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Second Order Symmetric and Maxmin Symmetric Duality with Cone Constraints

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Abstract—In this paper, a pair of second order symmetric dual programs with cone constraints is formulated. For this pair of programs, weak, strong, converse and self duality theorems are validated under bonvexity-boncavity condition. Further, a pair of second order maxmin mixed integer symmetric dual programs involving cones are constructed and for this pair of programs, symmetric as well as self duality is investigate. Some particular cases are derived from our results.

Keywords—Second order symmetric and self duality, Cone constraints, Duality theorems, Bonvexity and boncavity, Maxmin symmetric duality, Mixed integer symmetric dual programs

1. INTRODUCTION

Symmetric duality in nonlinear programming in which the dual of the dual in the primal was first studied by Dorn (1960). Motivated with the results of Dorn (1960), the notion of symmetric duality was developed significantly by Dantzig et al. (1965), Chandra and Husain (1981), and Mond and Weir (1989). Dantzig et al. (1965) formulated a pair of Wolfe type symmetric dual programs with the non-negative orthants as cones under convexity-convavity on the kernel function that occurs in the programs. The same results were subsequently generalized by Bazaraa and Goode (1973) to arbitrary cones. Nanda (1988) studied symmetric duality for a pair of nonlinear mixed integer programs involving arbitrary cone under invexity.

Mond (1974) initiated second order symmetric duality of Wolfe type in nonlinear programming and also indicated possible computational advantages of second order dual over the first order dual. Later, Bector and Chandra (1986) presented a pair of Mond-Weir type second order dual programs and proved weak, strong and self duality theorems under pseudobonvexity – pseudoboncarity. Devi (1998) constructed a pair of second order symmetric dual programs over cones and studied duality for the same; but this formulation of second order symmetric dual programs seems quite strange and apparently different from the traditional Wolfe type second order symmetric dual programs of Mond (1974) as well as Mond-Weir type second order symmetric dual programs formulated by Bector and Chandra (1986).

In (1969) Balas presented a pair of Wolfe type first order minimax mixed integer symmetric dual programs as a generalization of the results of Dantzig et al. (1965), while in (1995) Kumar et al., Husian and Chandra (1981) dealt with Mond-Weir type first order maximin mixed integer symmetric dual programs. Later, Gulati and Ahmed (1997) formulated second order maximin mixed integer symmetric dual programs and proved various duality theorems including self duality theorem.

In this research, we formulate Wolfe type second order dual programs with cone constraints and prove weak, strong, converse and self duality theorems under bonvexity – boncavity condition. Further, we generalize these Wolfe type dual programs to maximin second order dual programs by constraining some of the components of the two variables of the programs to belong to arbitrary sets. For integers of these programs also, symmetric as well as self duality is incorporated. Particular cases are generated from our results.

2. NOTATIONS AND PRE-REQUISITES

Let R^k denote the k-dimensional Euclidian space. Let Γ be a closed convex cone with nonempty interior in R^k.

Definition 1. The positive polar cone Γ^* of Γ is defined by

$$\Gamma^* = \left\{ \xi \in R \,\middle|\, x^T \xi \geq 0, \text{ for all } x \in \Gamma \right\}$$

where x^T denotes the transpose of x.

Let $f(x, y)$ be a twice differentiable real valued function on an open set in $R^n \times R^m$. Then $\nabla_x f(\bar{x}, \bar{y})$ and

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$\nabla_y f(\bar{x},\bar{y})$ denote gradient vectors with respect to x and y respectively evaluated at $(\bar{x},\bar{y})$. $\nabla_x^2 f(\bar{x},\bar{y})$ and $\nabla_y^2 f(\bar{x},\bar{y})$ are respectively the $n \times n$ and $m \times m$ symmetric Hessian matrices. $\frac{\partial}{\partial y_i}(\nabla_y^2 f(\bar{x},\bar{y}))$ is the $m \times m$ matrix obtained by differentiating the elements of $\nabla_y^2 f(\bar{x},\bar{y})$ with respect to y_i and $(\nabla_y^2 f(\bar{x},\bar{y})p)_{\bar{y}}$ denotes the matrix whose (i, j) the element is $\frac{\partial}{\partial y_i}(\nabla_y^2 f(\bar{x},\bar{y})p)_j$.

