International Journal of
Operations Research

International Journal of Operations Research Vol. 5, No. 3, 169-179 (2008)

$M^X/G/1$ Queue with Bernoulli Service Schedule under Both Classical and Constant Retrial policies

Madhu Jain[1, *], G.C. Sharma[1], and Sapna Chakrawarti[2]

[1]Institute of Basic Science, Khandari, Agra-282002 (India)

[2]St. John's College, Agra-282002 (India)

Received March 2007; *Revised September 2007*; *Accepted May 2008*

Abstract—Retrial queues have been widely used to model the problems in telephone switching systems, telecommunications networks, etc., wherein a job receiving incomplete service may seek service repeatedly, until served successfully. This paper deals with $M^X/G/1$ bulk retrial queue with Bernoulli service schedule. The server is being subjected to active breakdowns. The investigation is made by taking the concept of the impatient customers under both classical and constant retrial policy. Chapman-Kolmogrov equations are constructed by using supplementary variable technique and the queue size distribution by using the probability generating method has been obtained. We also analyze the stochastic decomposition property for retrial queue under consideration. Some performance characteristics and special cases are established.

Keywords—$M^X/G/1$, Retrial, Bernoulli feedback, Supplementary variable, Stochastic decomposition, Generating function, Queue size

1. INTRODUCTION

Retrial queue is characterized by the feature that a customer arriving when all servers accessible to him are busy, then it leaves the service area and after some random times repeats its demand. This feature plays a special role in several computer and communication networks. Other applications include stacked aircraft waiting to land, a message in a packet network switching, etc. Thus the area of possible applications of such queues is wide. At present the theory of retrial queues is recognized as an important part of queueing theory. A comprehensive survey on the retrial queues was done by Yang and Templeton (1987), Falin (1990). Atenica et al. (2003) studied a retrial queue with starting failures, feedback and general retrial policy. Atenica and Moreno (2006) considered a discrete time $Geo/G/1$ retrial queue with second optional service and obtained the distributions for orbit size and the system size. A single server infinite capacity retrial queue with Poisson input and deterministic bulk service rule was analyzed by Chang (2006).

In the present paper, we are concerned with the bulk retrial $M^X/G/1$ queue with Bernoulli service schedule, wherein the server is subjected to the active breakdowns. Queueing problems with batch retrial queue are common in a number of real situations as its applicability is connected with the performance evaluation of many real time systems; to illustrate we refer a local area network operating under transmission protocols like CSMA/CD (cf. Choi et al. (1992), Artalejo et al. (2005)). A detail study on the bulk arrival queue and their applicability was done by Choudhary and Templeton (1983). Kumar and Madheswari (2003) discussed some more complicated queueing situations with retrials and batch arrivals. Yechiali (2004) investigated the batch arrival queue with server vacations. Dudin et al. (2004) gave the analysis of $BMAP/G/1$ retrial system in which the customers arrive according to a batch markovian arrival process and on finding the server busy, enters into the orbit. Choudhary et al. (2007) suggested the steady state analysis of a batch arrival queueing problem with two-phase service along with the Bernoulli schedule and vacation. Furthermore, Choudhary and Madan (2007) addressed a batch arrival queue with Bernoulli vacation schedule and a random setup time under a restricted admissibility policy.

In the practical situations the service interruptions due to server breakdown is quite common. Wang et al. (2001) gave the reliability analysis of retrial queues with server breakdowns. Wu et al. (2005) considered the $M/G/1$ retrial queues with general retrial times in which the customers may balk or renege at particular times. By using the probability generating function method they obtained the queue length. Recently, Ke (2006) have analyzed the retrial queues with server breakdowns under the concepts of GSPN analyzer and NT policies, respectively.

In the present investigation, we analyze the two types of retrial policies, i.e. the classical retrial policy and the constant retrial policy. In the classical retrial policy, the probability of a repeated attempt during a given time

* Corresponding author's email: madhujain@sancharnet.in

interval is $n\theta dt + O(dt)$ (cf. Falin and Templeton (1997)) when 'n' customers are in the orbit and in case of constant retrial policy it is independent of the number of customers and is given as $\theta dt + O(dt)$. The constant retrial policy was first introduced by Fayolle (1986) who modeled a telephone exchange system.

Stochastic decomposition is a major result for the vacation models; the pioneer work in this direction was due to Furhamann and Cooper (1985). Artalejo and Falin (1994) applied this property for the retrial queues. Further, Natalia (2006) suggested the stochastic decomposition property for the retrial queues with breakdown by taking an exponential assumption for the retrial times as an approximation in the non-exponential case. In this context, recently the stochastic decomposition structures of the queue length and the waiting times in an $M/M/1$ queue have been demonstrated by Liu et al. (2007).

In this investigation, an attempt has been made to analyze Bernoulli service schedule for a bulk retrial queue with impatience customers. Using the supplementary variable technique, the steady state equations are constructed. Employing the probability generating function and Laplace transform, the steady state distributions of the server state and orbit length are derived. We establish a general stochastic decomposition law for $M^X/G/1$ retrial queueing system under consideration. The rest of the paper is structured as follows. The model description is given in section 2. The probability generating functions of queue size distributions for both classical and constant types of the retrial policies are obtained in section 3. Further, the performance measures are given in section 4. In section 5, the special cases are deduced. Section 6 provides the stochastic decomposition property. In section 7, we facilitate the numerical results. Section 8 concludes the paper by highlighting the noble features of the study done.

2. MODEL DISCRIPTION

Consider a single unreliable server retrial batch arrival queue where the customers arrive according to a Poisson process with rate λ. The server is subjected to failure and is sent for an immediate repair. Let 'X' be the random variable denoting the batch size and $P(X = k) = c_k$, k = 1, 2, 3, ... with $\sum_{k=1}^{\infty} c_k = 1$. Let $L(t)$ be the number of repeated customers and $S(t)$ is the number of customers present in the system at time t. Also let $\xi(t)$ be the server's state such that $\xi(t)$ = 0, 1, 2 or 3, according as server is idle, busy, broken-down with a customer waiting and broken-down state without a customer waiting with the server, respectively.

