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An Inventory Model with Inventory Level-Dependent Demand Rate, Deterioration, Partial Backlogging and Decrease in Demand

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Abstract—A deterministic inventory model for infinite time-horizon incorporating inventory level-dependent demand rate, deterioration begins after a certain time, partial backlogging and decrease in demand is developed. The salient feature of the developed model is the introduction of the concept of fractional decrease in demand due to ageing of inventory. Demand at any instant depends linearly on the on-hand inventory level at that instant. Deterioration of items begins after a certain time from the instant of their arrival in stock. A numerical example is presented to illustrate the application of developed model.

Keywords—EOQ, Deterioration, Partial backlogging, Fractional decrease in demand

1. INTRODUCTION

Traditional inventory models were formed under the assumption of constant demand or time-dependent demand. Recently a number of inventory models are formed considering the demand to be dependent on inventory level viz. initial inventory-level-dependent and instantaneous stock-level dependent.

The pioneer researcher who formed inventory models taking initial stock-level dependent demand is Gupta and Vrat (1986). Mandal and Phaujdar (1989a) corrected the flaw in Gupta and Vrat (1986) model using profit maximization rather than cost minimization as the objective. Baker and Urban (1988) developed an inventory model taking demand rate in polynomial functional form; dependent on inventory level. The same functional form was used by Datta and Pal (1990a). Datta and Pal (1990b) proposed an inventory model for deteriorating items, inventory-level dependent demand and shortages. Mandal and Phaujdar (1989n) proposed an inventory model in which shortages are allowed and demand is dependent on stock-level. In this model the rate of deterioration is assumed to be variable. Sarker et al. (1997) took demand to be dependent on inventory level incorporating an entirely new concept of decrease in demand (due to ageing of inventory or products reaching closer to their expiry date).

Montgomery et al. (1973) developed both deterministic and stochastic models considering the situation in which a fraction of demand during the stock out period is backordered and remaining is lost forever. Rosenberg (1979) developed a lot-size inventory model with partial backlogging taking "fictitious demand rate" that simplifies the analysis. Padmanabhan and Vrat (1995) proposed an inventory model for perishable items taking constant rate of deterioration, incorporating the three cases of complete, partial and no backlogging. Zeng (2001) developed an inventory model using partial backordering approach and minimizing the total cost function. This model identifies the conditions for partial backordering policy to be feasible. Recently Dye (2007) developed an inventory policy taking demand to be a function of selling price together with partial backlogging.

There can be certain products, which start deteriorating after a certain period of time rather than their immediate arrival in the stock. Also there are a number of products for which the demand decreases due to ageing of these products. Here we develop a model incorporating above two realistic features and partial backlogging that is more generalized than the model given by Dye and Ouyang (2005).

2. ASSUMPTIONS AND NOTATIONS

The inventory model is developed under following assumptions and notations:

2.1 Assumptions

1. Replenishment rate is infinite.
2. The lead-time is zero.
3. The demand function is deterministic and is a known function of instantaneous-stock-level $I(t)$ given by,

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$$D(t)=\begin{cases}\alpha+\beta I(t) & 0\le t<t_1\\ \alpha & t_1\le t<T\end{cases}$$

where $\alpha>0,\ 0<\beta<1$. The demand function defined above in (0, t_1) for the proposed model becomes

$$D(t)=\begin{cases}\alpha+\beta I(t), & 0<t\le\mu\\ \alpha+\beta I(t)-\gamma I(t), & \mu<t\le t_1\end{cases}$$

where γ is defined in notation section.

4. The deterioration of the items begins after a time μ from the instant of their arrival in stock. Hence the deterioration of the items is assumed to be governed by the function,

$$\theta(t')=\theta_0 H(t'-\mu)=\begin{cases}\theta_0, & t'>\mu\\ 0, & t'<\mu\end{cases}$$

where t' is the time measured from the instant of arrival of replenishment, θ_0 ($0<\theta_0<1$) is a constant and $H(t'-\mu)$ is Heaviside's function.
5. The time-horizon of the system is infinite.
6. Inventory level remains non-negative for a time t_1 in each cycle after which shortages are allowed and unsatisfied demand is backlogged at the rate of $\dfrac{1}{[1+\delta\ (T-t)]}$. The backlogging parameter δ is a positive constant, and $t_1<t<T$.
7. Lot size q raises the initial inventory level at the beginning of each cycle to S after fulfilling the backorder quantity ($q-S$).