Definition 2. Let C_1 and C_2 be closed convex cones in R^n and R^m respectively. A twice differentiable function f: $C_1 \times C_2 \to R$ is said to be

(i) Bonvex in x, if for all $x, q, u \in C_1$ and fixed y

$$f(x,v)-f(u,v)$$
$$\geq (x,u)^T\left[\nabla_x f(u,v)+\nabla^2{}_x f(u,v)q\right]-\frac{1}{2}q^T\nabla_x^2 f(u,v)q$$

(ii) Boncave in y, if for fixed x and for all $y, p, v \in C_2$

$$f(x,v)-f(x,y)$$
$$\leq (v-y)^T\left[\nabla_y f(x,y)+\nabla_y^2 f(x,y)p\right]$$
$$-\frac{1}{2}p^T\nabla_y^2 f(x,y)p$$

(iii) Skew-symmetric, when both C_1 and C_2 are in R^n and $C_1 = C_2 = C$ (say), and

$$f(x,y)=-f(x,y), \text{ for all } x \in C \text{ and } y \in C.$$

In the sequel, we shall require the Fritz John type necessary optimality conditions derived by Bazaraa and Goode (1973) and which are embodied in the following proposition.

Proposition 1. Let X be a convex set with nonempty interior in R^n and C be a closed convex cone in R^m. Let F be real valued function and G be a vector valued function, both defined on X.

Consider the problem:

(P_0): Minimize $F(z)$
Subject to $G(z) \in C$ and $z \in X$

If z solves the problem (P_0), then there exist $\alpha_0 \in R$ and $\delta \in C^*$ such that

$$\left[\alpha_0\nabla F(z_0)+\nabla\delta^T G(z_0)\right]^T(z-z_0)\geq 0 \text{ for all } z \in X,$$
$$\delta^T G(z_0)=0$$
$$(\alpha_0,\delta)\geq 0$$
$$(\alpha_0,\delta)\neq 0$$

The following concept of separability (Balas (1969)) is also needed in the subsequent analysis of this research.

Definition 3. Let $s^1, s^2, ..., s^p$ be elements of an elementary vector space. A real valued function $H_0(s^1, s^2, ..., s^p)$ will be called separable with respect to s^1 if there exist real-valued function $H_1(s^1)$ (independent of $s^2, ..., s^p$) and $H_2(s^2, ..., s^p)$ (independent of s^1), such that

$$H_0(s^1,s^2,...,s^p)=H_1(s^1)+H_2(s^2,...,s^p).$$

3. SECOND ORDER SYMMETRIC AND SELF DUALITY

In this section, we consider a pair of second order symmetric dual nonlinear programs with cone constraints and establish appropriate duality theorems.

Consider the following two programs:
Primal Problem

(SP): Minimize
$$G(x,y,p)=f(x,y)-y^T\left(\nabla_y f(x,y)+\nabla_y^2 f(x,y)p\right)-\frac{1}{2}p^T\nabla_y^2 f(x,y)p$$

Subject to
$$-\nabla_y f(x,y)-\nabla_y^2 f(x,y)p\in C_2^* \tag{1}$$
$$(x,y)\in C_1\times C_2 \tag{2}$$

and Dual Problem

(SD): Maximize
$$H(x,y,q)=f(x,y)-x^T\left(\nabla_x f(x,y)+\nabla_x^2 f(x,y)q\right)-\frac{1}{2}q^T\nabla_x^2 f(x,y)q$$

Subject to
$$\nabla_x f(x,y)+\nabla_x^2 f(x,y)q\in C_1^* \tag{3}$$
$$(x,y)\in C_1\times C_2 \tag{4}$$

where

(i) f: $C_1 \times C_2 \to R$ is a twice differentiable function,
(ii) C_1 and C_2 are closed convex cones with nonempty interior in R^n and R^m, respectively,
(iii) C_1^* and C_2^* are positive polar cones of C_1 and C_2 respectively.

Theorem 1 (Weak duality). Let (x, y, p) and (u, v, q) be feasible solutions of (SP) and (SD) respectively. Assume that $f(\cdot, y)$ is bonvex with respect to x for fixed y and $f(x, \cdot)$ is boncave with respect to y for fixed x for all feasible (x, y, p, u, v, q).

Then

infimum (SP) ≥ supremum (SD).

Proof. By bonvexity of $f(\cdot, y)$, we have

$$f(x,v)-f(u,v)$$
$$\geq (x-u)^T\left[\nabla_x f(u,v)+\nabla_x^2 f(u,v)q\right]-\frac{1}{2}q^T\nabla_x^2 f(u,v)q \qquad (5)$$

and by boncavity of $f(x, \cdot)$, we have

$$f(x,v)-f(x,y)\leq (v-y)^T\left[\nabla_y f(x,y)+\nabla_y^2 f(x,y)p\right]$$
$$-\frac{1}{2}p^T\nabla_x^2 f(x,y)p \qquad (6)$$

