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Redundancy Optimization of Reliability Models Subject to Imperfect Fault Coverage

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Abstract—In this paper we develop a search procedure for redundancy optimization of reliability models. We consider those systems where the reliability cannot be evaluated exactly but must be estimated through Monte Carlo simulation. At each iteration, two neighboring configurations are compared and the one that appears to be better is passed on to the next iteration. The search procedure uses an increasing sequence of observation at each iteration. The acceptance of a new configuration depends on the iteration number, therefore the search process turns out to be time-inhomogeneous Markov chain. We show that if the increase occurs slower than a certain rate, the search process will converge to the optimal set with probability one. The proposed procedure is illustrated through numerical examples of redundancy optimization for reliability systems subject to imperfect fault coverage.

Keywords—System reliability, Simulation, Stochastic optimization, Markov chains, Imperfect fault coverage

1. INTRODUCTION

The redundancy allocation problem involves the determination of system level configuration to maximize system reliability or some other system performance characteristic. This problem is known to be *NP*-hard even with the assumption that component failures are s-independent. Many researchers have proposed a variety of approaches to solve the redundancy allocation problem using, for example, genetic algorithm (Coit and Smith(1996)), dynamic programming (Nakagawa and Miyazaki (1981)) and integer programming (Gen et al. (1990)). Their optimization methods require the assumption that system reliability could be found analytically. There are two major drawbacks in the analytical solution of system reliability. The first is that the individual component reliabilities are assumed to be known, and the second is that the component lifetimes are judged to be conditionally independent. Therefore, Monte Carlo simulation has become the most effective tool for performing realistic reliability analysis, because it allows accounting for realistic operating aspects such as, multi-state systems subject to imperfect fault coverage and allowing behavior that precludes analytical solutions. Further, simulation may be a more efficient approach for high-redundancy situations.

Consider a network of n components $C=\{c_1, \ldots, c_n\}$ with an arbitrary structure function ϕ. The indicator random variable $I_t(i)$, $i = 1, \ldots, n$, represents the state of component c_i at time t, i.e.,

$$I_t(i)=\begin{cases}1 & \text{if component } c_i \text{ is functioning at time } t,\\ 0 & \text{if component } c_i \text{ is not functioning at time } t.\end{cases}$$

Let us assume that the lifetime of the ith component, T_i, is a continuous random variable. Then, $T_i > t \Leftrightarrow I_t(i)=1$, and the reliability of the ith component at time t is, $r_t(i)=\Pr\{I_t(i)=1\}$. The state of the system, or the network, at time t is represented by the vector $I_t=\{I_t(1), \ldots, I_t(n)\}$ of component states. The structure function ϕ is a deterministic binary function of the vector I_t as follows:

$$\phi(I_t)=\begin{cases}1 & \text{if the network is functioning at time } t,\\ 0 & \text{if the network is not functioning at time } t.\end{cases}$$

Let τ be the random variable representing the lifetime of the system, then $\tau > t \Leftrightarrow \phi(I_t)=1$. For a system with configuration s, the reliability at time t is the probability that $\tau > t$ and can be expressed as, $R_t(s)=\Pr\{\phi_s(I_t)=1\}$.

Reliability optimization can be described as the process of determining the optimal design configuration that maximizes network reliability. Let $S=\{1, 2, \ldots, s\}$ be a finite set of system configurations, then we want to find the optimal configurations s^* with maximum values of $R_t(s)$ $\forall s \in S$. We will assume that S has multiple optima and S^* is the set containing them, i.e., $S^* = \{s \in S \mid R_t(s) \geq R_t(s'), \ \forall s' \in S\}$. If there is a unique optimum then $S^*=\{i^*\}$, where $R_t(i^*) > R_t(j) \quad \forall j \in S$,

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$j \neq i^*$.

A general problem formulation of redundancy optimization of a system is defined as follows.

$$\max_{s \in S} \ R_t(s) \tag{1}$$

where $R_t(s) = \Pr\{\phi_s(I_t) = 1\} = \Pr\{\text{system is functioning at time } t \mid \text{configuration } s\}$.

We focus on the case where the objective function is evaluated through simulation. In such a situation, all the function evaluations will include noise, so conventional (deterministic) optimization methods cannot be used to solve this problem.

Yan and Mukai (1992) present a method for simulation optimization. In their approach they compare the observations of the system performance with the observations of a fixed random variable (called a stochastic ruler (SR)) whose range covers all possible values of the objective function samples. They show that under fairly general conditions the SR algorithm will converge with probability one to the global optima.