2.2 Notations:

T	=	The fixed length of each ordering cycle.
$P(t_1, T)$	=	Profit function per unit time.
p	=	The selling price per unit.
A	=	The ordering cost per order.
C	=	The cost price per unit.
i	=	The inventory carrying cost as fraction, per unit per unit time.
R	=	The fixed opportunity cost of lost sales.
C_2	=	The shortage cost, per unit per unit time.
S	=	The inventory level at time $t=0$.
θ_0	=	The rate of deterioration ($0<\theta_0<1$).
γ	=	The demand decrease rate factor.

3. MODEL FORMULATION

Let q be the number of items received at the beginning of the cycle and $(q-S)$ items be delivered for the fulfillment of backorder, leaving a balance of S items as the initial inventory at time $t=0$. The inventory level falls to level S_1 ($<S$) at time $t=\mu$ due to demand. After time $t=\mu$ the demand decrease rate γ becomes effective and the inventory further depletes due to demand and deterioration $\theta_0 I$. At $t=t_1$ the inventory level falls to zero and shortages are backlogged up to time $t=T$, when next lot arrives. The diagrammatic representation of the system is as Figure 1.

The variation of inventory level $I(t)$, with respect to time can be described by the following differential equations:

$$\frac{dI(t)}{dt}=-\left[\alpha+\beta I(t)\right],\ 0\le t\le\mu \tag{1}$$

$$\frac{dI(t)}{dt}+\theta_0 I(t)=-\left[\alpha+\beta I(t)-\gamma I(t)\right],\ \mu<t\le t_1 \tag{2}$$

$$\frac{dI(t)}{dt}=-\frac{\alpha}{\left[1+\delta(T-t)\right]},\ t_1<t\le T \tag{3}$$

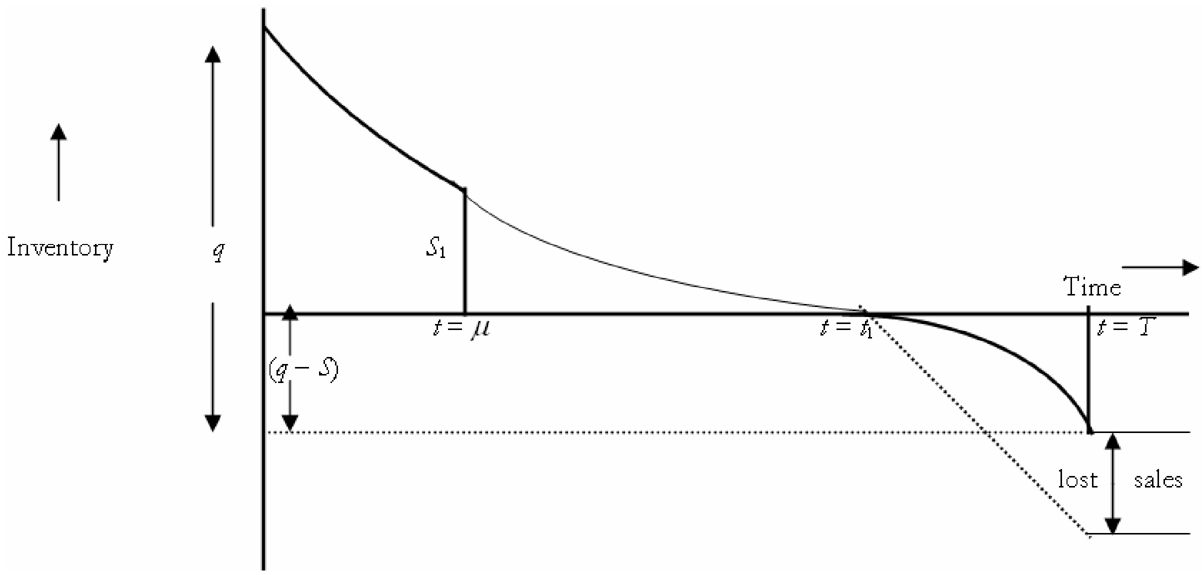

Figure1. Inventory level versus time relationship.

