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On Non-Differentiable Multiobjective Second Order Symmetric Duality

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Abstract—In this paper, we construct a pair of Mond-Weir type second order symmetric dual problems, in which the objective function contains support function and is, therefore, nondifferentiable. For this pair of problems, we validate weak, strong and converse duality theorems under pseudobonvexity – pseudoboncavity assumption on the kernel function that appears in the problems. A second order self duality theorem is also proved under additional appropriate conditions. Discussion on some particular cases shows that our results generalize earlier results in the related domain.

Keywords—Mond-Weir type second order symmetric dual, Support functions, Duality theorems, Pseudobonvexity, Pseudoboncavity, A second order self dual

1. INTRODUCTION

Duality theory has played an important role in development of optimization theory. Inception of duality theory in linear programming may be traced to classical minmax theory of Von Neumann (1959) and was first explicitly given by Gale et al. (1951). Duality results have proved to be useful in the growth of numerical algorithms for solving certain classes of optimization problems. For linear programming problems, this fact is well established. Duality in linear programming has applications in economics, for example, a linear programming problem dealing a production plan for the maximization of profit within limitations on the availability of resources, then the dual problem enables to offload some available resources so as to minimize the expected cost from each of the resources. This is known as shadow price of resources in economics. For non-linear programming problem, the existence of duality theory helps to develop numerical algorithms, as it provides suitable stopping rule for primal and dual problems. Applications of duality are prominent in physics, management science, economics and engineering.

Following Dorn (1960), first order symmetric and self duality results in mathematical programming has been derived by a number of authors, notably, Dantzig et al. (1965), Mond (1965), Bazaraa and Goode (1973), Mond and Weir (1981). Later Weir and Mond (1989) discussed symmetric duality in multiobjective programming by using the concept proper efficiency. Chandra and Prasad (1993) presented a pair of multiobjective programming problem by associating a vector valued infinite game to this pair. Gulati et al. (1997) also established duality results for multiobjective symmetric dual problem without non-negativity constraints.

The study of second order dual is significant because it has computational advantage over first order duality, for it provides tighter bound for the value of the objective function when approximations are used (Mangasarin (1975)). Motivated with Mangasarian (1975), Mond (1974) was the first to study Wolfe type second order symmetric duality bonvexity – boncavity. Subsequently, Bector and Chandra (1986) established second order symmetric and self duality results for a pair of non-linear programs under pseudobonvexity – pseudoboncavity condition. Devi (1998) formulate a pair of second order symmetric dual programs and established corresponding duality results involving η-bonvex functions and Mishra (2000) extended the results of Devi (1998) to multiobjective non-linear programming. Recently, Sunjeja et al. (2003) presented a pair of Mond-Weir type multiobjective second order symmetric and self dual program without non negativity constraint and proved vairous duality results under bonvexity and pseudobonvexity.

In this paper, we construct in the spirit of Mond and Schechtor (1996) a pair of Mond-Weir type multiobjective second order symmetric dual programs in which objective a support function occurred in the objective function and hence non-differentiable. We validate various duality results under pseudobonvexity – pseudoboncavity assumption. A self duality theorem is also proved. Some special cases are also derived form our results. The importance of this kind

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of programs containing $\sqrt{x^T Bx}$ or a support function lies in the fact that even though objective function and/or constraint functions are nonsmooth, a simple representation for the dual may be found.

2. NOTATIONS AND PRE-REQUISITES

The following conventions for vectors x and y in n-dimensional Euclidian space R^n will be used:

$x < y \Leftrightarrow x_i < y_i,\ i = 1, 2, ..., n,$
$x \leqq y \Leftrightarrow x_i \leqq y_i,\ i = 1, 2, ..., n,$
$x \leq y \Leftrightarrow x_i \leq y_i,\ i = 1, 2, ..., n,$ but $x \neq y,$
$x \not\leq y$ is the negation of $x \leq y.$

For $x, y \in R,\ x \leq y$ and $x < y$ have the usual meaning.

Let ϕ be a twice differentiable from $R^n \times R^m \to R.$ Then $\nabla_1\phi$ and $\nabla_2\phi$ denote gradient vectors with respect to x and y, respectively; $\nabla_1^2\phi$ and $\nabla_2^2\phi$ are respectively, the $n \times n$ and $m \times m$ symmetric Hessian matrices. $\frac{\partial}{\partial y_i}(\nabla_2^2\phi)$ is the $m \times m$ matrix obtained by differentiating the elements of $\nabla_2^2\phi$ with respect to y_i and $\nabla_2(\nabla_1^2\phi(x, y)q)$ denotes the matrix whose (i, j)th element is $\frac{\partial}{\partial y_i}(\nabla_1^2\phi(x, y)q)_j$, where $q \in R^n$.