We assume that the successive attempts made by the same customer are exponentially distributed with rate θ_n, given that there are 'n' customers in the orbit. Upon return from the retrial group if the customer finds the server busy, then it always rejoins the retrial orbit and continues till it is completely served. We have taken into consideration the two policies of retrial, i.e. classical retrial policy in which the repeated customers reattempt for the service at a fixed rate $\theta_n = \theta$, whereas in the constant retrial policy the repeated customers become discouraged and rejoin at the reduced rate $\theta_n = \frac{\theta}{n}$, due to the more number of customers present in the orbit. The customer who was under service during the server breakdown gets lost with probability $q \in (0,1]$, and thus regarded as the impatient customer. On the contrary, the customer who waits for the server with probability p ($p = 1 - q$) till its repair is over is regarded as the patient customer.

The underlying process is defined by $X(t) = [\xi(t), N(t), \theta_1(t), \theta_2(t)]$, which is Markov. We introduce a supplementary variable $\theta_1(t)$ as the elapsed time variable of the customer when $\xi(t) \in \{1, 2\}$; and if $\xi(t) \in \{2, 3\}$, then $\theta_2(t)$ represents the elapsed repair time. The service time of the customer is independent random variable with common distribution function $B_1(x)$, density function $b_1(x)$. Laplace-Stieltjes transform of density function and n-th moments of service time are denoted by as $\beta_1(s)$ and $\beta_{1,n}$, respectively. Moreover, the repair time distribution function is $\beta_2(y)$ with its density function $b_2(y)$. Also $\beta_2(s)$ and $\beta_{2,n}$ denote its Laplace-Stieltjes transform and n-th moments, respectively. Then the conditional completion rates $b_1(x)$ and $b_2(y)$ for the service and repair respectively, are

$$b_1(x) = \frac{B_1'(x)}{1 - B_1(x)} \quad \text{and} \quad b_2(y) = \frac{B_2'(y)}{1 - B_2(y)}$$

Now, we define the limiting probabilities as:

$$P_{0,n} = \lim_{t \to \infty} P\left[\xi(t) = 0, N(t) = n; n \geq 0\right]$$

Also the limiting probability densities are defined as given below:

$$P_{1,n}(x)$$
$$= \lim_{t \to \infty} P\left[\xi(t) = 1, N(t) = n, x < \theta_1(t) \leq x + dx; n \geq 0, x \geq 0\right]$$
$$P_{2,n}(x, y) = \lim_{t \to \infty} P\left[\xi(t) = 2, N(t) = n, \theta_1(t) = x, y \leq \theta_2(t)\right]$$
$$P_{3,n}(y)$$
$$= \lim_{t \to \infty} P\left[\xi(t) = 3, N(t) = n, y < \theta_2(t) \leq y + dy; n > 1, y \geq 0\right]$$

Using the supplementary variable technique, we obtain the following equations at equilibrium:

$$0 = -(\lambda + n\theta_n)P_{0,n} + \int_0^\infty p_{1,n}(x) b_1(x) dx + (1 - \delta_{0,n}) \int_0^\infty p_{3,n}(y) b_1(y) dy, \; n \geq 0 \quad (1)$$

$$\frac{d}{dx} p_{1,n}(x) = -\left(\lambda + \alpha + b_1(x)\right) p_{1,n}(x)$$

$$+(1-\delta_{0,n})\lambda\sum_{k=1}^{\infty}c_k p_{1,n-k}(x)$$

$$+\int_0^{\infty} p_{2,n}(x,y)b_2(y)dxdy,\ n\geq 0 \tag{2}$$

$$\frac{\partial}{\partial y}p_{2,n}(x,y)=-\left(\lambda+b_2(y)\right)p_{2,n}(x,y)$$

$$+(1-\delta_{0,n})\sum_{k=1}^{\infty}c_k p_{2,n-k}(x,y),\ n\geq 0 \tag{3}$$

$$\frac{d}{dy}p_{3,n}(y)=-\left(\lambda+b_2(y)\right)p_{3,n}(y)$$

$$+(1-\delta_{1,n})\lambda\sum_{k=1}^{n}c_k p_{3,n-k}(y),\ n\geq 1 \tag{4}$$

Boundary conditions are given as:

$$p_{1,n}(0)=\lambda\sum_{k=1}^{n+1}c_k p_{0,n-k+1}+(n+1)\theta_{n+1}p_{0,n+1},\ n\geq 0 \tag{5}$$

$$p_{2,n}(x,0)=p\alpha\ p_{1,n}(x),\ n\geq 0 \tag{6}$$

$$p_{3,n}(0)=q\alpha\int_0^{\infty}p_{1,n-1}(x)dx,\ n\geq 1 \tag{7}$$

The normalization condition is given as:

$$\sum_{n=0}^{\infty}p_{0,n}+\sum_{n=0}^{\infty}\int_0^{\infty}p_{1,n}(x)\,dx+\sum_{n=0}^{\infty}\int_0^{\infty}\int_0^{\infty}p_{2,n}(x,y)\,dxdy$$

$$+\sum_{n=0}^{\infty}\int_0^{\infty}p_{3,n}(y)\,dy=1$$

To solve the Eq. (1)-(4), we define the following generating functions:

$$P_0(z)=\sum_{n=0}^{\infty}p_{0,n}z^n,\quad P_1(x,z)=\sum_{n=0}^{\infty}p_{1,n}(x)z^n,$$

$$P_2(x,y,z)=\sum_{n=0}^{\infty}p_{2,n}(x,y)z^n,\ P_3(y,z)=\sum_{n=0}^{\infty}p_{3,n}(y)z^n,$$

$$c(z)=\sum_{n=0}^{\infty}c_n z^n,\quad |z|\leq 1$$

3. QUEUE SIZE DISTRIBUTION

The performance of queueing system can be quantified by predicting performance indices of interest; to derive various measures of performance, the probability generating function approach can be employed. In this section, the probability generating functions and marginal generating functions of the queue size distributions are established. For this purpose, Eq. (1)-(4) and (6)-(7) are solved along with the boundary conditions for both classical and constant retrial case as follows.