Solutions of (1), (2) and (3) with the conditions $I(0) = S$ and $I(t_1) = 0$ are respectively

$$I(t) = \left(S + \frac{\alpha}{\beta}\right)e^{-\beta t} - \frac{\alpha}{\beta} \tag{4}$$

$$I(t) = \frac{\alpha}{(\theta_0 + \beta - \gamma)}\left[e^{(\theta_0+\beta-\gamma)(t_1-t)} - 1\right] \tag{5}$$

$$I(t) = -\frac{\alpha}{\delta}\left\{\ln\left[1+\delta(T-t_1)\right] - \ln\left[1+\delta(T-t)\right]\right\};\quad t_1 < t \le T \tag{6}$$

Using conditions $I(\mu) = S_1$ in (4) and (5) it gives

$$S_1 = \left(S + \frac{\alpha}{\beta}\right)e^{-\beta\mu} - \frac{\alpha}{\beta} \tag{7}$$

$$S_1 = \frac{\alpha}{(\theta_0 + \beta - \gamma)}\left[e^{(\theta_0+\beta-\gamma)(t_1-\mu)} - 1\right] \tag{8}$$

Elimination of S_1 from (7) and (8) gives

$$S = \frac{\alpha e^{\beta\mu}}{(\theta_0+\beta-\gamma)}\left[e^{(\theta_0+\beta-\gamma)(t_1-\mu)} - 1\right] + \frac{\alpha}{\beta}(e^{\beta\mu} - 1) \tag{9}$$

Holding cost per cycle

$$\begin{aligned}&= C \times i\left[\int_0^{\mu} I(t)dt + \int_{\mu}^{t_1} I(t)dt\right]\\ &= C \times i \times L - C \times i \times N \times t_1 + C \times i \\ &\quad \times M\left[e^{(\theta_0+\beta-\gamma)(t_1-\mu)} - 1\right]\end{aligned} \tag{10}$$

(Using Eq. (4) and (5))

where

$$L = \frac{\alpha\left(e^{\beta\mu} - 1\right)}{\beta^2} - \frac{\alpha\mu(\theta_0 - \gamma)}{\beta(\theta_0+\beta-\gamma)}$$

$$M = \frac{\alpha}{\beta(\theta_0+\beta-\gamma)}\left[\left(e^{\beta\mu} - 1\right) + \frac{\beta}{(\theta_0+\beta-\gamma)}\right]$$

and $N = \dfrac{\beta}{(\theta_0+\beta-\gamma)}$

Shortage cost per cycle

$$\begin{aligned}&= -C_2\int_{t_1}^{T} I(t)dt\\ &= \frac{C_2\,\alpha}{\delta}\left[(T-t_1) - \frac{1}{\delta}\ln\left[1+\delta(T-t_1)\right]\right]\end{aligned} \tag{11}$$

(Using Eq. (6))

Opportunity cost due to lost sales

$$\begin{aligned}&= \alpha R\int_{t_1}^{T}\left[1 - \frac{1}{\left[1+\delta(T-t)\right]}\right]dt\\ &= \alpha R\left[(T-t_1) - \frac{1}{\delta}\ln\left[1+\delta(T-t_1)\right]\right]\end{aligned} \tag{12}$$

Purchase cost per cycle

$$\begin{aligned}&= C \times S + C \times \text{Amount backordered (at } t = T)\\ &= C\left[\frac{\alpha e^{\beta\mu}}{(\theta_0+\beta-\gamma)}\left[e^{(\theta_0+\beta-\gamma)} - 1\right] + \frac{\alpha}{\beta}\left(e^{\beta\mu} - 1\right)\right]\\ &\quad + C\frac{\alpha}{\delta}\ln\left[1+\delta(T-t_1)\right]\end{aligned} \tag{13}$$

Sales revenue per cycle

$$\begin{aligned}&= p\left\{\int_0^{\mu}\text{demand in } [0,\mu]\, dt + \int_{\mu}^{t_1}\text{demand in } [\mu, t_1]\, dt\right.\\ &\quad \left. + \int_{t_1}^{T}\text{demand in } [t_1, T]\right\}\end{aligned}$$

Using (1), (2), (3), (9) and solving we get

$$\begin{aligned}&= p\left\{\frac{\alpha}{(\theta_0+\beta-\gamma)}\left[\left(e^{\beta\mu} - 1\right) + \frac{(\beta-\gamma)}{(\theta_0+\beta-\gamma)}\right]\times\right.\\ &\quad \left[e^{(\beta-\gamma)(t_1-\mu)} - 1\right] + \frac{\alpha}{\beta}\left(e^{\beta\mu} - 1\right) + \frac{\alpha\theta_0}{(\theta_0+\beta-\gamma)}(t_1 - \mu)\\ &\quad \left. + \frac{\alpha}{\delta}\ln\left[1+\delta(T-t_1)\right]\right\}\end{aligned} \tag{14}$$