Definition 1. Let C be compact convex set in R^n The support function of C is defined by

$$s(x|C) = Max\{x^T y : y \in C\}.$$

Definition 2. Let Q be a nonempty convex set in R^n, and let $\psi : Q \to R$ be convex. Then z is called a subgradient of ψ at $\bar{x} \in Q$ if

$$\psi(x) \geq \psi'(\bar{x}) + z^T(x - \bar{x}), \quad \text{for all } x \in Q.$$

A support function, being convex and every where finite, has a subdifferential, i.e.; there exists z such that

$$s(y|C) \geq s(x|C) + z^T(y - x), \quad \text{for all } x \in C.$$

The set of all subdifferential of $s(y|C)$ is given by

$$\partial s(x|C) = \{z \in C : z^T x = s(x|C)\}.$$

For a set Γ, the normal cone to Γ at a point $x \in \Gamma$ is defined by

$$N_\Gamma(x) = \{y \mid y^T(z - x) \leqq 0, \text{ for all } z \in \Gamma\}.$$

When C is a compact convex set, y is in $N_C(x)$ if and only if $s(y|C) = x^T y$, i.e., x is the subdifferentiable of s at y.

Consider the following multiobjective program:

(VP) Minimize $\phi(x)$
Subject to $x \in X_0$

where $f : R^n \to R^n$ and $X_0 \subseteq R^n$.

Definition 3. A feasible point $\bar{x}$ is said to be a weak minimum of (VP), if there does not exist any $x \in X_0$ such that $f(x) < f(\bar{x})$

Definition 4. A feasible point $\bar{x}$ is said to be efficient solution of (VP), if there does not exist any feasible x such that $f(x) \leq f(\bar{x})$.

An efficient solution of (VP) is obviously a weak minimum to (VP).

Definition 5. A feasible point $\bar{x}$ is said to be properly efficient solution of (VP), if it is an efficient solution of (VP) and if there exists a scalar $M > 0$ such that for each i and $x \in X_0$ satisfying $\psi_i(x) < \psi(\bar{x})$, we have

$\psi_i(\bar{x}) - \psi_i(x) \leq M(\psi_j(x) - \psi_j(\bar{x})),$
for some j, satisfying $\psi_j(x) > \psi_j(\bar{x})$.

Definition 6. A twice differentiable real function ϕ defined on $R^n \times R^m$ is said to be any $y \in R^m$

(i) Pseudobonvex in x, if for all $x, q \in R^n$ and $y \in R^n$, and for fixed y,

$$(x - u)^T[\nabla_1\phi(u, y) + \nabla_2^2\phi(u, y)q] \geq 0$$
$$\Rightarrow \phi(x, y) \geq \phi(u, y) - \frac{1}{2}q^T\nabla_1\phi(u, y)q$$

(ii) Pseudoboncave in y, if for all $x, q \in R^n$ and y and $v \in R^m$,

$$(v - y)^T[\nabla_2\phi(x, y) + \nabla_2^2\phi(x, y)p] \leq 0$$
$$\Rightarrow \phi(x, v) \leq \phi(x, y) - \frac{1}{2}p^T\nabla_2\phi(x, y)p$$

(iii) Skew symmetric, when both x and $y \in R^n$, and $\phi(x, y) = -\phi(y, x)$, for all in the domain of ϕ.

3. SECOND ORDER MULTIOBJECTIVE SYMMETRIC DUALITY

Consider the following pair of nondifferentiable second order symmetric dual programs:

(SVP):
Minimize $F(x, y, z, p) = (F_1(x, y, z_1, p), \ldots, F_k(x, y, z_k, p))$

Subject to $\sum_{i=1}^{k} \lambda_i (\nabla_2 f_i(x, y) - z_i + \nabla_2^2 f_i(x, y) p) \leqq 0,$ (1)

$y^T \sum_{i=1}^{k} \lambda_i (\nabla_2 f_i(x, y) - z_i + \nabla_2^2 f_i(x, y) p) \geqq 0,$ (2)

$\lambda > 0,$ (3)

$x \geq 0,\ z_i \in D_i,\ i = 1, 2, \ldots, k$ (4)

and

(SVD):
Maximize $G(u, v, w, q) = (G_1(u, v, w_1, q), \ldots, G_k(u, v, w_k, q))$

Subject to $\sum_{i=1}^{k} \lambda_i (\nabla_1 f_i(u, v) + w_i + \nabla_1^2 f_i(u, v) q) \geqq 0,$ (5)

$u^T \sum_{i=1}^{k} \lambda_i (\nabla_1 f_i(u, v) + w_i + \nabla_1^2 f_i(u, v) q) \leqq 0,$ (6)

$\lambda > 0,$ (7)

$v \geq 0,\ w_i \in C_i,\ i = 1, 2, \ldots, k,$ (8)

where

(i) $F_i(x, y, z_i, p)$
$= f_i(x, y) + s(x \mid C_i) + y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x, y) p$
$G_i(u, v, w_i, q)$
$= f_i(u, v) + s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u, v) q$

(ii) $w = (w_1, \ldots, w_k)$ with $w_i \in C_i$ for $i \in \{1, 2, \ldots, k\}$, $z = (z_1, \ldots, z_k)$ with $z_i \in D_i$ for $i \in \{1, 2, \ldots, k\}$, and $\lambda = (\lambda_1, \ldots, \lambda_k)^T$ with $\lambda_i \in R$ for $i \in \{1, 2, \ldots, k\}$; and

(iii) for each $i \in \{1, 2, \ldots, k\}$, $s(x \mid C_i)$ and $s(y \mid D_i)$ represent support functions of compact convex set C_i in R^n and compact convex set D_i in R^m, respectively.