Andradottir (1995, 1996) proposes two methods for solving discrete simulation optimization. These two methods involve generating Markov chains. The state that is visited most often by these Markov chains is used as an estimate of the solution. She has shown that these two methods converge to an optimal solution almost surely.

Ahmed et al. (1997) present a heuristic integrated approach of simulated annealing with simulation to determine the design parameters of multi-echelon repairable-item inventory system. Ahmed and Alkhamis (2002) integrate the method of simulated annealing with ranking and selection for simulation optimization.

In this paper we consider the problem of optimal allocation of components to the various positions in systems subjected to imperfect fault coverage where uncovered component failures cause immediate system failure, even in the presence of adequate redundancy. Many emergency application of digital systems, especially safety oriented systems, that are used in life critical applications are often designed with sufficient redundancy to be fault tolerant: they are able to detect and locate failures and then reconfigure the system in order to minimize the effects of faults on the service. However, if the system cannot adequately detect, locate, and recover from a fault, then system failure can result even when there exists adequate redundancy. Such an uncovered failure, that is, a fault which is not covered by the automatic recovery mechanisms, leads to global system failure, regardless of the state of the system. As an example, in computing systems, an undetected fault may affect the subsequent calculations and operations, and then operate on incorrect data, possibly leading to overall system failure (Amari et al. (1999)). Therefore, an accurate analysis must account for not only the complex system-structure, but the system fault and error recovery behavior as well. An accurate analysis is needed in fixing the optimal level of redundancy, otherwise, an increase in redundancy could decrease the system reliability due to imperfect fault coverage (Amari et al. (1999, 2004)).

Amari et al. (2004) study the optimal allocation of components to maximize the reliability of special classes of systems subjected to imperfect fault coverage such as parallel, *k*-out-of-*n* series-parallel systems. They show that the reliability of systems subjected to imperfect fault coverage decreases after a certain level of redundancy. Therefore, there exists an optimal level of redundancy that maximizes the system reliability.

This paper is organized as follows: Section 2 presents problem structure. In section 3, we present our search strategy using increasing sample size and prove that it converges to the optimal set with probability one. Then in section 4, we present computational experience for two test cases. Finally, section 5 contains some concluding remarks.

2. PROBLEM STRUCTURE

In this section we propose a stochastic algorithm for solving the discrete optimization problem discussed in section 1. Our goal is to find the configuration that has maximum reliability $R_t(s)$. Assume $S = \{1, 2, ..., s\}$ is a non-empty discrete finite set of configurations and the search is conducted by picking an initial point in S and then comparing a neighboring point according to the following definition.

Definition 2.1. For each $s \in S$ there exists a subset $N(s)$ of $S - \{s\}$ which is called the set of neighbors of s, such that each point in $N(s)$ can be reached from s in a single transition.

Assumption 2.1. For any pair $(i, j) \in S \times S$, j is reachable from i, i.e. there exists a finite sequence, $\{n_m\}_{m=0}^{l}$ for some l, such that $i_{n_0} = i$, $i_{n_\ell} = j$ and $i_{n_{m+1}} \in N(i_{n_m})$, $m = 0, 1, 2, ..., \ell - 1$.

Our search is organized in such a way that the next solution candidate is found among the neighbors of the present candidate. Now we impose a stochastic structure to the selection of a candidate among the neighbors by the following function Q. Given $s \in S$, a candidate is selected from $N(s)$ such that the probability of selecting a neighbor $s' \in N(i)$ is equal to $Q(s, s')$ which is defined as follows.

Definition 2.2. A function Q: $S \times S \to [0, 1]$ is said to be a generating probability function for S and N if

1. $Q(s, s') > 0 \Leftrightarrow s' \in N(s)$ and
2. $\sum_{s' \in S} Q(s, s') = 1$ for all $s \in S$.

$Q(s, s')$ is the probability of generating solution point s' as a candidate for the next solution point, when the system is in solution point s. We will consider $Q(s, s')$

such that the probability is distributed uniformly over $N(s)$. Given $s \in S$, a candidate solution is selected among $N(s)$ such that the probability of selecting a neighbor $s' \in N(s)$ is equal to $Q(s, s')$, given by

$$Q(s, s') = \begin{cases} 1/|N(s)| & \text{for } s' \in N(s), \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $|N(s)|$ represents the cardinality of $N(s)$.