Profit per unit time is given by

$$\begin{aligned}\xi\,(t_1, T) = \frac{1}{T}\{&[\text{sales revenue} - \text{ordering cost}\\ &- \text{holding cost} - \text{shortage cost}\\ &- \text{opportunity cost} - \text{purchase cost}\}\end{aligned} \tag{15}$$

Using (10) to (14) in (15), we get

$$\begin{aligned}\xi\,(t_1, T) = \frac{1}{T}\{&X e^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Z + Yt_1 - X_1(T-t_1)\\ &+ Y_1\ln\left[1+\delta(T-t_1)\right]\}\end{aligned} \tag{16}$$

where

$$X = \frac{\alpha e^{\beta\mu}(p - C)}{(\theta_0+\beta-\gamma)} - \frac{\alpha p\theta_0}{(\theta_0+\beta-\gamma)^2} - C \times i \times M$$

$$Y = \frac{\alpha p\theta_0}{(\theta_0+\beta-\gamma)} + C \times i \times N$$

$$Z = -X + \frac{\alpha}{\beta}(p-C)\left(e^{\beta\mu}-1\right) - \frac{\alpha p \theta_0 \mu}{\left(\theta_0+\beta-\gamma\right)} - C_1 L - C_3$$

$$X_1 = \frac{C_2\alpha}{\delta} + \alpha R \quad \text{and} \quad Y_1 = \frac{1}{\delta}\left[X_1 + \alpha(p-C)\right]$$

For the maximization of profit we set,

$$\frac{\partial \xi\ (t_1, T)}{\partial t_1} = 0 \ \text{ and } \ \frac{\partial \xi\ (t_1, T)}{\partial T} = 0$$

$$\frac{\partial \xi}{\partial t_1} = 0 \Rightarrow T = t_1 + \frac{Y_1}{\eta(t_1)} - \frac{1}{\delta} \quad \text{(Using Eq. (16))} \tag{17}$$

where,

$$\eta(t_1) = X\left(\theta_0+\beta-\gamma\right)e^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Y + X_1$$

and

$$\frac{\partial \xi}{\partial T} = 0 \Rightarrow \frac{-1}{T^2}\Big\{X\, e^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Z + Y t_1 - X_1(T-t_1) + Y_1 \ln\left[1+\delta(T-t_1)\right]\Big\} + \frac{1}{T}\left\{-X_1 + \frac{Y_1\delta}{\left[1+\delta(T-t_1)\right]}\right\} = 0 \tag{18}$$

On eliminating T from (17) and (18), we get an equation in a single variable t_1 as,

$$Xe^{(\theta_0+\beta-\gamma)(t_1-\mu)}\left[1+\frac{\left(\theta_0+\beta-\gamma\right)}{\delta} - \left(\theta_0+\beta-\gamma\right)t_1\right] + Z - Y_1 + \frac{1}{\delta}\left[Y+X_1\right] + Y_1 \ln Y_1\delta - Y_1 \ln\Big[Y + X_1 + X\left(\theta_0+\beta-\gamma\right)e^{(\theta_0+\beta-\gamma)(t_1-\mu)}\Big] = 0 \tag{19}$$

Eq. (19) can be solved by using any iterative method, say Newton Raphson method as given by Chu (1999). Let t_1^* be the optimal root obtained we then get T^* by using (17). Hence t_1^* and T^* jointly constitute the optimal solution provided the following conditions are satisfied,

$$\left(\frac{\partial^2 \xi(t_1, T)}{\partial t_1^2}\right)_{t=t_1^*} < 0, \quad \left(\frac{\partial^2 \xi(t_1, T)}{\partial T^2}\right)_{T=T^*} < 0$$

and

$$\left(\frac{\partial^2 \xi(t_1, T)}{\partial t_1^2}\right)_{t=t_1^*}\left(\frac{\partial^2 \xi(t_1, T)}{\partial T^2}\right)_{T=T^*} > \left(\frac{\partial^2 \xi(t_1, T)}{\partial t_1 \partial T}\right)^2_{\substack{t_1=t_1^* \\ T=T^*}}$$

By using the optimal values of t_1^* and T^* in (16), optimal profit $\xi(t_1^*, T^*)$ can be obtained.

Case I. Complete backlogging ($\delta = 0$).