Multiplying (6) by (–1) and adding the resulting inequality to (5), we obtain

$$\left[f(x,v)-y^T\left(\nabla_y f(x,y)+\nabla_y^2 f(x,y)p\right)-\frac{1}{2}p^T\nabla_y^2 f(x,y)p\right]$$
$$-\left[f(u,v)-u^T\left(\nabla_x f(u,v)+\nabla_x^2 f(u,v)q\right)-\frac{1}{2}q^T\nabla_x^2 f(u,v)q\right]$$
$$\geq x^T\left[\nabla_x f(u,v)+\nabla_x^2 f(u,v)q\right]$$
$$-v^T\left[\nabla_y f(x,y)+\nabla_y^2 f(x,y)p\right]. \qquad (7)$$

Now since $x \in C_1$ and $\nabla_x f(u,v)+\nabla_x^2 f(u,v)q \in C_1^*$, we have

$$x^T\left[\nabla_x f(u,v)+\nabla_x^2 f(u,v)q\right]\geq 0 \qquad (8)$$

and since $v \in C_2$ and $-[\nabla_y f(x,y)+\nabla_y^2 f(x,y)]\in C_2^*$, we have

$$-v^T\left[\nabla_y f(x,y)+\nabla_y^2 f(x,y)p\right]\geq 0 \qquad (9)$$

The inequality (7) together with (8) and (9), yields,

$$f(x,y)-y^T[\nabla_y f(x,y)+\nabla_y^2 f(x,y)p]-\frac{1}{2}p^T\nabla_y^2 f(x,y)p$$
$$\geq f(u,v)-u^T[\nabla_x f(u,v)+\nabla_x^2 f(u,v)q]-\frac{1}{2}q^T\nabla_x^2 f(u,v)q$$

This implies

infimum (SP) ≥ supremum (SD).

Theorem 2 (Strong duality). Let $(\bar{x},\bar{y},\bar{p})$ be an optimal solution of (SP). Also let

(A_1): the matrix $\nabla_y^2 f(\bar{x},\bar{y})$ is non singular, and

(A_2): $\nabla_y(\nabla_y^2 f(\bar{x},\bar{y})\bar{p})$ be negative definite.

Then $(\bar{x},\bar{y},\ \bar{q}=0)$ is feasible for (SD) and the objective values of the programs (SP) and (SD) are equal. Moreover, if the requirements of Theorem 1 are fulfilled, then $(\bar{x},\bar{y},\ \bar{q})$ is an optimal solution of (SD).

Proof. We use Proposition 1 to prove this theorem. Here $z=(x,y,p)$, $\bar{z}=(\bar{x},\bar{y},\bar{p})$, $x \in C_1, p \in R^m$ and $y \in C_2$

$$F(\bar{z})=f(\bar{x},\bar{y})-\bar{y}^T\left(\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\right)$$
$$-\frac{1}{2}\bar{p}^T\nabla_y^2 f(\bar{x},\bar{y})\bar{p}$$

$$G(\bar{z})=-\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{p} \text{ and } C=C_2^*$$

Since $(\bar{x},\bar{y},\bar{p})$ is an optimal solution of (SP), by Proposition 1, there exist $\alpha \in R$ and $\beta \in C_2^*$ such that

$$\Big[\alpha\nabla_x f(\bar{x},\bar{y})-(\alpha\bar{y}+\beta)\nabla_x\nabla_y f(\bar{x},\bar{y})$$
$$-(\alpha\bar{y}+\frac{\alpha\bar{p}}{2}+\beta)\nabla_x\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\Big](x-\bar{x})$$
$$-\Big[(\alpha\bar{y}+\alpha\bar{p}+\beta)\nabla_y^2 f(\bar{x},\bar{y})$$
$$+(\alpha\bar{y}+\frac{\alpha\bar{p}}{2}+\beta)\nabla_x\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\Big](y-\bar{y})\geq 0 \qquad (10)$$
$$(\alpha y+\alpha p+\beta)\nabla_y^2 f(\bar{x},\bar{y})=0 \qquad (11)$$
$$\beta^T\left[\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\right]=0 \qquad (12)$$
$$(\alpha,\beta)\geq 0 \qquad (13)$$
$$(\alpha,\beta)\neq 0 \qquad (14)$$

The relation (11), in view of the hypothesis (A_1), gives

$$\beta=-\alpha(\bar{y}+\bar{p}). \qquad (15)$$

It follows that $\alpha \neq 0$, for if $\alpha = 0$, (15) implies $\beta = 0$. Hence $(\alpha, \beta) = 0$ contradicts (14). Thus $\alpha > 0$.