3. STOCHASTIC ALGORITHM FOR OPTIMAL SYSTEM RELIABILITY

Our goal is to find the solution point that has maximum reliability $R_t(s)$. Given we start our search at solution point $i \in S$, a candidate j is selected from $N(i)$ with probability $Q(i, j)$ and a move is performed from i to j if $\phi_j(I_t) > \phi_i(I_t)$, where

$$\phi_\ell(I_t) = \begin{cases} 1 & \text{with probability } R_t(\ell), \\ 0 & \text{with probability } 1 - R_t(\ell), \end{cases} \quad \text{for all } \ell \in S \tag{3}$$

At each iteration k a sample of n_k pairs of observations is taken from i and j, (ϕ_{i1}, ϕ_{j1}), (ϕ_{i2}, ϕ_{j2}), ..., (ϕ_{in}, ϕ_{jn}). A candidate neighbor $j \in N(i)$ is accepted and a move is performed from i to j if the observations from j dominates the observations from i in each pair, i.e. if $\bigcap_{m=1}^{n_k} (\phi_{im} < \phi_{jm})$ occurs, where $\bigcap_{m=1}^{n_k} \{\phi_{im} < \phi_{jm}\} \equiv (\phi_{i1} < \phi_{j1}) \cap (\phi_{i2} < \phi_{j2}) \cap \ldots \cap (\phi_{in_k} < \phi_{jn_k})$. Accordingly the acceptance probability, $A_{ij}(k)$, the probability of accepting solution point j once it is generated from solution point i, is defined as follows:

$$\begin{aligned} A_{ij}(k) &= [\Pr\{\phi_i(I_t) < \phi_j(I_t)\}]^{n_k} \\ &= [\Pr(\phi_i(I_t) = 0, \ \phi_j(I_t) = 1)]^{n_k} \\ &= [(1 - R_t(i)) R_t(j)]^{n_k} \end{aligned} \tag{4}$$

Define $\{X_k, \ k = 0, 1, 2, \ldots\}$ to be the states of the search process at each iteration k where X_k is the current state of the search process at iteration k. The details of the algorithm, Rel-Opt, for finding the configuration with optimal $R_t(s)$ are given below.

Rel-Opt Algorithm:

1. Select a starting point $X_0 \in S$. Let $k = 0$. Go to *Step* 2.
2. Given $X_k = i$, choose a candidate Z_k from $N(i)$ with probability distribution; $\Pr[Z_k = j \mid X_k = i] = Q_{ij}, j \in N(i)$.
3. Given $Z_k = j$, generate n_k pairs of independent observations from i and j (ϕ_{i1}, ϕ_{j1}), (ϕ_{i2}, ϕ_{j2}), ..., $(\phi_{in_k}, \phi_{jn_k})$.
4. Given $Z_k = j$, set $X_{k+1} = \begin{cases} Z_k & \text{if } \bigcap_{m=1}^{n_k} \{\phi_{im} < \phi_{jm}\}, \\ X_k & \text{otherwise.} \end{cases}$
5. Set $k = k + 1$, update n_k as defined in Theorem 3.1. Go to *Step* 2.

The Stochastic process $\{X_k, \ k = 0, 1, 2, \ldots\}$ produced by the above algorithm is a discrete-time inhomogenous Markov chain with transition matrices P_1, P_2, ..., where

$$\begin{aligned} P_k &= \left(p_{ij}^{(k,k+1)}\right), \ i, j \in S \\ &= \left(\frac{1}{|N(i)|}[(1 - R_t(i)) \ R_t(j)]^{n_k}\right), \ i, j \in S. \end{aligned} \tag{5}$$

Here $p_{ij}^{(k,k+1)}$ is the probability of going from state i at time k to state j at time k+1 which depends on k, where

$$p_{ij}^{(k,k+1)} = \Pr[X_{k+1} = j \mid X_k = i] = Q(i, j) A_{ij}(k)$$
$$= \begin{cases} \dfrac{1}{|N(i)|}[(1 - R_t(i)) R_t(j)]^{n_k} & \text{if } j \in N(i), \\ 1 - \displaystyle\sum_{l \in N(i)} \frac{1}{|N(l)|}[(1 - R_t(i)) R_t(l)]^{n_k} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

3.1 Convergence analysis of Rel-Opt algorithm

Convergence analysis for $\{X_k, \ k = 1, 2, \ldots\}$ to the limit probability vector π^∞, where $\pi_i^\infty = \lim_{k \to \infty} \Pr(X_k = i)$, (i.e., $\sum_{i \in S^*} \pi_i^\infty = 1$ and $\pi_i^\infty = 0$ for $i \notin S^*$) can be proved using the strong ergodicity theory of inhomogeneous Markov chains (Isaacson and Madsen (1976), Iosifescu (1980)). To prove ergodicity we need the following definitions:

Definition 3.1. Let P be a stochastic matrix. The ergodic coefficient of P, denoted by $\alpha(P)$, is defined by $\alpha(P) = \min_{i,k} \sum_{j=1}^{\infty} \min \ (p_{ij}, p_{kj})$.