Multiplying Eq. (2)-(4) and (6)-(7) by appropriate powers of z and summing over 'n', we get

$$\frac{\partial}{\partial x}P_1(x,z)=-\left(\lambda+\alpha+b_1(x)\right)P_1(x,z)$$

$$+\lambda c(z)P_1(x,z)+\int_0^{\infty}P_2(x,y,z)b_2(y)dxdy \tag{8}$$

$$\frac{\partial}{\partial y}P_2(x,y,z)=-\left(\lambda+b_2(y)\right)P_2(x,y,z)+\lambda c(z)P_2(x,y,z) \tag{9}$$

$$\frac{\partial}{\partial y}P_3(y,z)=-\left(\lambda+b_2(y)\right)P_3(y,z)+\lambda c(z)P_3(y,z) \tag{10}$$

$$P_2(x,0,z)=p\alpha P_1(x,z) \tag{11}$$

$$P_3(0,z)=q\alpha\, z\int_0^{\infty}P_1(x,z)dx \tag{12}$$

On solving Eq. (9) and (10), we have

$$P_2(x,y,z)=P_2(x,0,z)\left[1-B_2(y)\right].e^{-[\lambda(1-c(z))]y} \tag{13}$$

$$P_3(y,z)=P_3(0,z)\left[1-B_2(y)\right].e^{-[\lambda(1-c(z))]y} \tag{14}$$

Using Eq. (11) and (13) in (8), we obtain

$$\frac{\partial}{\partial x}P_1(x,z)=-\left[\lambda\left(1-c(z)\right)+\alpha+b_1(x)\right]P_1(x,z)$$

$$+p\alpha\beta_2\left(\lambda\left(1-c(z)\right)\right)P_1(x,z) \tag{15}$$

which yields

$$P_1(x,z)=P_1(0,z).e^{-a(z)x}.\left[1-B_1(x)\right] \tag{16}$$

where

$$a(z)=\lambda\left(1-c(z)\right)+\alpha\left[1-p\beta_2\left(\lambda\left(1-c(z)\right)\right)\right],$$

with $a(1)=q\alpha$ and $a'(1)=-\lambda\left(1+p\alpha\beta_{2,1}\right)$

3.1 Classical retrial policy

In the classical retrial policy, the probability of a repeated attempt during a given time interval is dependent on the number of customers present in the system and is given by $n\theta dt+O(dt)$, as such $\theta_n=\theta$.

Multiplying Eq. (1) and (5) by appropriate powers of z and summing over n, we get

$$\lambda P_0(z)+P_0'(z)z\theta_n=\int_0^{\infty}P_1(x,z)b_1(x)dx+\int_0^{\infty}P_3(y,z)b_1(y)dy \tag{17}$$

and

$$P_1(0,z)=\lambda c(z)P_0(z)+P_0'(z)\theta_{n+1} \tag{18}$$

Thus using Eq. (14) and (16) in (17) and solving, we obtain

$$\lambda P_0(z)+P_0'(z).z\theta$$
$$=\beta_1(a(z))P_1(0,z)+\beta_2\left[\lambda\left(1-c(z)\right)\right]P_3(0,z)$$

Also from Eq. (12), we have

$$P_3(0,z)=q\alpha z\frac{1-\beta_1(a(z))}{a(z)}P_1(0,z) \tag{19}$$

Now using Eq. (16), (18) and (19), we have

$$P_0'(z)$$
$$=\frac{\lambda}{\theta}\times\left(a(z)-q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_2\left(a(z)\right)\right]\right)$$
$$\div\left(q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]\right.$$
$$\left.-a(z)\left[z-\beta_1\left(a(z)\right)\right]\right)$$
$$\times P_0(z) \tag{20}$$

The solution of the above equation is given by

$$P_0(z)$$
$$=P_0(1).\exp\left\{\int_1^z\frac{\lambda}{\theta}\right.$$
$$\times\left[a(u)-q\alpha u\beta_2\lambda\left(1-c(u)\right)\right]\left[1-\beta_1\left(a(u)\right)\right]$$
$$\div\left(q\alpha u\beta_2\left[\lambda\left(1-c(u)\right)\right]\left[1-\beta_1\left(a(u)\right)\right]\right.$$
$$\left.\left.-a(u)\left[u-\beta_1\left(a(u)\right)\right]\right)du\right\} \tag{21}$$

On substituting values of $P_0(z)$, from Eq. (18) and (20), we get

$$P_1(0,z)$$
$$=\frac{\lambda\left(1-c(z)\right)a(z)}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]}$$
$$\times P_0(z) \tag{22}$$

Also, using Eqs. (19) and (22), we find

$$P_3(0,z)$$
$$=\frac{\lambda q\alpha z\left(1-c(z)\right)\left[1-\beta_1\left(a(z)\right)\right]a(z)}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]}$$
$$\times P_0(z) \tag{23}$$

Using the normalizing condition, the value of the unknown constant $P_0(1)$ can be found as

$$P_0(1)=1-\frac{\lambda\left(1+\alpha\,\beta_{2,1}\right)\left[1-q\beta_1(q\alpha)\right]}{q\alpha\beta_1(q\alpha)}$$

3.2 Constant retrial policy

The constant retrial policy can be treated by considering queue dependent reduced retrial rate of the repeated customers. According to this policy, the repeated customers are discouraged when more number of customers join the orbit as such the retrial rate θ_n is taken to be $\frac{\theta}{n}$, when there are n repeated customers in the system.

Multiplying Eq. (1) and (5) by appropriate powers of z and summing over n, we get

$$(\lambda+\theta)P_0(z)-\theta P_{0,0}$$
$$=\int_0^\infty P_1(x,z)\,b_1(x)dx+\int_0^\infty P_3(y,z)\,b_1(y)dy \tag{24}$$

$$P_1(0,z)=P_0(z)\left(\lambda\,c(z)+\frac{\theta}{z}\right)+\frac{\theta}{z}p_{0,0} \tag{25}$$

where $p_{0,0}$ denotes the probability of system being empty.