It is to be remarked here that unlike the formulation of the Mond-Weir type second order symmetric dual programs in Suneja et al. (2003), here we have chosen for each $i \in \{1, 2, \ldots, k\}$, $p_i = p \in R^m$ and $q_i = q \in R^n$ as this choice seems to be in conformity with the analysis for identification of second order dual in nonlinear programming by Mangasarian (1975).

Theorem 1. (Weak Duality): For feasible solutions (x, y, λ, z, p) and (u, v, λ, w, q) for the programs (SVP) and (SVD), let $\sum_{i=1}^{k} \lambda_i (f_i(\cdot, y) + (\cdot)^T w_i)$, for each $w_i \in C_i$, $i \in \{1, 2, \ldots, k\}$ be pseudobonvex at u for fixed y and $\sum_{i=1}^{k} \lambda_i (f_i(x, \cdot) + (\cdot)^T z_i)$, for each $z_i \in D_i$, $i \in \{1, 2, \ldots, k\}$ be pseudoboncave at y. Then

$F(x, y, \lambda, z, p) \nleq G(u, v, \lambda, w, q).$

Proof. By multiplying (5) by x^T and subtracting (6), we have

$$(x - u)^T \sum_{i=1}^{k} \lambda_i \left(\nabla_1 f_i(u, v) + w_i + \nabla_1^2 f_i(u, v) q \right) \geqq 0.$$

This, because of pseudobonvexity of $\sum_{i=1}^{k} \lambda_i (f_i(\cdot, y) + (\cdot)^T w_i)$, implies

$$\sum_{i=1}^{k} \lambda_i (f_i(x, v) + x^T w_i - f_i(u, v) - u^T w_i + \frac{1}{2} q^T \nabla_1^2 f_i(u, v) q) \geqq 0 \tag{9}$$

From (1), (2) and $v \geq 0$, we have

$$(v - y)^T \sum_{i=1}^{k} \lambda_i (\nabla_2 f(x, y) - z_i + \nabla_2^2 f(x, y) p) \leqq 0.$$

By pseudoboncavity of $\sum_{i=1}^{k} \lambda_i (f_i(x, \cdot) + (\cdot)^T z_i)$, from this we get,

$$\sum_{i=1}^{k} \lambda_i (-f_i(x, v) + v^T z_i + f_i(x, y) + y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x, y) p) \geqq 0 \tag{10}$$

On adding (9) and (10), we have

$$\sum_{i=1}^{k} \lambda_i \left(f_i(x, y) + x^T w_i + y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x, y) p \right) - \sum_{i=1}^{k} \lambda_i \left(f_i(u, v) + u^T w_i - v^T z_i - \frac{1}{2} q^T \nabla_1^2 f_i(u, v) q \right) \geqq 0.$$

Since for each $w_i \in C_i$, $x^T w_i \leq s(x \mid C_i)$ and each $z_i \in D_i$, $v^T z_i \leq s(v \mid D_i)$, the above inequality gives

$$\sum_{i=1}^{k} \lambda_i \left(f_i(x, y) + s(x \mid C_i) + y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(x, y) p \right) \geqq \sum_{i=1}^{k} \lambda_i \left(f_i(u, v) - s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u, v) q \right).$$

or

$$\sum_{i=1}^{k} \lambda_i F_i(x, y, z_i, p) \geqq \sum_{i=1}^{k} \lambda_i G_i(u, v, w_i, q).$$

That is,

$F(x, y, z, p) \geqq G(u, v, w, q)$.

This implies

$F(x, y, z, p) \nleq G(u, v, w, q)$

Theorem 2. (Strong Duality): Let f_i, $i \in (1, 2, ..., k)$ be thrice differentiable on $R^n \times R^m$. Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ be a properly efficient solution of (SVP); fix $\lambda = \bar{\lambda}$ in (SVD) and assume that (H1): The set $\{\nabla_2^2 f_1, ..., \nabla_2^2 f_k\}$ is linearly independent, (H2): $\nabla_2(\nabla_2^2(\lambda^T f)\bar{p})$ is positive or negative definite, and (H3): The set $\{\nabla_2 f_1 - \bar{z} + \nabla_2^2 f_1 \bar{p}, ..., \nabla_2 f_k - \bar{z}_k + \nabla_2^2 f_k \bar{p}\}$ is linearly independent. Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q} = 0)$ is feasible for (SVD) and $F(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q})$.