Now putting $\bar{x}=x$ and using (15) in (10), we obtain,

$$(\frac{\alpha\bar{p}}{2})^T\left[\nabla_y(\nabla_y^2 f(\bar{x},\bar{y})\bar{p})\right](y-\bar{y})\geq 0 \text{ , for all } y \in C_2.$$

Putting $y=\bar{p}+\bar{y}$ and using $\alpha > 0$, from the above inequality

$$p^T\left[\nabla_y(\nabla_y^2 f(\bar{x},\bar{y})\bar{p})\right]\bar{p}\geq 0$$

which, because of (A_2), yields,

$$\bar{p}=0 \qquad (16)$$

Using (15) and (16) along with $\alpha > 0$ in (10) we have

$$\nabla_x f(\bar{x},\bar{y})(x-\bar{x})\geq 0, \text{ for all } x \in C_1 \qquad (17)$$

Since C_1 is closed convex cone, therefore, for each $x \in C_1$ and $\bar{x} \in C_1$, it implies $x + \bar{x} \in C_1$. Now, replacing x by $x + \bar{x}$ in (17), we have

$$x^T \left(\nabla_x f(\bar{x}, \bar{y}) + \nabla_x^2 f(\bar{x}, \bar{y}) \cdot 0 \right) \geq 0 \tag{18}$$

This implies

$$\nabla_x f(\bar{x}, \bar{y}) + \nabla_x^2 f(\bar{x}, \bar{y}) \cdot 0 \in C_1^*.$$

Thus $(\bar{x}, \bar{y}, \bar{q} = 0)$ is feasible for (SD).

Putting $x = 0$ in (17) and $x = \bar{x}$ in (18), we have respectively

$$\bar{x}^T \left(\nabla_x f(\bar{x}, \bar{y}) + \nabla_x^2 (\bar{x}, \bar{y}) \cdot 0 \right) \leq 0$$

and

$$\bar{x}^T \left(\nabla_x f(\bar{x}, \bar{y}) + \nabla_x^2 (\bar{x}, \bar{y}) \cdot 0 \right) \geq 0.$$

These together implies

$$\bar{x}^T \left(\nabla_x f(\bar{x}, \bar{y}) + \nabla_x^2 (\bar{x}, \bar{y}) \cdot 0 \right) = 0. \tag{19}$$

Using $\beta = \alpha \bar{y}$ and $\bar{p} = 0$ along with $\alpha > 0$ in (12), we have

$$\bar{y}^T \left(\nabla_y f(\bar{x}, \bar{y}) + \nabla_y^2 (\bar{x}, \bar{y}) \cdot 0 \right) = 0 \tag{20}$$

Consequently, we obviously have,

$$\begin{aligned} G(\bar{x}, \bar{y}, \bar{p}) &= f(\bar{x}, \bar{y}) - \bar{y}^T \left(\nabla_y f(\bar{x}, \bar{y}) \right. \\ &\quad \left. + \nabla_y^2 (\bar{x}, \bar{y}) \bar{p} \right) - \frac{1}{2} \bar{p}^T \nabla_y^2 f(\bar{x}, \bar{y}) \bar{p} \\ &= f(\bar{x}, \bar{y}) - \bar{x}^T \left(\nabla_x f(\bar{x}, \bar{y}) \right. \\ &\quad \left. + \nabla_x^2 (\bar{x}, \bar{y}) \bar{q} \right) - \frac{1}{2} \bar{q}^T \nabla_x^2 f(\bar{x}, \bar{y}) \bar{q} \\ &= H(\bar{x}, \bar{y}, \bar{q}) \end{aligned}$$

That is, the objective values of (SP) and (SD) are equal. By Theorem 1, the optimality of $(\bar{x}, \bar{y}, \bar{z})$ for (SD) follows.

We will only state a converse duality theorem (Theorem 3) as the proof of this theorem would follow analogously to that of Theorem 2.

Theorem 3 (Converse duality). Let $(\bar{x}, \bar{y}, \bar{q})$ be an optimal solution of (SD). Also let

(C_1): the matrix $\nabla_x^2 f(\bar{x}, \bar{y})$ is nonsingular, and

(C_2): $\nabla_x (\nabla_x^2 f(\bar{x}, \bar{y}) \bar{q})$ be a positive definite.

Then $(\bar{x}, \bar{y}, \bar{p} = 0)$ is feasible for (SP) and the objective values of (SP) and (SD) are equal. Furthermore, if the hypotheses of Theorem 1 are met, then $(\bar{x}, \bar{y}, \bar{p})$ is an optimal solution of (SP).