On solving Eq. (12), (14) and (16) and making use of Eq. (24), we get

$$\left(\lambda\,c(z)+\theta\right)P_0(z)-\theta\,p_{0,0}$$
$$=\int_0^\infty P_1(0,z).e^{-a(z)x}.\left[1-B_1(x)\right]b_1(x)dx$$
$$+\int_0^\infty P_3(0,z)\left[1-B_2(y)\right].e^{-\left[\lambda\left(1-c(z)\right)\right]y}\,b_1(y)dy$$
$$=\beta_1\left(a(z)\right)P_1(0,z)+\beta_2\left[\lambda\left(1-c(z)\right)\right]P_3(0,z) \tag{26}$$

Also from Eq. (19) and (24)-(26), we get

$$P_0(z)$$
$$=\left(q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]\right.$$
$$\left.-a(z)\left[z-\beta_1\left(a(z)\right)\right]\lambda\left(1-c(z)\right)a(z)\right)$$
$$/\left(z\left[\left(\lambda z+\theta\right)q\alpha\beta_2\left(1-c(z)\right)-\lambda\,a(z)\right]\right.$$
$$\left.\times\left[1-\beta_1\left(a(z)\right)\right]-\theta\,a(z)\left[z-\beta_1\left(a(z)\right)\right]\right)$$
$$\times\theta\,p_{0,0} \tag{27}$$

where $p_{0,0}$ is determined by using the normalizing condition as

$$p_{0,0}=1-\lambda\frac{q\alpha+(\lambda+\theta)\left(1+\alpha\,\beta_{2,1}\right)\left[1-q\beta_1(q\alpha)\right]}{\theta\,q\alpha\beta_1(q\alpha)}$$

Above Eq. (27) together with (25) give

$$P_1(0,z)$$

$$= \frac{\lambda\left(1-c(z)\right)a(z)}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times P_0(z) \quad (28)$$

Using Eq. (28) and (19), we have

$$P_3(0,z) = \frac{\lambda q\alpha z\left(1-c(z)\right)\left[1-\beta_1\left(a(z)\right)\right]a(z)}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times P_0(z) \quad (29)$$

3.3 Probability generating functions

The stationary distribution of the queue size distribution can be established in terms of the generating functions, which are obtained as follows:

Using Eq. (16) and (22), we obtain

$$P_1(x,z) = \frac{\lambda\left(1-c(z)\right)a(z)\left[1-B_1(x)\right].e^{-a(z)x}}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times P_0(z) \quad (30)$$

Making use of Eq. (16) and (22) in Eq. (11), we have

$$P_2(x,y,z) = \left(\lambda p\alpha\left(1-c(z)\right)a(z)\left[1-B_1(x)\right] \times e^{-a(z)x}\left[1-B_2(y)\right].e^{-\left[\lambda(1-c(z))\right].y}\right) \div \left(q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right] - a(z)\left[z-\beta_1\left(a(z)\right)\right]\right) \times P_0(z) \quad (31)$$

Again using Eq. (12) and (23), we obtain

$$P_3(y,z) = \left(\lambda q\alpha z\left(1-c(z)\right)\left[1-\beta_1\left(a(z)\right)\right] \times e^{-a(z)x}\left[1-B_2(y)\right].e^{-\left[\lambda(1-c(z))\right].y}\right) \div \left(q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right] - a(z)\left[z-\beta_1\left(a(z)\right)\right]\right) \times P_0(z) \quad (32)$$

For both the classical and constant retrial cases, we have already determined $P_0(z)$ as given in Eq. (21).

Now, we obtain the marginal generating functions in different states of the server as follows:

(i) When the server is busy then

$$P_1(z) = \int_0^\infty P_1(x,z)\,dx = \frac{\lambda\left(1-c(z)\right)a(z)\left[1-\beta_1\left(a(z)\right)\right]}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times P_0(z) \quad (33)$$

(ii) When the server is broken-down with a customer waiting in the system, we obtain

$$P_2(z) = \int_0^\infty P_2(x,y,z)\,dx\,dy = \frac{p\alpha\left[1-\beta_1\left(a(z)\right)\right]\left[1-\beta_2\left(\lambda\left(1-c(z)\right)\right)\right]}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times P_0(z) \quad (34)$$

(iii) When the server is in broken-down state without a customer waiting in the system, then

$$P_3(z) = \int_0^\infty P_3(y,z)\,dy = \frac{q\alpha z\left[1-\beta_1\left(a(z)\right)\right]\left[1-\beta_2\left(\lambda\left(1-c(z)\right)\right)\right]}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times P_0(z) \quad (35)$$

In order to derive average orbit size and the average system size, we obtain the corresponding generating functions as follows:

(a) The probability generating function of the orbit size is given by

$$G(z) = P_0(z)+P_1(z)+P_2(z)+P_3(z) = \frac{\lambda\left(1-c(z)\right)+p\alpha\left[1-\beta_2\left(\lambda\left(1-c(z)\right)\right)+q\alpha\beta_1\left(a(z)\right)\right]}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times\left(1-c(z)\right)P_0(z) \quad (36)$$

(b) The probability generating function of the system size is obtained as

$$Q(z) = P_0(z)+zP_1(z)+zP_2(z)+P_3(z) = \frac{a(z)\beta_1\left(a(z)\right)}{q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times\left(1-c(z)\right)P_0(z) \quad (37)$$

The availability of the server is an important index for any queueing system. The server is available when it is either idle or in working state. Thus, the marginal generating functions corresponding to the availability of the server in different states are given as below:

(a) The marginal generating function of the orbit size when the server is available, is

$$\begin{aligned}&G_A(z)\\&=P_0(z)+P_1(z)\\&=\Big(\lambda\left(1-c(z)\right)+q\alpha z\beta_2\left(\lambda\left(1-c(z)\right)\right.\\&\qquad\left.\times\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]\right)\\&\quad\div\Big(q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]\\&\qquad\qquad-a(z)\left[z-\beta_1\left(a(z)\right)\right]\Big)\\&\quad\times P_0(z)\end{aligned}\tag{38}$$

(b) The marginal generating function of the system size when the server is available

$$\begin{aligned}&Q_A(z)=P_0(z)+zP_1(z)\\&=\Big(\left(1-c(z)\right)a(z)\,\beta_1\left(a(z)\right)\\&\qquad-\alpha z\left[1-\beta_1\left(a(z)\right)\right]\left[1-\beta_2\left(\lambda\left(1-c(z)\right)\right)\right]\Big)\\&\quad\div\Big(q\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]\\&\qquad\qquad-a(z)\left[z-\beta_1\left(a(z)\right)\right]\Big)\\&\quad\times P_0(z)\end{aligned}\tag{39}$$

4. PERFORMANCE MEASURES

By using generating functions obtained in previous section, various performance measures can be derived as follows:

(a) **Long run probabilities of the system states:**

- The probability that the server being idle is given by

$$\begin{aligned}P_I&=\lim_{z\to1}P_0(z)\\&=1-\frac{\lambda\left(1+\alpha\,\beta_{2,1}\right)\left[1-q\beta_1\left(q\alpha\right)\right]}{q\alpha\beta_1\left(q\alpha\right)}\end{aligned}\tag{40}$$

- The probability that the server is in busy state is given as

$$P_B=\lim_{z\to1}P_1(z)=\frac{\lambda\left[1-\beta_1\left(q\alpha\right)\right]}{q\alpha\beta_1\left(q\alpha\right)}\tag{41}$$

- The probability that the server is in broken-down state with a customer waiting in the system, is given by

$$P_D=\lim_{z\to1}P_2(z)=\frac{\lambda p\left[1-\beta_1\left(q\alpha\right)\right]}{q\beta_1\left(q\alpha\right)}\tag{42}$$

- The probability that the server is in broken-down state without a customer waiting in the system, is obtained as

$$P_W=\lim_{z\to1}P_3(z)=\frac{\lambda\beta_{2,1}\left[1-\beta_1\left(q\alpha\right)\right]}{\beta_1(q\alpha)}\tag{43}$$

- The probability of server being available is given by

$$P_A=P_I+P_B=1-\frac{\lambda\beta_{2,1}\left[1-\beta_1\left(q\alpha\right)\right]}{q\beta_1(q\alpha)}\tag{44}$$

(b) **Average queue length:**

- The number of customers in the orbit is obtained as

$$\begin{aligned}E[L]&=\lim_{z\to1}G'(z)\\&=\left\{\lambda h(1-p\alpha\nu_1)-q\alpha(\gamma_1\,a-2\beta_1(q\alpha))\right\}\psi\end{aligned}\tag{45}$$

- The number of customers in the system is given by

$$E[S]=\lim_{z\to1}Q'(z)=2q\alpha\,\beta_1(q\alpha)\,h\psi\tag{46}$$

The value of ψ for both classical and constant retrial policy is given below:

(i) **For classical retrial case**

$$\psi=\frac{\lambda\left(a-q\alpha\left(1-\lambda h\nu_1\right)\right)}{\theta\left\{q\alpha\left(\left(1-\lambda h\nu_1\right)-1\right)-a\right\}}\tag{47}$$

(ii) **For constant retrial case**

$$\psi=\frac{N_2D_1-D_2N_1}{2(D_1)^2}\tag{48}$$

where

$$N_1=\left\{q\alpha\left(\delta\left(1-\lambda h\nu_1\right)-\gamma_1a\right)-\left(a\delta+q\alpha\left(1-\gamma_1a\right)\right)\right\}\theta\;p_{0,0}$$

$$\begin{aligned}D_1=\theta\,q\alpha&\left\{\left(\delta-a\gamma_1\right)+\left(\lambda\,hq\alpha\left(1-(\lambda+\theta)\nu_1\right)-\lambda a\right)\delta\right\}\\&-\theta\left(a\delta+q\alpha\left(1-a\gamma_{1_1}\right)\right)\end{aligned}$$

$$\begin{aligned}N_2=&\left\{q\alpha\left(2\lambda h\nu_1\left(\gamma_1a-\delta\right)-\lambda\left(\delta\left(h_1\nu_1-\lambda^2h^2\,\nu_2\right)\right)\right.\right.\\&\qquad\left.\left.-\gamma_1\left(b+2a-a^2\right)\right)-a\left(2(1-\gamma_1)+\left(1-\gamma_1a\right)+b\delta\right)\right\}\\&\times\theta.p_{0,0}\end{aligned}$$

$$\begin{aligned}D_2=&\left\{\delta\left(\lambda hq\alpha-\left(\lambda h+\theta\right)\lambda hq\alpha\nu_1-\lambda a\right)-\theta\,q\alpha\gamma_1a\right.\\&\qquad\left.+\eta\gamma_1(1+a)+(\eta+\eta_1)\delta-\theta\,q\alpha\left(\gamma_1(a+b)+\gamma_2\right)\right\}\\&-\theta\left\{a\left(2(1-a\gamma_1)-\gamma_1-a\gamma_2q\alpha\right)+b\delta\right\}\end{aligned}$$

with

$a = a'(1) = -\left(\lambda(1+p\alpha\beta_{2,1})\right),\ b = a''(1)$

$\gamma_1 = \beta_1'(q\alpha),\ \gamma_2 = \beta_1''(q\alpha)$

$h = c'(1),\ h_1 = c''(1),\ h_2 = c'''(1),\ \nu_1 = \beta_2'(0),\ \nu_2 = \beta_2''(0)$

$\delta = 1-\beta_1(q\alpha),\ \eta = \lambda h q\alpha\left(1-(\lambda+\theta)\nu_1\right)-\lambda a$

$\eta_1 = \lambda q\alpha\left\{-2\lambda h^2\nu_1 + (\lambda+\theta)\left(\lambda h^2\nu_2 - h_1\nu_1\right)+h_1\right\}-\lambda a$

5. SPECIAL CASES

In this section, we deduce some special cases by assigning appropriate parameter values to the system performance metrics in order to verify our results with the existing results.

Case I: $M/G/1$ retrial queue with active breakdowns and Bernoulli schedule

In this case when customers arrive singly, i.e. when $c(z) = z$, and $h = 1$, $h_1 = h_2 = 0$, then Eq. (36) and (37) reduce to

$$G(z) = \frac{\lambda(1-z)+p\alpha\left[1-\beta_2\left(\lambda(1-z)\right)+q\alpha\beta_1\left(a(z)\right)\right]}{q\alpha z\beta_2\left[\lambda(1-z)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times (1-z)P_0(z) \tag{49}$$

and

$$Q(z) = \frac{a(z)\beta_1\left(a(z)\right)}{q\alpha z\beta_2\left[\lambda(1-z)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times (1-z)P_0(z) \tag{50}$$

respectively, which coincide with the results given by Atenica et al. (2006) with both types of retrial policies.