Moreover, if the hypotheses of Theorem 1 are satisfied for all feasible solutions of (SVP) and (SVD), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q})$ is a properly efficient solution of (SVD).

Proof. Since $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}, \bar{p})$ is a properly efficient solution of (SVP), it is weak minimum of (SVP). Hence there exists $\alpha \in R^n$, $\beta \in R^m$, $\mu \in R^k$, $\eta \in R^k$, $\gamma \in R^k$ and $\theta_i \in R^n$, (i = 1, 2, ..., k) such that the following Fritz John optimality condition (Mangasarian and Fromovitz (1967)) are satisfied at $(\bar{x}, \bar{y}, \bar{z}, \bar{\lambda}, \bar{p})$, (suppressing the arguments):

$$\sum_{i=1}^{k} \alpha_i(\nabla_1 f_i + \theta_i) + \sum_{i=1}^{k} \bar{\lambda}_i(\beta - \gamma \bar{y})^T \nabla_1 \nabla_2^2 f_i + \sum_{i=1}^{k} \left\{ (\beta - \gamma \bar{y})\bar{\lambda}_i - \frac{\alpha_i \bar{p}}{2} \right\}^T \nabla_1(\nabla_1^2 f_i \bar{p}) = \eta, \tag{11}$$

$$\sum_{i=1}^{k} (\alpha_i - \gamma \bar{\lambda}_i)(\nabla_2 f_i - \bar{z}_i) + \sum_{i=1}^{k} (\beta - \gamma \bar{y} - \gamma \bar{p})\bar{\lambda} \nabla_2^2 f_i + \sum_{i=1}^{k} \left\{ (\beta - \gamma \bar{y})\bar{\lambda}_i - \frac{\alpha_i \bar{p}}{2} \right\}^T \nabla_2(\nabla_2^2 f_i \bar{p}) = 0, \tag{12}$$

$$\sum_{i=1}^{k} \{(\beta - \gamma \bar{y})\bar{\lambda}_i - \alpha_i \bar{p}\}^T \nabla_2^2 f_i = 0, \tag{13}$$

$$(\beta - \gamma \bar{y})^T \{\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}\} - \mu_i = 0, \tag{14}$$

$$\alpha_1 \bar{y} + (\beta - \gamma \bar{y})\lambda_1 \in N_{D_1}(\bar{z}_i), \quad i = 1, 2, ..., k, \tag{15}$$

$$\theta_i \in C_i, \ \theta_i^T x = s(\bar{x} \mid C_i), \quad i = 1, 2, ..., k, \tag{16}$$

$$\beta^T \sum_{k=1}^{k} \bar{\lambda}_1(\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}) = 0, \tag{17}$$

$$\gamma \bar{y}^T \sum_{k=1}^{k} \bar{\lambda}_1(\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}) = 0, \tag{18}$$

$$\mu^T \bar{\lambda} = 0, \tag{19}$$

$$\eta^T \bar{x} = 0, \tag{20}$$

$$(\alpha, \beta, \gamma, \mu, \eta) \geqq 0, \tag{21}$$

$$(\alpha, \beta, \gamma, \mu, \eta) \neq 0. \tag{22}$$

Since $\bar{\lambda} > 0$, from (19), it follows that $\mu = 0$. Consequently, from (14), we obtain

$$(\beta - \gamma \bar{y})^T (\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}) = 0 \tag{23}$$

In view of (H1), (13) yields

$$(\beta - \gamma \bar{y})\bar{\lambda}_i = \alpha_i \bar{p}, \ i = 1, 2, ..., k. \tag{24}$$

Using (24) in (12), we have

$$\sum_{i=1}^{k} (\alpha_i - \gamma \bar{\lambda}_i)\left\{ (\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}) \right\} + \frac{1}{2} \sum_{i=1}^{k} \bar{\lambda}_i \nabla_2(\nabla_2^2 f_i \bar{p})(\beta - \gamma \bar{y}) = 0 \tag{25}$$

Pre-multiplying (25) by $(\beta - \gamma \bar{y})^T$ and then using (23), we get

$$(\beta - \gamma \bar{y})^T \nabla_2(\nabla_2(\bar{\lambda}^T f)\bar{p}(\beta - \gamma \bar{y}) = 0.$$

In view of (H3), this yields

$$\beta - \gamma \bar{y} = 0. \tag{26}$$

Using (26) in (25), we obtain

$$\sum_{i=1}^{k} (\alpha_1 - \gamma \bar{\lambda}_i)(\nabla_2 f_i - \bar{z}_i + \nabla_2^2 f_i \bar{p}) = 0.$$