Theorem 4 (Self duality). Let f: $R^n \times R^m \to R$ be skew symmetric and $C_1 = C_2$, then (SP) is self dual. Furthermore, if (SP) and (SD) are dual programs and $(\bar{x}, \bar{y}, \bar{s})$ is an optimal solution for (SP), then $(\bar{x}, \bar{y}, \bar{p} = 0)$ and $(\bar{y}, \bar{x}, \bar{q} = 0)$ are optimal solutions for (SP) and (SD), and $G(\bar{x}, \bar{y}, \bar{p}) = 0 = H(\bar{x}, \bar{y}, \bar{q})$.

Proof. Recasting the problem (SD) as a minimization problem, we have

(SD)$_1$:

$$\text{Minimize } -\left\{ f(x, y) - x^T \left(\nabla_x f(x, y) + \nabla_x^2 f(x, y) q \right) - \frac{1}{2} q^T \nabla_x^2 f(x, y) q \right\}$$

$$\text{Subject to } \nabla_x f(x, y) + \nabla_x^2 f(x, y) q \in C_1^*$$
$$(x, y) \in C_1 \times C_2.$$

Since f is skew symmetric,

$$\nabla_x f(x, y) = -\nabla_y f(y, x)$$

and

$$\nabla_x^2 f(x, y) = -\nabla_y^2 f(y, x);$$

and $C_1 = C_2$, the problem (SD)$_1$ becomes

$$\text{Minimize } \left\{ f(x, y) - x^T (\nabla_y f(x, y) + \nabla_y^2 f(x, y) q) - \frac{1}{2} q^T \nabla_y^2 f(x, y) q \right\}$$

$$\text{Subject to } -\nabla_y f(y, x) - \nabla_y^2 f(y, x) q \in C_2^*$$
$$(x, y) \in C_1 \times C_2$$

which is just the primal problem (SP). Thus (SP) is self dual. Hence if $(\bar{x}, \bar{y}, \bar{q})$ is an optimal solution for (SP), then and conversely, Also, $G(\bar{x}, \bar{y}, \bar{p}) = H(\bar{x}, \bar{y}, \bar{q})$.

Now we shall show that $G(\bar{x}, \bar{y}, \bar{p}) = 0$.

$$G(\bar{x}, \bar{y}, \bar{p}) = f(\bar{x}, \bar{y}) - \bar{y}^T \left(\nabla_y f(\bar{x}, \bar{y}) + \nabla_y^2 f(\bar{x}, \bar{y}) \bar{p} \right) - \frac{1}{2} \bar{p}^T \nabla_y^2 f(\bar{x}, \bar{y}) \bar{p} \tag{21}$$

Since $\bar{y} \in C_2$ and $-\nabla_y f(\bar{x}, \bar{y}) - \nabla_y^2 f(\bar{x}, \bar{y}) \bar{p} \in C_2^*$, therefore, we have

$$-\bar{y}^T\left(\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{p}\right)\geq 0. \tag{22}$$

Using (22) in (21), we have

$$G(\bar{x},\bar{y},\bar{p})\geq f(\bar{x},\bar{y})-\frac{1}{2}p^T\nabla_y^2 f(\bar{x},\bar{y})\bar{p}.$$

Using the conclusion $\bar{p}=0$ of Theorem 2, we get

$$G(\bar{x},\bar{y},\bar{p})\geq f(\bar{x},\bar{y}). \tag{23}$$

Similarly, in view of $x \in C_1$ together with $\nabla_y f(\bar{x},\bar{y})+\nabla_y^2 f(\bar{x},\bar{y})\bar{q}\in C_1^*$, and $\bar{q}=0$, we have

$$H(\bar{x},\bar{y},\bar{q})\leq f(\bar{x},\bar{y}). \tag{24}$$

By Theorem 2, we have

$$f(\bar{x},\bar{y})\leq G(\bar{x},\bar{y},\bar{p})=H(\bar{x},\bar{y},\bar{q})\leq f(\bar{x},\bar{y}).$$

This implies

$$G(\bar{x},\bar{y},\bar{p})=H(\bar{y},\bar{x},\bar{q})=f(\bar{x},\bar{y})=f(y,x)=-f(x,y).$$

Consequently, we have

$$G(\bar{x},\bar{y},\bar{p})=0.$$

4. MAXMIN SYMMETRIC AND SELF DUALITY

Let U and V be two arbitrary sets of integers in R^{n_1} and R^{m_1} respectively. Let K_1 and K_2 be closed convex cones with nonempty interiors in R^{n-n_1}, and R^{m-m_1}, respectively. Let $f(x, y)$ be a real valued function defined on an open set in $R^n\times R^m$ containing $S \times T$ where $S = U \times K_1$ and $T = V \times K_2$. Let K_i^* $(i = 1, 2)$ be the polars of K_i.