Case II: $M/G/1$ retrial queue with unreliable server

When $\theta_n = \theta$, $c(z) = z$, $h_1 = h_2 = 0$ and $q \to 0$, then

$$E(S) = \frac{\lambda}{\theta}\left(\frac{\lambda\beta_1\left(1+\alpha\beta_{2,1}\right)}{1-\lambda\beta_1\left(1+\alpha\beta_{2,1}\right)}\right) + \frac{\lambda^2}{2\left(1-\lambda\beta_1\left(1+\alpha\beta_{2,1}\right)\right)}\left(\alpha\beta_1\beta_{2,2}+\beta_2\left(1+\alpha\beta_{2,1}\right)^2\right) \tag{51}$$

which agrees with the results obtained by Wang et al. (2001), wherein the server is subjected to active breakdowns and the customer waits for the server till its repair is completed.

Case III: $M/G/1$ queue with Bernoulli feedback

Setting $\theta = 0$, $c(z) = z$, $h_1 = h_2 = 0$ and $\alpha = 0$, we obtain

$$E[S] = \lambda\beta_1 + \frac{(\lambda\beta_1)^2}{2\left(q-\lambda\beta_1\right)} + \frac{\lambda\beta_1(1-q)}{\left(q-\lambda\beta_1\right)} \tag{52}$$

which matches with the result obtained by Takagi (1996).

Case IV: $M/G/1$ queue without Bernoulli feedback

When $\theta = 0$, $c(z) = z$, $h_1 = h_2 = 0$ and $\alpha = 0$, then Eq. (46) gives

$$E[S] = \lambda\beta_1 + \frac{(\lambda\beta_1)^2}{2\left(1-\lambda\beta_1\right)} \tag{53}$$

The above results tallies with the results for the classical $M/G/1$ model (cf. Takagi (1991), pp. 7).

Case V: $M^X/G/1$ unreliable queue without Bernoulli feedback

In this case, $q \to 0$, so that Eq. (37) yields the following result

$$Q(z) = \frac{a(z)\beta_1\left(a(z)\right)}{\alpha z\beta_2\left[\lambda\left(1-c(z)\right)\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times \left(1-c(z)\right)P_0(z) \tag{54}$$

Case VI: $M^X/G/1$ retrial queue without active breakdown and no Bernoulli feedback

On taking $\alpha = 0$ and $q \to 0$, Eq. (37) reduces to

$$Q(z) = \frac{a(z)\beta_1\left(a(z)\right)}{a(z)\left[\beta_1\left(a(z)\right)-z\right]}\left(1-c(z)\right)P_0(z) \tag{55}$$

Case VII: $M^X/G/1$ queue with active breakdown without retrial

On substituting $\theta = 0$ and $q = 0$ in Eq. (46), the following result corresponding to the bulk arrival queueing system under unreliable server is obtained

$$E[S] = \frac{\lambda\left(1+\alpha\beta_2\right)^2(h+h_1)+(\lambda h)^2\alpha\beta_2^2}{2(1-\lambda h)\left(1+\alpha\beta_2\right)} \tag{56}$$

6. STOCHASTIC DECOMPOSITION

Retrial queues have been widely applicable in many practical problems in computer and communication networks. In this section, we are concerned with the stochastic decomposition property of the system size distribution. Stochastic decomposition property of our model shows that the system size distribution in the steady state can be decomposed into two random variables; one corresponding to the system size of the ordinary queue and the other random variable which can be interpreted as the system size of the queueing model under consideration when the server is idle.

In retrial queueing model under consideration, the probability generating function of the number of the customers in the system can be expressed as

$$P(z) = \frac{a(z)\beta_1\left(a(z)\right)\left(1-c(z)\right)\left(1-\frac{\lambda\left(1+\alpha\,\beta_{2,1}\right)\left[1-q\beta_1(q\alpha)\right]}{q\alpha\beta_1(q\alpha)}\right)}{q\alpha z\beta_2\left[\lambda((1-c(z))\right]\left[1-\beta_1\left(a(z)\right)\right]-a(z)\left[z-\beta_1\left(a(z)\right)\right]} \times\left\{\frac{P_0(z)}{P_0(1)}\right\} \quad (57)$$

The fraction in the right hand side of Eq. (57) corresponds to the probability generating function of the system size when the server is idle. Here we observe that the system size distribution of our queueing model decomposes into the distributions of the two random variables; (i) the system size distribution for the $M^X/G/1$ retrial queue with batch arrivals under active breakdowns and Bernoulli schedule and (ii) the conditional distribution of the number of the customers in the orbit, given that the server is idle.

7. NUMERICAL RESULTS AND SENSITIVITY ANALYSIS

In order to facilitate the numerical results for the performance measures of the retrial queueing system, we develop a program in software Matlab and run on Pentium IV. By taking illustration, the computational results of the system performance measures corresponding to the different varying parameters are provided. The distribution of the service time is taken as *k*-Erlangian. Figures 1-4 depict the graphs corresponding to the system queue size and the orbit size by varying various input parameters namely retrial rate (θ), the breakdown rate (α), and the orbit joining rate (*q*) and different values of '*k*'. The default input parameters chosen are $\lambda = 1$, $\alpha = 0.1$, $\mu = 3$ and $b = 5$.

Figures 1 (a-c) and 2 (a-c) display the results for the classical retrial policy. It is clear from these that the average queue size and the average orbit size both first decrease and then almost become constant with the increasing values of '*k*'. Figure 1 (a) and 2 (a) show the influence of the retrial rate on the queue size and the average orbit size, respectively; it is observed that the system queue size increases sharply while the orbit size decreases with the increasing values of retrial rate θ. The patterns so obtained illustrate the fact that more retrial attempts give rise to a large number of customers in the queue but has reverse effect on the orbit size.

Figures 1 (b) and 2 (b) depict the variation of the queue size and the orbit size with the increasing failure rate of the server; the increasing trends of both the queue size and the orbit size with respect to the increasing values of the failure rate (α) are noticed. This is due to the fact that more often failures of server cause more accumulation of the customers in the queue as well as in the orbit. Figures 1 (c) and 2 (c) exhibit the effect of the orbit joining probability '*q*'. In this case also the increasing behavior of both queue size and the orbit size for the increasing values of *q* is noticed; however the effect is more prominent for higher values of *q*.