This, because of (H3), implies

$$\alpha_i - \gamma \bar{\lambda}_i = 0, \ i = 1, 2, ..., k. \tag{27}$$

If γ = 0, from (11), (26) and (27), we have η = 0, β = 0 and α = 0 respectively. Hence $(\alpha, \beta, \gamma, \mu, \eta) = 0$, contradicting (22). Thus $\gamma > 0$ and from (27), it implies $\alpha_i > 0$ (i = 1, 2, ..., k). From (24) along with (26), we have $\bar{p}$ = 0. Consequently from (11) together with (26) and (21), we obtain

$$\sum_{i=1}^{k} \alpha_i(\nabla_1 f_i(\bar{x}, \bar{y}) + \theta_i) = \eta.$$

By (27) it implies

$$\gamma \sum_{i=1}^{k} \bar{\lambda}_i(\nabla_1 f_i(\bar{x}, \bar{y}) + \theta_i) = \eta$$

which from (20) and (21) along implies

$$\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_1 f(\bar{x}, \bar{y})_i + \theta_i) \geqq 0 \tag{28}$$

and $\quad \bar{x}^T \sum_{i=1}^{k} \bar{\lambda}_i (\nabla_1 f_i(\bar{x}, \bar{y}) + \theta_i) = 0. \qquad (29)$

From (16) and (26) respectively we have

$$w_i \in C_i,\ i = 1, 2, ..., k,\ y \geq 0. \qquad (30)$$

From (29) and (30), it follows that ($\bar{x}$, $\bar{y}$, $\bar{\lambda}$, $\bar{w}$, $\bar{q}$ $= 0$) = ($\bar{x}$, $\bar{y}$, $\bar{\lambda}$, $\bar{\theta}$, $\bar{q}$ $= 0$) where $\theta = (\theta_1, ..., \theta_k)$ is feasible for (SVD).

From (15) along with (26) and $\alpha_i > 0$, it implies $\bar{y} \in N_{D_i}(\bar{z}_i)$, $i \in \{1, 2, ..., k\}$ and this gives

$$\bar{y}^T \bar{z}_i \leq s(\bar{y} \mid D_i),\ i \in \{1, 2, ..., k\}. \qquad (31)$$

Now, using (17), (29) and (31) along with $\bar{p} = \bar{w} = \bar{q}$, we have

$$f_i(\bar{x}, \bar{y}) + s(\bar{x} \mid C_i) - \bar{y}^T \bar{z}_i - \frac{1}{2} \bar{p}^T \nabla_2^2 f_i(\bar{x}, \bar{y}) \bar{p}$$
$$= f_i(\bar{x}, \bar{y}) + s(y \mid D_i) - \bar{x}^T \bar{w}_i - \frac{1}{2} \bar{q}^T \nabla_1^2 f_i(\bar{x}, \bar{y}) \bar{q}$$

for $i \in \{1, 2, ..., k\}$,

or

$$F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}) = G_i(\bar{x}, \bar{y}, \bar{w}_i, \bar{q}) \quad \text{for each } i \in \{1, 2, ..., k\}.$$

This implies

$$F(\bar{x}, \bar{y}, \bar{z}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{w}, \bar{q}) \quad \text{for each } i \in \{1, 2, ..., k\}. \qquad (32)$$

We claim that $(\bar{x}, \bar{y}, \bar{w}, \bar{q})$ is efficient for (SVD). If this would not be the case, then there would exist a feasible solution $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}, \bar{q})$ of (SVD) such that

$$G(\bar{x}, \bar{y}, \bar{w}, \bar{q}) \leq G(\bar{u}, \bar{v}, \bar{w}, \bar{q}),$$

which by (32) gives

$$F(\bar{x}, \bar{y}, \bar{z}, \bar{p}) \leq G(\bar{u}, \bar{v}, \bar{w}, \bar{q}).$$

This is a contradiction to Theorem 1. If $(\bar{x}, \bar{y}, \bar{w}, \bar{q})$ were improperly efficient for (SVD), then for some feasible $(u, v, \bar{\lambda}, w, q)$ of (SVD) and some i

$$\left(f_i(u, v) - s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(\bar{x}, \bar{y}) \bar{q} \right)$$
$$- \left(f_i(\bar{x}, \bar{y}) - s(\bar{y} \mid D_i) + \bar{x}^T \bar{w}_i - \frac{1}{2} q^T \nabla_1^2 f_i(\bar{x}, \bar{y}) \bar{q} \right) > M,$$

for any $M > 0$. Using (32), we have

$$\left[f_i(u, v) - s(v \mid D_i) + u^T w_i - \frac{1}{2} q^T \nabla_1^2 f_i(u, v) q \right]$$
$$- \left[f_i(\bar{x}, \bar{y}) + s(\bar{x} \mid C_i) - y^T z_i - \frac{1}{2} p^T \nabla_2^2 f_i(\bar{x}, \bar{y}) \bar{p} \right] > M,$$

i.e.

$$G_i(u, v, w_i, q) - F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}) > M$$

and for any $\lambda > 0$, this yields

$$\sum_{i=1}^{k} \bar{\lambda}_i G_i(u, v, w_i, q) > \sum_{i=1}^{k} \bar{\lambda}_i F_i(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}),$$

i.e.,

$$\bar{\lambda}^T G(u, v, w_i, q) > \bar{\lambda}^T F(\bar{x}, \bar{y}, \bar{z}_i, \bar{p}).$$

This again contradicts Theorem 1.