We consider the following pair of nonlinear mixed integer programs:

Primal Problem

$$\text{(MSP)}: \underset{x^1}{Max}\ \underset{x^2,y,s}{Min}\ \phi(x,y,s)$$
$$=f(x,y)-(y^2)^T\left(\nabla_{y^2}f(x,y)+\nabla_{y^2}^2 f(x,y)s\right)$$
$$-\frac{1}{2}s^T\nabla_{y^2}^2\ f(x,y)s$$

Subject to $-\nabla_{y^2}f(x,y)-\nabla_{y^2}^2 f(x,y)s\in K_2^*$

$x^1\in U,\ (x^2,y)\in K_1\times T.$

and

Dual Problem

$$\text{(MSD)}: \underset{y^1}{Min}\ \underset{x,y^2,r}{Max}\ \psi(x,y,r)$$
$$=f(x,y)-(x^2)^T\left(\nabla_{x^2}f(x,y)\right.$$
$$\left.+\nabla_{x^2}^2 f(x,y)r\right)f(x,y)-\frac{1}{2}(r^T)^T\nabla_{x^2}^2 f(x,y)r$$

Subject to $\nabla_{x^2}f(x,y)+\nabla_{x^2}^2 f(x,y)r\in K_1^*$

$y^1\in V,\ (x,y^2)\in S\times K_2$

where $s\in R^{m-m_1}$ and $r\in R^{n-n_1}$.

Also their feasible solutions will be denoted by

$$A=\{(x,y,s)\mid x^1\in U,(x^2,y)\in K_1\times T,$$
$$\nabla_{x^2}f(x,y)+\nabla_{x^2}^2 f(x,y)r\in K_1^*\}$$

$$B=\{x,y,r\}\mid y^1\in V,(x,y^2)\in S\times K_2,$$
$$-\nabla_{y^2}f(x,y)-\nabla^2{}_{y^2}f(x,y)s\in K_2{}^*\}.$$

Theorem 5 (Symmetric duality). Let $(\bar{x},\bar{y},\bar{s})$ be an optimal solution of (MSP). Also, Let

(i) $f(x, y)$ be separable with respect to x^1 or y^1,
(ii) $f(x, y)$ be bonvex in x^2 for every (x^1, y), and boncave in y^2 for every (x, y^1),
(iii) $f(x, y)$ be thrice differentiable in x^2 and y^2,
(iv) $\nabla_{y^2}^2 f(x,y)$ is non singular, and
(v) $\nabla_{y^2}(\nabla_{y^2}^2 f(\bar{x},\bar{y})\bar{s})$ is negative definite.

Then

(a) $\bar{s}=0$
(b) $(x^2)^T\nabla_{x^2}f(\bar{x},\bar{y})=0$
(c) $\phi(\bar{x},\bar{y},\bar{s}=0)=\psi(\bar{x},\bar{y},\bar{r}=0)$, and
(d) $(\bar{x},\bar{y},\bar{r})$ is an optimal solution of (MSD)

Proof. Let

$$Z=\underset{x^1}{Max}\ \underset{x^2,y,s}{Min}\ \{\phi(x,y,s):(x,y,s)\in A\}$$

and

$$W=\underset{y^1}{Min}\ \underset{x,y^2,r}{Max}\{\psi(x,y,r):(x,y,r)\in B\}$$

Since $f(x, y)$ is separable with respect to x^1 or y^1 (say, with respect to x^1), it follows that

$$f(x,y)=f^1(x^1)+f^2(x^2,y). \tag{25}$$

Therefore, $\nabla_{y^2}f(x,y) = \nabla_{y^2}f^2(x^2,y)$ and $\nabla_{y^2}^2 f(x,y) = \nabla_{y^2}^2 f^2(x^2,y)$.

Now Z can be rewritten as

$$Z = \underset{x^1}{Max}\ \underset{x^2,y,s}{Min}\Big\{f^1(x^1)+f^2(x^2,y)-(y^2)^T\Big(\nabla^2_{y^2}f(x^2,y)+\nabla^2_{y^2}f(x^2,y)s\Big)-\frac{1}{2}s^T\nabla^2_{y^2}f(x^2,y)s\Big\}$$

subject to $-\nabla_{y^2}f^2(x^2,y)-\nabla^2_{y^2}f^2(x^2,y)s\in K_2^*$

$(x^2,y^2)\in K_1K_2, x^1\in U$ and $y^1\in V\}$

$$= \underset{x^1}{Max}\ \underset{y^1}{Min}\ \underset{x^2,y^2,s}{Min}\Big\{f^1(x^1)+f^2(x^2,y)-(y^2)^T\Big(\nabla_{y^2}f^2(x^2,y)+\nabla^2_{y^2}f^2(x^2,y)s\Big)-\frac{1}{2}s^T\nabla^2_{y^2}f^2(x^2,y)s\Big\},$$