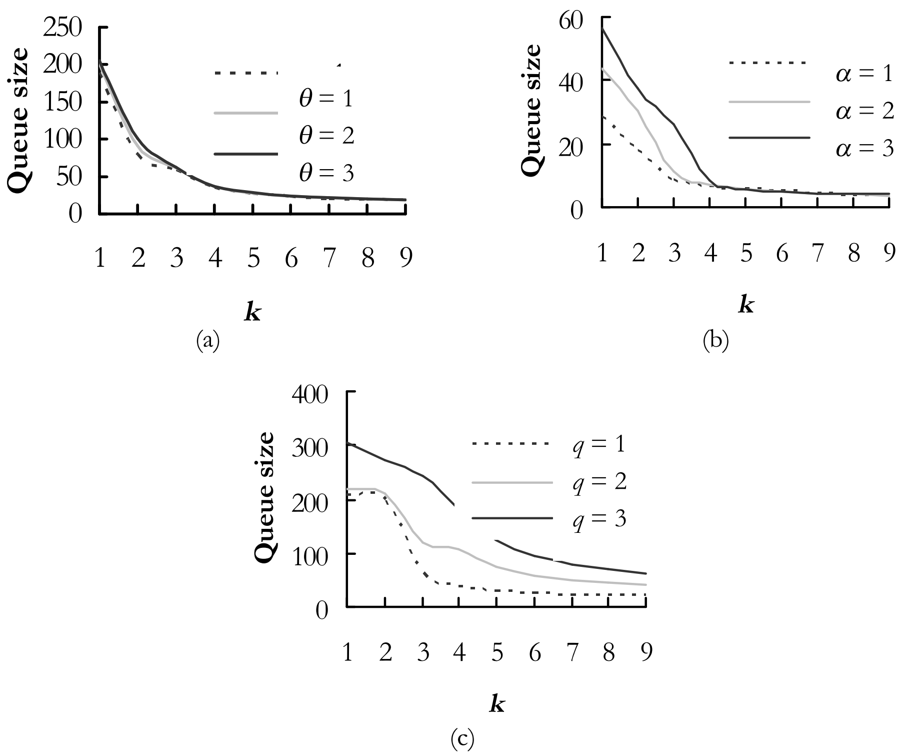

Figure 1. (a-c): Expected queue size vs. *k* for classical retrial case by varying (a)θ (b)α (c)q.

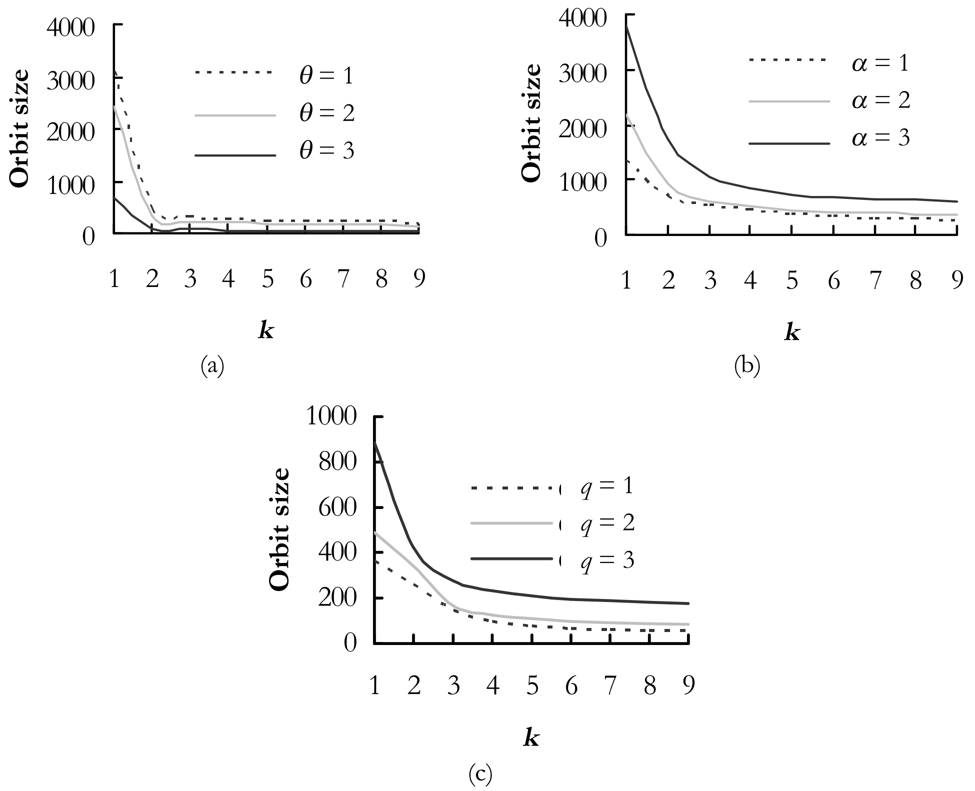

Figure 2. (a-c): Expected orbit size vs. k for classical retrial case by varying (a)θ (b)α (c)q.