Definition 3.2. A finite inhomogeneous Markov chain is weakly ergodic if $\forall i, j, \ \ell \in S, \ \forall m > 0$, $\lim_{k \to \infty} (p_{il}^{(m,k)} - p_{jl}^{(m,k)}) = 0$, where $p_{ij}^{(m,k)}$ is the (i, j)th element of $\prod_{r=m}^{k} P_r$.

Definition 3.3. A finite inhomogeneous Markov chain is strongly ergodic if there exists a stochastic vector q^*, such that $\forall i, j \in S,\ \forall m > 0: \lim_{k\to\infty} p_{il}^{(m,k)} = q_j^*$.

It is usually difficult to show that an inhomogeneous Markov chain is strongly ergodic directly from the definition. In the next section, we first show that the search process $\{X_k,\ k = 0,\ 1,\ ...\}$ is weakly ergodic.

3.2 Weak ergodicity of Rel-Opt algorithm

If $0 < R_t(i) < 1$, then $0 < A_{ij}(k) < 1$; therefore all states will communicate if they are connected by neighborhoods. Assume for any $i, j \in S = \{1, 2, ..., s\}$ there exists a path say, $\{i = i_0 \to i_1 \to i_2 \to ... \to i_{l-1} \to i_l = j\}$ such that $i_{m+1} \in N(i_m)$, $m = 0, 1, ..., \ell$. Let $L = \max_{i,j}\{\ell_{i,j}\}$ where $\ell_{i,j}$ = minimum length path from i to j. For fixed sample size n, the transition probability p_{ij} is given by $p_{ij} = \frac{1}{|N(i)|}[(1-R_t(i))R_t(j)]^n$ for $j \in N(i)$. Let P^L denotes the L-step transition matrix. Now we find a bound on all the off-diagonal elements of P^L.

Lemma 3.1. For P^L based on sample size n, all the off-diagonal elements are $\geq \xi^n$ and $\alpha(P^L) \geq |S|\xi^n$, where $\xi = \prod_{m=0}^{l-1} \frac{1}{|N(i_m)|}[(1-R_t(i_m))]$.

Proof.

$$P\{path\ i = i_0 \to i_1 \to ... \to i_{\ell-1} \to i_\ell = j\}$$

$$= \prod_{m=0}^{l-1} \frac{1}{|N(i_m)|}[(1-R_t(i_m))R_t(i_{m+1})]^n$$

$$= \frac{1}{\prod_{m=0}^{\ell-1}|N(i_m)|}\left[\prod_{m=0}^{\ell-1}(1-R_t(i_m))\ R_t(i_{m+1})\right]^n$$

$$\geq \frac{1}{[\max_i |N(i)|]^l}[(1-R_t(i_0))(1/4)^{l-1}\ R_t(i_l)]^n$$

(because $(1-R_t(i))\,R_t(i) > 1/4$)

$$\geq \left[\frac{\min_i(1-R_t(i))(1/4)^L \min_i R_t(i)}{[\max_i |N(i)|]^L}\right]^n = [\xi]^n > 0.$$

Because $0 < R_t(i)'s < 1$ and $|S| < \infty$, $\min R_t(i)$ and $\min(1 - R_t(i))$ exist. By definition 3.1 it follows that $\alpha(P^L) \geq |S|\xi^n$.

Theorem 3.1. Let the sample size at iteration k, n_k, satisfy $n_k \leq trunc\left[1+\frac{\log(1+\frac{k}{L})}{\log(\frac{1}{\xi})}\right]$, for $k = 0, 1, \dots$, where $trunc[x]$ denotes the greatest integer smaller or equal to x, then the Markov chain $\{X_k\}$ generated by search Rel-Opt Algorithm using these n_k is weakly ergodic.

Proof. A Markov chain for which $\sum_{j=0}^{\infty}\alpha(P^{(n_j,n_{j+1})})$ diverges for some sequence $n_1 < n_2 < ... < n_j < n_{j+1} < ...$ is weakly ergodic (Isaacson and Madsen (1976)). Consider the sequence $n_i = (i-1)L$, $i = 1, 2, 3, ...$ Let $k(n)$ denote the number of iterates which sample at size n in multiplies of L. Then $\sum_{j=0}^{\infty}\alpha(P^{(n_j,n_{j+1})}) = \sum_{n=1}^{\infty}k(n)\alpha((P)^L) \geq \sum_{n=1}^{\infty}k(n)\,|S|\,\xi^n = \infty$, if $k(n) \geq (1/\xi)^n$.

