -k(x, y).$$

Theorem 3. Let $k : R^n \times R^n \mapsto R$ be skew symmetric and $C_1 = C_2$. Then (*WP*) is a self dual. Furthermore, if (*WP*) and (*WD*) are dual programs and $(\bar{x}, \bar{y}, \bar{p})$ is an optimal solution for (*WP*), then $\bar{p} = 0$, $(\bar{y}, \bar{x}, \bar{r} = 0)$ is an optimal solution for (*WD*) and the values of the two objective functions are equal to zero.

Proof: (*WD*) can be written as

$$\begin{aligned} \text{Max}_{y^1} \text{Min}_{x, y^2} \; & -k(x, y) + (x^2)^T \nabla_{x^2} k(x, y) \\ & + (x^2)^T \nabla_{x^2 x^2} k(x, y) r + \frac{1}{2} r^T \nabla_{x^2 x^2} k(x, y) r \\ \text{subject to} \quad & -\nabla_{x^2} k(x, y) - \nabla_{x^2 x^2} k(x, y) r \in C_1^* \\ & x^1 \in U, y^1 \in V \\ & y^2 \in C_1 \end{aligned}$$

Since k is skew symmetric

$k(x, y) = -k(y, x)$, $\nabla_{x^2} k(x, y) = -\nabla_{y^2} k(y, x)$ and $\nabla_{x^2 x^2} k(x, y) = -\nabla_{y^2 y^2} k(y, x)$.

Therefore, the above problem becomes

$$\begin{aligned} \text{Max}_{y^1} \text{Min}_{x, y^2} \; & k(y, x) - (x^2)^T \nabla_{y^2} k(y, x) \\ & - (x^2)^T \nabla_{y^2 y^2} k(y, x) r - \frac{1}{2} r^T \nabla_{y^2 y^2} k(y, x) r \\ \text{subject to} \quad & \nabla_{y^2} k(y, x) + \nabla_{y^2 y^2} k(y, x) r \in C_1^* \\ & x^1 \in U, y^1 \in V \\ & y^2 \in C_1 \end{aligned}$$

which is (*WP*). Thus (*WP*) is a self dual.

Hence if $(\bar{x}, \bar{y}, \bar{p})$ is optimal for (*WP*), then $\bar{p} = 0$ and $(\bar{y}, \bar{x}, \bar{r} = 0)$ is optimal for (*WD*). Also, $F(\bar{x}, \bar{y}, \bar{p})$ $= G(\bar{y}, \bar{x}, \bar{r})$. Now we show that $F(\bar{x}, \bar{y}, \bar{p}) = 0$.

$$\begin{aligned} F(\bar{x}, \bar{y}, \bar{p}) &= k(\bar{x}, \bar{y}) - (y^2)^T \nabla_{y^2} k(\bar{x}, \bar{y}) \\ & \quad - (y^2)^T \nabla_{y^2 y^2} k(\bar{x}, \bar{y}) \bar{p} - \frac{1}{2} \bar{p}^T \nabla_{y^2 y^2} k(\bar{x}, \bar{y}) \bar{p} \\ & \geq k(\bar{x}, \bar{y}) - \frac{1}{2} \bar{p}^T \nabla_{y^2 y^2} k(\bar{x}, \bar{y}) \bar{p} \end{aligned}$$

(using (1), (6), and the definition of polar cone)
$= k(\bar{x}, \bar{y})$.
(by **Theorem 1**)

Similarly, $G(\bar{x}, \bar{y}, \bar{r}) \leq k(\bar{x}, \bar{y})$. Hence by **Theorem 1**,

$$k(\bar{x}, \bar{y}) \leq F(\bar{x}, \bar{y}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{r}) \leq k(\bar{x}, \bar{y}),$$

which implies that
$F(\bar{x},\bar{y},\bar{p}) = G(\bar{y},\bar{x},\bar{r}) = k(\bar{x},\bar{y}) = k(\bar{y},\bar{x}) = -k(\bar{x},\bar{y}),$

and therefore $F(\bar{x},\bar{y},\bar{p}) = 0.$

Theorem 4. Let $k : R^n \times R^n \mapsto R$ be skew symmetric and $C_1 = C_2$. Then (*MP*) is a self dual. Furthermore, if (*MP*) and (*MD*) are dual programs and $(\bar{x},\bar{y},\bar{p})$ is an optimal solution for (*MP*), then $\bar{p}=0$, $(\bar{y},\bar{x},\bar{r}=0)$ is an optimal solutions for (*MD*) and the value of the two objective functions are equal to zero.