or

$$Z = \underset{x^1}{Max}\,\underset{y^1}{Min}\{f^1(x^1)+\Theta^1(y^1)\,|\,x^1\in U, y^1\in V\} \tag{26}$$

where

$$(\text{MSP})_0: \Theta^1(y^1) = \underset{x^2,y^2,s}{Min}\Big\{f^2(x^2,y)-(y^2)^T\Big(\nabla_{y^2}f^2(x^2,y)+\nabla^2_{y^2}f^2(x^2,y)s\Big)-\frac{1}{2}s^T\nabla^2_{y^2}f^2(x^2,y)s\Big\}$$

Subject to $-\nabla_{y^2}f^2(x^2,y)-\nabla^2_{y^2}f^2(x^2,y)s\in K_2^*$

$(x^2,y^2)\in K_1\times K_2$.

Similarly,

$$W = \underset{y^1}{Min}\,\underset{x^1}{Max}\{f^1(x^1)+\Theta^2(y^1)\,|\,x^1\in U, y^1\in V\} \tag{27}$$

where

$$(\text{MSD})_0: \Theta^2(y^1) = \underset{x^2,y^2,r}{Min}\Big\{f^2(x^2,y)-(x^2)^T\Big(\nabla_{x^2}f^2(x^2,y)+\nabla^2_{x^2}f^2(x^2,y)r\Big)-\frac{1}{2}r^T\nabla^2_{x^2}f^2(x^2,y)r\Big\}$$

Subject to $\nabla_{x^2}f^2(x^2,y)+\nabla_{x^2}f^2(x^2,y)r\in K_1^*$

$(x^2,y^2)\in K_1\times K_2$.

For any given y^1, the program $(\text{MPS})_0$ and $(\text{MPD})_0$ are a pair of second order symmetric dual nonlinear program involving cone treated in the proceeding section and hence in view of assumptions (ii)-(v), Theorem 2 becomes applicable.

Therefore, for $y^1=\bar{y}^1$ we have

$$\bar{s}=0,\ (\bar{x}^2)^T\nabla_{x^2}f^2(\bar{x}^2,\bar{y})=0 \tag{28}$$

and

$$\Theta^1(\bar{y}^1)=\Theta^2(\bar{y}^1) \tag{29}$$

It remains to show that $(\bar{x},\bar{y},\ \bar{r}=0)$ is optimal for (MSD). If this is not the case, there exists $y^{*1}\in V$ such that $\Theta^2(y^{*1})<\Theta^2(\bar{y}^1)$. But then, in view of the assumptions (iv) and (v), we have

$$\Theta^1(\bar{y}^1)=\Theta^2(\bar{y}^1)>\Theta^2(y^{*1})=\Theta^1(y^{*1}),$$

which contradicts the optimality of $(\bar{x}^2,\bar{y}^2,\bar{s}=0)$ for (MSP). Hence $(\bar{x},\bar{y},\bar{r}=0)$ is an optimal solution for (MSD).

Also, (25) and (28) prove (b), whereas $\phi(\bar{x},\bar{y},\bar{s}=0)=\psi(\bar{x},\bar{y},\bar{r}=0)$ follows form (26), (27) and (29).

As earlier, here to, the converse duality theorem (Theorem 6) will be merely stated.

Theorem 6 (Converse duality). Let $(\bar{x},\bar{y},\bar{r})$ be an optimal solution of (MSD), also let

(i) $f(x,y)$ be separable with respect to x^1 and y^1
(ii) $f(\cdot,y)$ be bonvex in x^2 for every (x^1,y), and boncave in y^2 for every (x,y^1),
(iii) $f(x,y)$ be thrice differentiable in x^2 and y^2,
(iv) $\nabla^2_{x^2}f(\bar{x},\bar{y})$ is non singular,
(v) $\nabla_{x^2}(\nabla^2_{x^2}f(\bar{x},\bar{y})\bar{r})$ is positive definite.

Then

(e) $\bar{r}=0$
(f) $(y^2)^T\nabla_{y^2}f(\bar{x},\bar{y})=0$
(g) $\phi(\bar{x},\bar{y},\bar{s}=0)=\psi(\bar{x},\bar{y},\bar{r}=0)$ and
(h) $(\bar{x},\bar{y},\bar{s})$ is an optimal solution of (MSP).