Therefore the series diverges if $k(n) \geq (1/\xi)^n$. To find the condition on the sample size at each iterate such that the series diverges, we proceed as follows. Let B_i, $i \geq 2$ denote the total number of iterates in multiples of L performed with sample size $\leq (i-1)$, i.e., $B_i = \sum_{n=1}^{i-1}k(n)$.

Assume that $B_i \geq \sum_{j=1}^{i-1}(\frac{1}{\xi})^j = \frac{1-(\frac{1}{\xi})^i}{1-\xi} - 1$. Then $1 + B_i \geq \frac{1-(\frac{1}{\xi})^i}{1-\frac{1}{\xi}} \cong (\frac{1}{\xi})^{i-1}$, $\log(1+B_i) \geq (i-1)\log(\frac{1}{\xi})$, $i \leq 1+\frac{\log(1+B_i)}{\log(\frac{1}{\xi})}$. Since iterates are measured in units of L, for individual iterates we have $n_k \leq 1+\frac{\log(1+\frac{k}{L})}{\log(\frac{1}{\xi})}$, where $\xi = \left[\frac{\min_i(1-R_t(i))(1/4)^L \min_i R_t(i)}{[\max_i |N(i)|]^L}\right]$. In words, if the sample size in Rel-Opt Algorithm increases slower than $n_k = trunc\left[1+\frac{\log(1+\frac{k}{L})}{\log(\frac{1}{\xi})}\right]$, then the search process $\{X_k\}$ is weakly ergodic.

3.3 Strong ergodicity of Rel-Opt algorithm

As in the case of weak ergodicity, it is usually difficult to show that an inhomogeneous chain is strongly ergodic directly from the definition. In this section, we show that the search process generated by Rel-Opt Algorithm is strongly ergodic. For a fixed sample size n, the search process $\{X_k,\ k = 0,\ 1,\ ...\}$ becomes a time homogeneous Markov chain since the state transition probability p_{ij} becomes independent of k. In this case the equilibrium probabilities are (Ahmed et al. (1998)):

$$\pi_i(n) = \frac{|N(i)|[R_t(i)/(1-R_t(i))]^n}{\sum_{j\in S}|N(j)|[R_t(j)/(1-R_t(j))]^n} \quad i = 1, 2, ..., s \tag{7}$$

To prove the strong ergodicity of the Markov chain associated with Rel-Opt Algorithm, we have to prove the following two Lemmas.

Lemma 3.2. (Monotone property of the equilibrium probabilities). For a sequence of probability vectors $\pi(n)$, of the form

$$\pi_i(n) = \frac{|N(i)|[R_t(i)/(1-R_t(i))]^n}{\sum_{j\in S}|N(j)|[R_t(j)/(1-R_t(j))]^n} \quad i = 1, 2, ..., s.$$

The following hold,

1. For each $i \notin S^*$, if $n < n'$ then $\pi_i(n) \le \pi_i(n')$,
2. For each $i \notin S^*$, there exists an integer n_i such that if $n_i \le n < n'$ then $\pi_i(n) \ge \pi_i(n')$.

Proof. Our proof follows the proof of Proposition 6.1 in Yan and Mukai (1992). Consider $\pi_i(n) = a_i b_i^n \Big/ \sum_{j=1}^{S} a_j b_j^n$ as a function of a real variable n, where a_i = $|N(i)|$, and b_i = $R_t(i)/(1-R_t(i))$. Then $\pi_i(n)$ is differentiable with respect to n. Noting $\frac{\mathrm{d}\alpha^n}{\mathrm{d}n} = \alpha^n \ln\alpha$ then, we have,

$$\frac{\mathrm{d}\pi_i(n)}{\mathrm{d}n} = \frac{\left[\sum_{j=1}^{s} a_j b_j^n\right] a_i b_i^n \ln b_i - a_i b_i^n \left[\sum_{j=1}^{s}(a_j b_j^n \ln b_j)\right]}{\left[\sum_{j=1}^{s} a_j b_j^n\right]^2}$$

$$= \frac{a_i b_i^n \left\{\sum_{j=1}^{s}(a_j b_j^n \ln b_i - a_j b_j^n \ln b_j)\right\}}{\left[\sum_{j=1}^{s} a_j b_j^n\right]^2}$$