Theorem 3. (Converse Duality): Let f_i for $i \in \{1, 2, ..., k\}$ be thrice differentiable on $R^n \times R^n$. Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q})$ be properly efficient of (SVD); fix $\lambda = \bar{\lambda}$ in (SVP) and assume that

(C1): the set $\{\nabla_1^2 f_1, ..., \nabla_1^2 f_k\}$ is linearly independent

(C2): the set $\{\nabla_1^2 f_1 + \bar{w}_i + \nabla_1^2 f_1 \bar{q}, ..., \nabla_1^2 f_k + \bar{w}_k + \nabla_1^2 f_k \bar{q}\}$ is linearly independent, and

(C3): $\nabla_1(\nabla_1^2(\lambda^T f)\bar{q})$ is positive or negative definite.

Then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p} = 0)$ is feasible of (SVP), and $F(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{q})$.

Moreover, if the hypotheses of Theorem 1 are satisfied for all feasible solution of (SVP) and (SVD), then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ is a properly efficient of (SVP).

Proof. It follows on the lines of Theorem 2.

4. SECOND ORDER MULTIOBJECTIVE SELF DUALITY

In this section, we now prove the following self duality theorem for the primal (SVP) and the dual (SVD). We describe (SVP) and (SVD) as the dual programs if the conclusions of Theorem 2 hold.

Theorem 4. (Self Duality): Let for $i \in \{1, 2, ..., k\}$, f_i be skew symmetric and $C_i = D_i$. Then (SVP) is self dual. If also (SVP) and (SVD) are dual programs, and $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}, \bar{p})$ is a joint optimal solution, then so is $(\bar{y}, \bar{x}, \bar{\lambda}, \bar{z}, \bar{p})$ and $F(\bar{x}, \bar{y}, \bar{z}, \bar{p}) = 0$.

Proof. Recasting the dual (SVD) as a minimization program, we have

Minimize

$$\Big(-f_1(x,y)+s(y\,|\,D_1)-x^Tw_i-\frac{1}{2}q^T\nabla_1^2 f_1(x,y)$$
$$-f_k(x,y)+s(y\,|\,D_k)-x^Tw_k-\frac{1}{2}q^T\nabla_1^2 f_k(x,y)q\Big)$$

Subject to

$$-\sum_{i=1}^{k}\lambda_i(\nabla_1 f_i(x,y)+w_i+\nabla_1^2 f_i(x,y)q)\leq 0,$$
$$-x^T\sum_{i=1}^{k}\lambda_i(\nabla_1 f_i(x,y)+w_i+\nabla_1^2 f_i(x,y)q)\geq 0,$$
$$\lambda>0, w_i\in C_i,\ i=1,2,\ldots,k,$$
$$y\geqq 0.$$

Since f_i is skew symmetric, therefore, for each $i\in\{1, 2, \ldots, k\}$, $f_i(x,y)=-f_i(y,x), \nabla_1 f_i(x,y)=-\nabla_2 f_i(y,x)$ and $\nabla_1^2 f_i(x,y)=-\nabla_2^2 f_i(y,x)$.

Therefore, the above program becomes,

Minimize

$$\Big(f_1(y,x)+s(y\,|\,C_1)-x^Tw_i-\frac{1}{2}q^T\nabla_1^2 f_1(y,x)$$
$$-f_k(y,x)+s(y\,|\,C_k)-x^Tw_k-\frac{1}{2}q^T\nabla_1^2 f_k(y,x)q\Big)$$

Subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(y,x)-w_i+\nabla_2^2 f_i(y,x)q)\leqq 0,$$
$$x^T\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(y,x)-w_i+\nabla_2^2 f_i(y,x)q)\geqq 0,$$
$$\lambda>0, w_i\in D_i,\ i=1,2,\ldots,k,$$
$$y\geqq 0.$$

This is just (SVP). Thus $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{q})$ optimal for (SVP) implies $(\bar{y},\bar{x},\bar{\lambda},\bar{z},\bar{q})$ optimal for (SVD). By a similar argument, $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ optimal for (SVP) implies $(\bar{y},\bar{x},\bar{\lambda},\bar{z},\bar{p})$ optimal for (SVD).