In this case, shortages are completely backlogged and profit per unit time is given as

$$\xi(t_1, T) = \frac{1}{T}\Big\{Xe^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Yt_1 + Z + (p-C)(T-t_1)\alpha - \frac{C_2\alpha}{2}(T-t_1)^2\Big\}$$

In this case our model reduces to Jain and Kumar (2007).

Case II. Complete lost sales ($\delta \to \infty$).

For this case, from (17) we get $T^* = t_1$. In this situation optimal solution does not allow shortage.

In this case profit per unit time is given as,

$$\xi(t_1) = \frac{1}{t_1}\left[Xe^{(\theta_0+\beta-\gamma)(t_1-\mu)} + Yt_1 + Z\right]$$

4. EXAMPLE

For the numerical illustration of the developed model, we consider the following values of parameter in appropriate units:

α = 600
β = {0.2, 0.3, 0.4}
μ = {0.2, 0.3, 0.4}
δ = {1, 5, 10, 25, 50}
γ = 0.01
S = 7
A = 250
θ_0 = 0.05
C = 5
I = 0.35
C_2 = 3

The results obtained for constant and varying μ (and vice-versa) are shown in Tables 1 and 2. The tables also incorporate results of case I, δ = 0, (i.e. complete backlogging) and case II, $\delta \to \infty$ (i.e. complete lost sales) as a special case.

5. CONCLUSION

Most of the inventory models in the existing inventory management literature take deterioration of items into account. These models fall short, considering the situation where the quality of the product becomes susceptible to continuous deterioration. In such cases the demand of the items is supposed to decrease due to the aging of inventoried items. Therefore to sketch a more realistic policy, the inventory managers should take into account this realistic feature of decrease in demand for the product, which degrades in value as the time passes, and their demand tends to decrease.

Table 1. Sensitivity of β, δ and profit function for μ = 0.2

β		$\delta \to 0$	1	5	10	25	50	∞
	t_1	0.5659	0.63867	0.67245	0.6791	0.68353	0.68568	0.68668
0.2	T	0.87274	0.75667	0.70698	0.69774	0.69115	0.68955	0.68668
	ξ	841.87641	566.75268	528.62619	521.08117	516.05884	514.2117	512.47409
	t_1	0.60231	0.67106	0.70211	0.70814	0.71214	0.71354	0.71498
0.3	T	0.89697	0.78262	0.73448	0.7253	0.71927	0.71715	0.71498
	ξ	843.84668	597.79521	564.81034	558.36507	554.08376	552.57988	551.03494
	t_1	0.64692	0.71134	0.73952	0.74493	0.74849	0.74974	0.75103
0.4	T	0.92804	0.81598	0.7696	0.76085	0.75509	0.75308	0.75103
	ξ	847.27278	631.61539	603.76705	598.3832	594.82362	593.57566	592.29489

Table 2. Sensitivity of μ, δ and profit function for β = 0.2

μ		$\delta \to 0$	1	5	10	25	50	∞
	t_1	0.5659	0.63867	0.67245	0.6791	0.68353	0.68568	0.68668
0.2	T	0.87274	0.75667	0.70698	0.69774	0.69115	0.68955	0.68668
	ξ	841.87641	566.75268	528.62619	521.08117	516.05884	514.2117	512.47409
	t_1	0.57613	0.64634	0.6788	0.68518	0.68941	0.6909	0.69243
0.3	T	0.87896	0.76224	0.71263	0.70314	0.69687	0.69468	0.69243
	ξ	842.71407	576.82077	540.27405	533.06264	528.26645	526.57948	524.84518
	t_1	0.58854	0.65669	0.68802	0.69416	0.69824	0.69967	0.70114
0.4	T	0.88876	0.77112	0.72135	0.71185	0.70559	0.70339	0.70114
	ξ	843.55173	583.90669	548.68534	541.75345	537.14667	535.52657	533.86225

In this paper, an inventory model is developed for inventory-level-dependent demand, deterioration (beginning after a fixed time μ), shortages; incorporating two realistic features like decrease in demand (γ) and backlogging.

It is clear from Table 1, that for constant β, increase in the value of δ results in a decrease in the value of cycle length and the optimal profit. Also for a constant δ, increase in value of β increases both the cycle length and the optimal profit. A similar trend is observed from Table 2 for constant value of μ.

With the infiltration of the concept of decrease in demand due to ageing of inventory and deterioration starting after a certain time as the items are actually received in stock; a more precise inventory model is formed in this paper which considerably increases the profit as compared to the previously developed Dye and Ouyang (2005).

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