$$= \frac{a_i b_i^n \left\{\sum_{j=1}^{s}[a_j b_j^n(\ln b_i - \ln b_j)]\right\}}{\left[\sum_{j=1}^{s} a_j b_j^n\right]^2}$$

$$= \frac{\sum_{j=1}^{s} \frac{a_j b_j^n}{a_i b_i^n} \ln(\frac{b_i}{b_j})}{\left[\sum_{j=1}^{s} a_j b_j^n / a_i b_i^n\right]^2}$$

$$= \frac{\sum_{j\in S^*} \frac{a_j b_j^n}{a_i b_i^n} \ln(\frac{b_i}{b_j}) + \sum_{j\in S-S^*} \frac{a_j b_j^n}{a_i b_i^n} \ln(\frac{b_i}{b_j})}{\frac{1}{(a_i b_i^n)^2} \sum_{j\in S^*} a_j b_j^n + \frac{1}{(a_i b_i^n)^2} \sum_{j\in S-S^*} a_j b_j^n}$$

Suppose that $i \notin S^*$. Then $R_t(i) > R_t(j)$ for all $i \notin S^*$ and $j \notin S^*$. Therefore $\frac{R_t(i)}{1-R_t(i)} > \frac{R_t(j)}{1-R_t(j)}$, i.e., $b_i > b_j$, so that $\frac{\mathrm{d}\pi_i(n)}{\mathrm{d}n} > 0$ for any $n > 0$. This implies conclusion (1). Suppose that $i \notin S^*$, then $R_t(i) < R_t(j)$ for all $i \notin S^*$. In this case as n goes to infinity the first term of the numerator decreases monotonically to zero, while the second term monotonically decreases to $-\infty$. On the other hand the first term of the denominator decreases to zero, while the second term increases monotonically to $+\infty$ as n goes to infinity. Therefore, there exists a real n_i such that $\frac{d\pi_i(n)}{dn} < 0$ for any $n \ge n_i$. This implies conclusion (2).

Lemma 3.3. The probability vector $\pi(n)$ consisting of $\pi_i(n)$ in (7) satisfies $\sum_{k=1}^{\infty}\|\pi(n_{k+1}) - \pi(n_k)\| < \infty$, where $\|v\|$ represents the l_1- norm of vector v, i.e., $\|v\| = \sum_{i=1}^{s}|v_i|$.

Proof. (Yan and Mukai (1992)). It follows from the monotone property of π_i that there exists an integer k^* such that, for any $k > k^*$,

$$\pi_i(n_{k+1}) \ge \pi_i(n_{k'}) \ \forall i \in S^*$$
$$\pi_i(n_{k+1}) \le \pi_i(n_{k'}) \ \forall i \notin S^*$$

Hence, for any $k > k^*$,

$$\|\pi(n_{k+1}) - \pi(n_k)\|$$
$$= \sum_{i\in S^*}[\pi_i(n_{k+1}) - \pi_i(n_k)] - \sum_{i\notin S^*}[\pi_i(n_{k+1}) - \pi_i(n_k)].$$

Note that from $\sum_{i\in S^*}\pi_i(n_k) + \sum_{i\notin S^*}\pi_i(n_k) = \|\pi(n_k)\| = 1$, we conclude that, for any $k > k^*$,

$$\|\pi(n_{k+1}) - \pi(n_k)\| = 2\sum_{i\in S^*}[\pi_i(n_{k+1}) - \pi_i(n_k)].$$

Therefore, we have, for any $\ell \ge k^*$,

$$\sum_{k=k^*}^{\ell}\|\pi(n_{k+1}) - \pi(n_k)\|$$
$$= 2\sum_{i\in S^*}[\pi_i(n_{\ell+1}) - \pi_i(n_{k^*})] \le 2\sum_{i\in S^*}\pi_i(n_{\ell+1}) \le 2.$$

Theorem 3.2. Let n_k be as defined in Theorem 3.1. Then the Markov chain $\{X_k\}$ generated by Rel-Opt Algorithm is strongly ergodic. Furthermore, $\lim_{k\to\infty}\Pr(X_k \in S^*) = 1$.

Proof. It follows from Theorem 3.1 that the Markov chain $\{X_k\}$ is weakly ergodic. Then, using Theorem V.4.3. of (Isaacson and Madsen (1976)) the Markov chain $\{X_k\}$ is strongly ergodic.