If (SVP) and (SVD) are dual programs and $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$ is jointly optimal, then by Theorem 2, we have for each $i\in\{1, 2, \ldots, k\}$,

$$s(\bar{x}\,|\,C_i)-\bar{y}^T\bar{z}_i=-s(\bar{y}\,|\,D_i)+\bar{x}^T\bar{w}_i \text{ and } \bar{p}=\bar{q}=0 \tag{33}$$

For joint optimal solution $(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})$, we have for each $i\in\{1, 2, \ldots, k\}$

$$F_i(\bar{x},\bar{y},\bar{\lambda},\bar{z}_i,\bar{p})$$
$$=f_i(\bar{x},\bar{y})+s(\bar{x}\,|\,C_i)-\bar{y}^T\bar{z}_i-\frac{1}{2}\bar{p}^T\nabla_2^2 f_i(\bar{x},\bar{y})\bar{p}$$
$$=f_i(\bar{x},\bar{y})-s(\bar{y}\,|\,D_i)+\bar{x}^T\bar{w}_i-\frac{1}{2}\bar{q}^T\nabla_1^2 f_i(\bar{x},\bar{y})\bar{q}$$
$$=G_i(\bar{x},\bar{y},\bar{\lambda},\bar{w}_i,\bar{q})$$

This, in view of (33) yields,

$$F_i(\bar{x},\bar{y},\bar{\lambda},\bar{z}_i,\bar{p})=G_i(\bar{x},\bar{y},\bar{\lambda},w_i,\bar{q})=f_i(\bar{x},\bar{y}) \quad \text{for } i\in\{1,2,\ldots,k\} \tag{34}$$

Since $(\bar{y},\bar{x},\bar{\lambda},\bar{z}_i,\bar{p})$ is also a joint optimal solution, one can show, in a similar manner, that

$$F_i(\bar{x},\bar{y},\bar{\lambda},\bar{z}_i,\bar{p})=f_i(\bar{y},\bar{x})=G_i(\bar{y},\bar{x},\bar{\lambda},\bar{w}_i,\bar{q}) \quad \text{for } i\in\{1,2,\ldots,k\} \tag{35}$$

From (34) and (35), we have

$$F_i(\bar{x},\bar{y},\bar{\lambda},\bar{z}_i,\bar{p})=f_i(\bar{x},\bar{y})=f_i(\bar{y},\bar{x})=-f_i(x,y) \quad \text{for } i\in\{1,2,\ldots,k\}$$

Therefore, for each $i\in\{1, 2, \ldots, k\}$

$$F_i(\bar{x},\bar{y},\bar{\lambda},\bar{z}_i,\bar{p})=0 \quad \text{for each } i\in\{1, 2, \ldots, k\}.$$

That is,

$$F(\bar{x},\bar{y},\bar{\lambda},\bar{z},\bar{p})=0.$$

5. SPECIAL CASES

If we choose $C_i=\{0\}$ and $D_i=\{0\}$ for each $i\in\{1, 2, \ldots, k\}$ and p_i corresponding to each f_i instead of having $p=p_i$, for each $i\in\{1, 2, \ldots, k\}$ in the primal (SVP) and q_i corresponding to each f_i in the dual (SVD) instead of having $q=q_i$ for each $i\in\{1, 2, \ldots, k\}$, then these programs reduce to the following programs without non-negativity constraints, studied by Suneja et al. (2003):

Primal (SVP)$_0$:

Minimize $F(x,y,p)=(F_1(x,y,p),\ldots,F_k(x,y,p_k))$

Subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(x,y)+\nabla_2^2 f_i(x,y)p_i)\leqq 0,$$
$$y^T\sum_{i=1}^{k}\lambda_i(\nabla_2 f_i(x,y)+\nabla_2^2 f_i(x,y)p_i)\geqq 0,$$
$$\lambda>0,$$

and

Dual (SVD):

Maximize $G(u,v,q)=(G_1(u,v,q_1),\ldots,G_k(u,v,q_k))$

Subject to

$$\sum_{i=1}^{k}\lambda_i(\nabla_1 f_i(u,v)+\nabla_1^2 f_i(u,v)q_i)\geqq 0,$$

$$u^T \sum_{i=1}^{k} \lambda_i (\nabla_1 f_i(u,v) + \nabla_1^2 f_i(u,v) q_i) \leqq 0,$$
$$\lambda > 0,$$

where for each $i \in \{1, 2, ..., k\}$

$$F_i(x, y, p_i) = f_i(x, y) - \frac{1}{2} p_i^T \nabla_2 f_i(x, y) p_i,$$
$$G_i(u, v, q_i) = f_i(u, v) - \frac{1}{2} q_i^T \nabla_1 f_i(u, v) q_i,$$

where $p = (p_1, ..., p_k)$, $p_i \in R^m$ and $q = (q_1, ..., q_k)$ with $q_i \in R^n$, $\lambda = (\lambda_1, ..., \lambda_k)^T$ with $\lambda_i \in R$.