Theorem 7 (Self duality). Let f: $R^n\times R^m\to R$ be skew symmetric. Then (MSP) is self dual. Further, if (MSP) and (MSD) are dual programs and $(\bar{x},\bar{y},\bar{s})$ is an optimal solution for (MSP), then $(\bar{x},\bar{y},\bar{s}=0)$ and $(\bar{x},\bar{y},\bar{r}=0)$ are optimal solution for (MSP) and (MSD) respectively, and $\phi(\bar{x},\bar{y},\bar{s})=0=\psi(\bar{x},\bar{y},\bar{r})$.

Proof. The proof follows along the lines of proof of Theorem 4.

5. SPECIAL CASES

If $C_1=R^n_+$ and $C_2=R^m_+$ where R^n_+ and R^m_+ are nonnegative orthants in R^n and R^m, then the problems (SP) and (SD) will reduce to the following problems treated by Mond (1974):

Primal (P):

$$\text{Minimize } G_0(x, y, p) = f(x, y) - y^T\nabla_y f(x, y) + \nabla_y^2 f(x, y)p) - \frac{1}{2}p^T\nabla_y^2(x, y)p$$

$$\text{Subject to } \nabla_y f(x, y) + \nabla_y^2 + (x, y)p \le 0,$$
$$x \ge 0, y \ge 0,$$

and

Dual (D):

$$\text{Maximize } H_0(x, y, q) = f(x, y) - y^T\nabla_x f(x, y) + \nabla_x^2 f(x, y)p) - \frac{1}{2}q^T\nabla_x^2 + (x, y)q$$

$$\text{Subject to } \nabla_x f(x, y) + \nabla_x^2 f(x, y)q \ge 0$$
$$x \ge 0, y \ge 0 .$$

It is to be remarked that $y \ge 0$ and $x \ge 0$ can be deleted respectively from the problems (P) and (D) as these constraints are not essential.

If only p and q are required the zero vectors, then our problem (SP) and (SD) become the following (first order) symmetric dual programs over cones studied by Bazaraa and Goode (1973):

Primal (P_0)**:**

$$\text{Minimize } f(x, y) - y^T\nabla_y f(x, y)$$
$$\text{Subject to } -\nabla_y f(x, y) \in C_2^*,$$
$$(x, y) \in C_1 \times C_2,$$

Dual (D_0)**:**

$$\text{Maximize } f(x, y) - x^T\nabla_x f(x, y)$$
$$\text{Subject to } -\nabla_x f(x, y) \in C_1^*,$$
$$(x, y) \in C_1 \times C_2.$$

Finally, if U and V are empty sets and $p = s$ and $r = q$, Then (MSP) and (MSD) will become, the problems (SP) and (SD) considered in Section 3.

REFERENCES

1. Balas, E. (1969). Minimax and duality for linear and nonlinear mixed integer programming. In: J. Abadie (Ed.), *Integer and Nonlinear Programming*, North-Holland, Amsterdam.
2. Bazaraa, M.S. and Goode, J.J. (1973). On symmetric duality in nonlinear programming. *Operations Research*, 21(1): 1-9.
3. Bector, C.R. and Chandra, S. (1986). Second order symmetric and self dual programs. *Opsearch*, 23: 89-95.
4. Chandra, S. and Husain, I. (1981). Symmetric dual non differentiable programs. *Bulletin of the Australian Mathematical Society*, 24: 259-307.
5. Dantzig, G.B., Eisenberg, E., and Cottle, R.W. (1965). Symmetric dual nonlinear programs. *Pacific Journal of Mathematics*, 15: 809-812.
6. Devi, G. (1998). Symmetric duality for nonlinear programming problem involving η-convex functions. *European Journal of Operational Research*, 104: 615-621.
7. Dorn, W.S. (1960). A symmetric dual theorem for quadratic programs. *Journal of Operational Society of Japan*, 2: 93-97.
8. Gulati, T.R. and Ahmad, I. (1997). Second order symmetric duality for nonlinear mixed integer programs. *European Journal of Operational Research*, 101: 122-129.
9. Kumar, V., Husain, I., and Chandra, S. (1995). Symmetric duality for minimax mixed integer programming. *European Journal of Operational Research*, 80: 425-430.
10. Mond, B. (1974). Second order duality for nonlinear programs. *Opsearch*, 11: 90-99.
11. Mond, B. and Weir, T. (1989). Generalized convexity and duality. In: S. Schaible and W.T. Ziemba (Eds.), *Generalized Concavity in Optimization and Economics*, Academic Press, New York.
12. Nanda, S. (1988). Invex generalization of some duality results. *Opsearch*, 25(2): 105-111.