4. EXPERIMENTAL RESULTS

As stated before, Monte Carlo simulation allows accounting for realistic systems without simplified assumptions i.e., allowing behavior that preclude analytical solutions. In this section we implement Rel-Opt Algorithm for two test cases to find the optimal redundancy levels that maximize system reliability subject to imperfect fault coverage. We choose test case 1 where analytical solutions exist, so that we can easily compare the simulation optimization results with the analytical results. These test cases have been adapted from examples provided by Amari et al. (1999, 2004).

It is well known that for the precise Monte Carlo evaluation of a highly reliable system, the crude Monte Carlo requires a very large number of simulations. Therefore, in order to reduce the number of Monte Carlo runs, a variance reduction technique should be used. In this paper, we implement the geometric sampling technique developed by Konak et al. (2004). This is a new event-driven sampling technique for network reliability estimation. It provides variance reduction with minimum overheads and is most effective for highly reliable network.

4.1 Test case 1

4.1.1 *R*-out-of-*m* systems

Consider an r-out-of-m system with m i.i.d components where the system functions if and only if at least r of the m components function. Let p denote component reliability and p_c denote the probability that the system can recover given a fault has occurred. Furthermore, assume that p and p_c are given as fixed probabilities. We apply the simulation optimization procedure for two systems with $r = 1, 2$.

Table 1 presents the parameters used for the above systems with the optimal solutions obtained by Rel-Opt Algorithm and the analytical solutions. On the kth iteration with configuration i, and a candidate configuration j, we simulate the system for t units of time and let $\phi_l(I_t)$, $l = i, j$, be the indicator random variable that takes the value 1 if the system is functioning at time t and zero otherwise. For our search strategy, we let n_k grows at a slow rate. Figure 1 shows the performance of our algorithm when it is applied to solve the above systems. The x-axis shows the number of iterations that were used in our simulation, while the y-axis shows the average estimated optimal reliability value at the estimated optimal solution based on 100 replications. The optimal redundancy levels that maximize system 1 and 2 reliabilities are 3 and 7 with corresponding system reliabilities equal to 0.984 and 0.989, respectively. The convergence trajectories shown in Figure 1 indicate that our algorithm converges to the optimal values.

4.1.2 Series-Parallel system with non *s*-identical components

The redundancy allocation problem considered here pertains to a series-parallel system with non s-identical components subjected to imperfect fault-coverage. It is assumed that there exists m parallel subsystems connected in series. For each subsystem, the reliability of each component and the fault coverage are denoted by p_i and $p_{c,i}$, respectively. The objective is to find the optimal number of components in each subsystem simultaneously which maximize the overall system reliability.

We implement Rel-Opt Algorithm to this optimization problem with the following parameters: $m = 2$, $p_1 = 0.9$, $p_2 = 0.75$, $p_{c,1} = 0.9$, and $p_{c,2} = 0.95$. Using Rel-Opt Algorithm, the optimal number of components in subsystem 1 and 2 are 2 and 3, respectively with system reliability equal to 0.923 which are the same as the analytical solution.

4.2 Test case 2: Bridge network

Suppose we have a bridge network consisting of 5 subsystems where the subsystems are not identical. All subsystems are designed with components in a r-out-of-m configuration. The objective is to select the level of redundancy in each subsystem simultaneously to maximize the overall system reliability. Figure 2 presents a typical example of the system configuration being considered.

Table 2 shows the input parameters of the problem. Note that m_{min} is the lower bound for the level of redundancy in each subsystem. Finding the set of optimal number of components in each subsystem is an NP-hard problem. With exhaustive searching, the optimal vector for maximizing system reliability is (3, 5, 2, 5, 2) and the corresponding system reliability is 0.999. We let n_k grows at a slow rate and applying our search algorithm, the optimal redundancy levels that maximize system reliability are (3, 5, 2, 5, 2) with corresponding system reliability equal to 0.999 which are the same as the exhaustive search results.

Figure 3 shows the performance of Rel-Opt Algorithm when it is applied to solve the above systems. The x-axis shows the number of iterations that were used in our simulation, while the y-axis shows the average estimated optimal reliability value at the estimated optimal solution based on 100 replications. The convergence trajectories shown in Figure 3 indicate that our algorithm converges to the optimal value after about 700 iterations.