Proof. (*MD*) can be written as

$$\text{Max}_{y^1} \text{Min}_{x,y^2} -k(x,y) + \frac{1}{2} r^T \nabla_{x^2x^2} k(x,y) r$$

subject to $-\nabla_{x^2} k(x,y) - \nabla_{x^2x^2} k(x,y) r \in C_1^*$

$(x^2)^T \{\nabla_{x^2} k(x,y) + \nabla_{x^2x^2} k(x,y) r\} \leq 0$

$x^1 \in U, y^1 \in V$

$y^2 \in C_1$

Since k is skew symmetric

$$k(x,y) = -k(y,x),\ \nabla_{x^2} k(x,y) = -\nabla_{y^2} k(y,x)$$

and

$$\nabla_{x^2x^2} k(x,y) = -\nabla_{y^2y^2} k(y,x)$$

Therefore, the above problem becomes

$$\text{Max}_{y^1} \text{Min}_{x,y^2} k(y,x) - \frac{1}{2} r^T \nabla_{y^2y^2} k(y,x) r$$

subject to $\nabla_{y^2} k(y,x) + \nabla_{y^2y^2} k(y,x) r \in C_1^*$

$(x^2)^T \{\nabla_{y^2} k(y,x) + \nabla_{y^2y^2} k(y,x) r\} \geq 0$

$x^1 \in U, y^1 \in V$

$y^2 \in C_1$

which is (*MP*). Thus (*MP*) is a self dual. Hence if $(\bar{x},\bar{y},\bar{p})$ is optimal for (MP), then $\bar{p}=0$ and $(\bar{y},\bar{x},\bar{r}=0)$ is optimal for (*MD*). Also, $M(\bar{x},\bar{y},\bar{p}) = N(\bar{y},\bar{x},\bar{r})$. Now we show that $M(\bar{x},\bar{y},\bar{p}) = 0$.

$$M(\bar{x},\bar{y},\bar{p}) = k(\bar{x},\bar{y}) - \frac{1}{2} \bar{p}^T \nabla_{y^2y^2} k(\bar{x},\bar{y}) \bar{p}$$
$$= k(\bar{x},\bar{y}) \text{ (by **Theorem 2**)}$$

Similarly, $N(\bar{y},\bar{x},\bar{r}) = k(\bar{y},\bar{x})$. Therefore,

$$M(\bar{x},\bar{y},\bar{p}) = N(\bar{y},\bar{x},\bar{r}) = k(\bar{x},\bar{y}) = k(\bar{y},\bar{x}) = -k(\bar{x},\bar{y})$$

and hence $M(\bar{x},\bar{y},\bar{p}) = 0.$

6. SPECIAL CASES

We now consider some of the special cases of the problems considered above. For these cases $C_1 = R_+^{n-n_1}$ and $C_2 = R_+^{m-m_1}$.

(a) Over these specific cones, our problems are reduced to the programs of Gulati et al. (2006).
(b) If $\eta_1(x,u) = x - u$ and $\eta_2(v,y) = v - y$, then we get the programs studied in Gulati and Ahmad (1997).
(c) If $U = \phi$ and $V = \phi$, then (*WP*) and (*WD*) are reduced to the problems studied in Mond (1974). Also, if $p = 0$ and $r = 0$, then (*WP*) and (*WD*) become the symmetric dual programs of Dantzig et al. (1965).
(d) If $p = 0$ and $r = 0$, then we get the programs considered in Kumar et al. (1995).

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