If only $p = q = 0$, then our programs reduce to the following pair of first order Mond-Weir type symmetric dual programs.

Primal (VP):

Minimize $F(x, y, z) = (F_1(x, y, z_1), ..., F_k(x, y, z_k))$
Subject to

$$\sum_{i=1}^{k} \lambda_i (\nabla_2^2 f_i(x, y) - z_i) \leqq 0,$$
$$y^T \sum_{i=1}^{k} \lambda_i (\nabla_2^2 f_i(x, y) - z_i) \geqq 0,$$
$$x \geqq 0, \quad \lambda > 0,$$
$$z_i \in D_i, \quad i = 1, 2, ..., k,$$

and

Dual (VD):

Maximize $G(u, v, w) = (G_1(u, v, w_1), ..., G_k(u, v, w_k))$
Subject to

$$\sum_{i=1}^{k} \lambda_i (\nabla_1^2 f_i(u, v) + w_i) \geqq 0,$$
$$u^T \sum_{i=1}^{k} \lambda_i (\nabla_1^2 f_i(u, v) + w_i) \geqq 0,$$
$$y \geqq 0, \quad \lambda > 0,$$
$$w_i \in C_i, \quad i = 1, 2, ..., k,$$

where

$$F_i(x, y, z_i) = f_i(x, y) + s(x \mid C_i) - y^z z_i$$

and

$$G_i(u, v, w_i) = f_i(u, v) - s(v \mid C_i) + u^T w_i.$$

For these programs, the duality and self duality results can be proved analogously to those of the preceding sections.

REFERENCES

1. Bazarra, M.S. and Goode, J.J. (1973). On symmetric duality in nonlinearly programming. *Operations Research*, 21(1): 1-9.
2. Bector, C.R. and Chandra, S. (1986). Second order symmetric and self-dual programs. *Opsearch*, 23(2): 98-95.
3. Chandra, S. and Prasad, D. (1993). Symmetric duality in multiobjective programming. *Journal of Australian Mathematical Society*, 35: 198-206.
4. Craven, B.D. (1977). Lagrangian conditions and quasiduality. *Bulletin of Australian Mathematical Society*, 16: 325-339.
5. Dantzig, G.B., Eisenberg, E., and Cottle, R.W. (1965). Symmetric dual nonlinear programs. *Pacific Journal of Mathematics*, 15: 809-812.
6. Devi, G. (1998). Symmetric duality for nonlinear programming problem involving η-convex functions. *European Journal of Operational Research*, 104: 615-621.
7. Dorn, W.S. (1960). A symmetric dual theorem for quadratic programs. *Journal of Operations Research Society of Japan*, 2: 93-97.
8. Gale, D., Kuhn, H.W., and Tucker, A.W. (1951). Linear programming and the theory of games. In: T.C. Koopmans (Ed.), *Activity Analysis of Production and Allocation*, John Wiley & Sons, New York, pp. 317-329.
9. Gulati, T.R., Husain, I., and Ahmed, A. (1997). Multiobjective symmetric duality with invexity. *Bulletin of the Australian Mathematical Society*, 56:25-36.
10. Mangasarian, O.L. and Fromovitz, S. (1967). The Fritz John necessary optimality conditions in the presence of equality and inequality constraints. *Journal of Mathematical Analysis and Applications*, 17: 37-47.
11. Mangasarin, O.L. (1975). Second order higher order duality in nonlinear programming. *Journal of Mathematical Analysis and Applications*, 51: 607-620.
12. Mishra, S.K. (2000). Multiobjective second order symmetric duality with cone constraints. *European Journal of Operational Research*, 126: 675-682.
13. Mond, B. (1965). A symmetric dual theorem for nonlinear programs. *Quarterly Journal of Applied Mathematics*, 23: 265-269.
14. Mond, B. (1974). Second order duality for nonlinear programs. *Opsearch*, 51: 90-99.
15. Mond, B. and Schechter, M. (1996). Non differentiable symmetric duality. *Bulletin of Australian Mathematical Society*, 53: 177-188.
16. Mond, B. and Weir, T. (1981). Generalized concavity and duality. In: S. Schaible and W.T. Ziemba (Eds.), *Generalized Concavity in Optimization and Economics*, Academic Press, New York, pp. 263-279.
17. Von. Neumann, J. (1959). On the theory of games and strategy. In: R.D. Luce and A.W. Tucker (Eds), *Contributions to the Theory of Games Vol. IV, Annals of Mathematics Studies Number 40*, Princeton University Press, Princeton, N.J., pp. 13-42.
18. Suneja, S.K., Lalitha, C.S., and Seema, K. (2003). Second order symmetric duality in multiobjective

programming. *European Journal of Operational Research*, 144: 492-500.
19. Weir, T. and Mond, B. (1989). Generalized convexity and duality in multiobjective programming. *Bulletin of Australian Mathematical Society*, 39: 287-299.