Table 1. System parameters and optimal solution for test case 1

System	p	p_c	r	m (analytical solution)	m (Rel-Opt algorithm)
1	0.90	0.950	1	3	3
2	0.75	0.995	2	7	7

Table 2. System parameters

Subsystem	r	m_{min}	p	p_c
S_1	2	3	.95	.9999
S_2	3	3	.995	.9999
S_3	2	2	.99	.9995
S_4	3	4	.995	.9999
S_5	2	2	.99	.999995

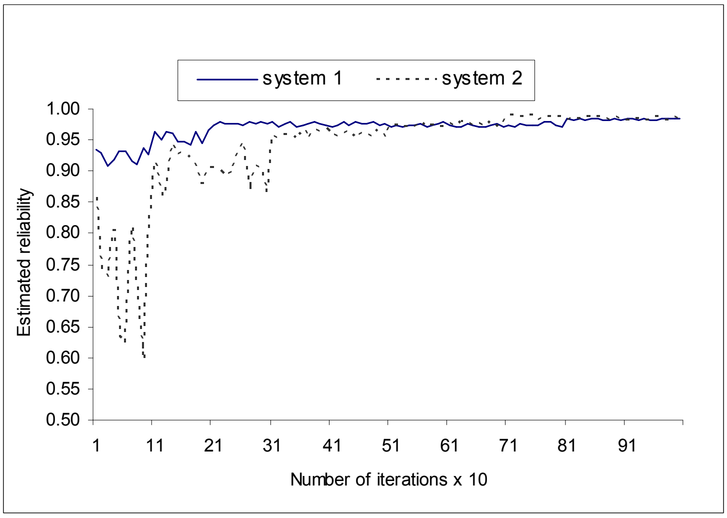

Figure 1. Performance of Rel-Opt algorithm for test case 1 (r-out-of m systems).

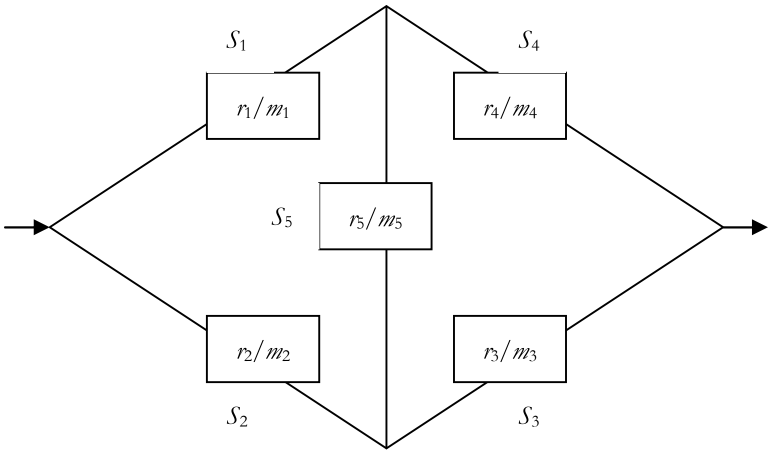

Figure 2. A bridge structure.

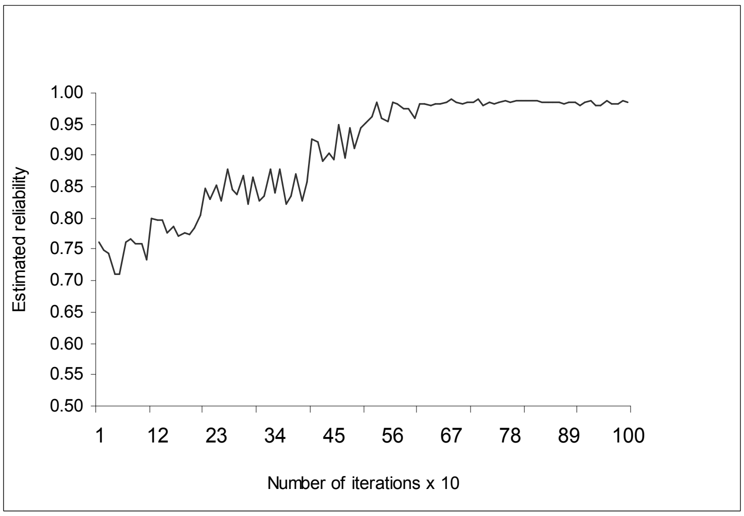

Figure 3. Performance of Rel-Opt Algorithm for test case 2.

5. CONCLUSION

We have presented an efficient search method for finding the optimal allocation of system components that maximize system reliability subject to imperfect fault coverage. We considered systems where reliability cannot be obtained analytically and has to be estimated through Monte Carlo simulation. In each iteration, two neighboring configurations are compared and the one that appears to be better is passed on to the next iteration. The search procedure uses an increaseing sequence of observation at each iteration. We show that if the increase occurs slower than a certain rate, the Markov chain is strongly ergodic and that the search process converges to the optimal set with probability one. Computational experience shows the efficiency of the proposed search method.

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