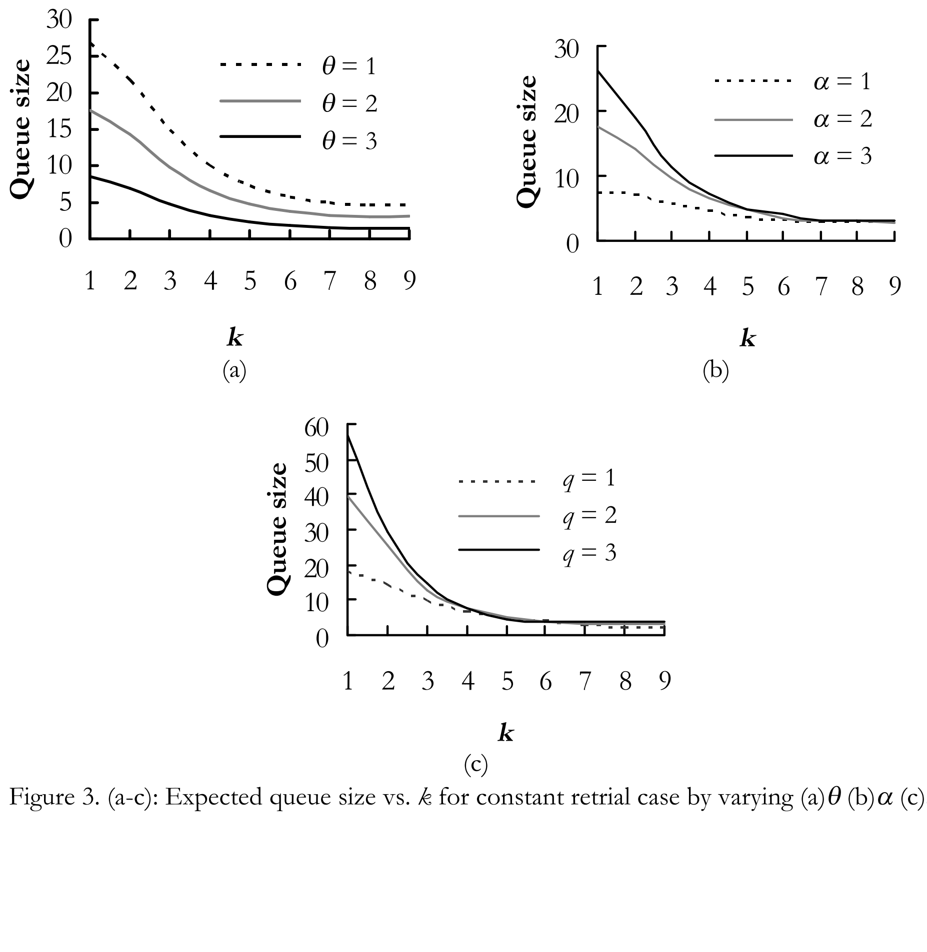

Figure 3. (a-c): Expected queue size vs. k for constant retrial case by varying (a)θ (b)α (c)q.

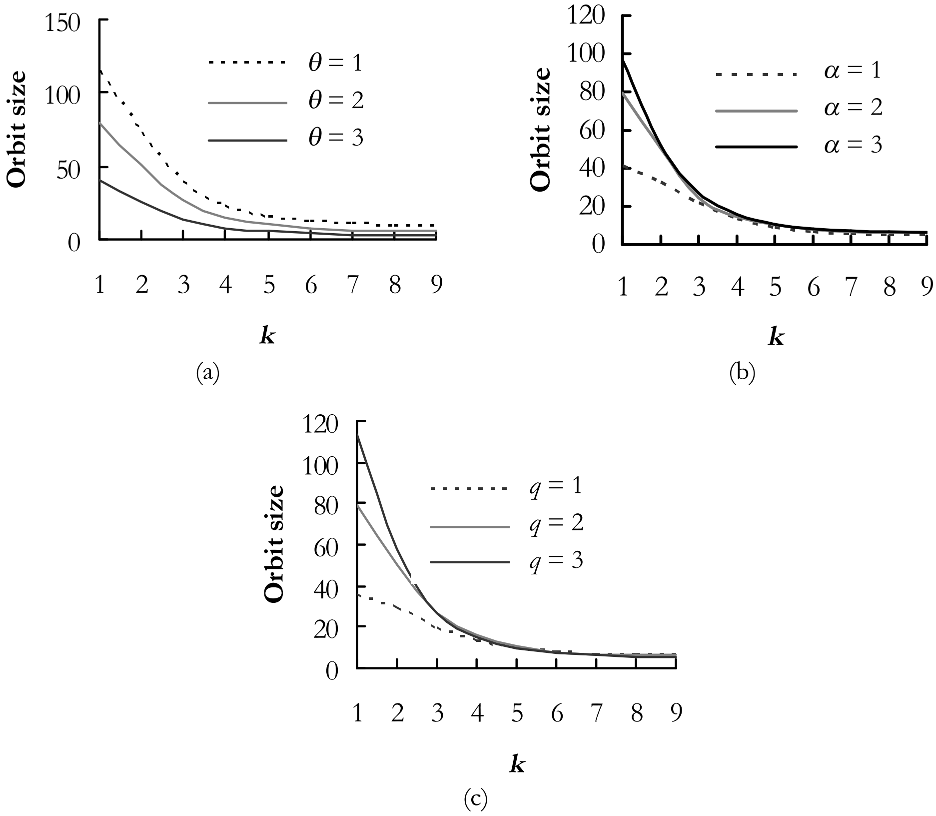

Figure 4. (a-c): Expected orbit size vs. *k* for constant retrial case by varying (a)θ (b)α (c)q.

Figures 3-4 show the varying trends of the system queue size and the orbit size, respectively with the variation in k for the constant retrial policy. In Figure 3 (a-c) and 4 (a-c), we plot the variation of the system size and the orbit size both for different values of *k* by varying θ, α and q. Figures 3 (a) and 4 (a) depict the graphs corresponding to the different values of retrial rate θ. It is observed that the queue size increases but the orbit size decreases with the increased values of θ which is quite obvious as the customers are retrying from the orbit to the queue. Figure 3 (b) and 4 (b) show the effect of failure rate on the system size and the orbit size. It is observed that as the failure rate goes on increasing, the system size also increases; this pattern is in the agreement with the practical situation, as the server is more prone to the breakdowns then it in turn increases the system size. The effect of the variation of the customer's joining probability q on the system size and the orbit size is observed from Figs. 3 (c) and 4 (c), respectively. It is noted that as the value of q increases, the queue size and the orbit size both increase; this is due to the fact that by increasing the joining probability, more customers join the system/orbit.

Based on the numerical illustrations, we overall conclude that

- The average system size and the average orbit size decrease with the increasing values of *k* by varying different parameters such as θ, α and q. It is also observed that whenever the failure rate of the server and the joining probability of the customers increase, the average system size and the average orbit size also tend to increase. Further, the higher values of retrial rate lead to decrease (increase) in the average orbit size (average queue size).
- There are significant higher values of orbit length and the system size in case of classical retrial policy case as compared to the constant retrial one.
- It is noted that the queue length initially decreases gradually for the increasing values of *k* and there after shows a linear increment on further increase in *k*.

The sensitivity analysis based on numerical results is of significance as it gives insight to the decision makers and industrial engineers to improve the quality of service provided based on the performance of the system with the variation of different parameters. In addition, the ready wrecker of the results can be prepared easily, which may also be helpful to the organizers and system analysts to examine the alternative ways to reduce congestion.

8. CONCLUSION

The retrial bulk queueing model with breakdowns under Bernoulli service schedule suggested can be widely used to model the congestion problems in telephone systems, computer and communication systems, etc. We have proposed the policy for the retrial queueing systems wherein the customers do not know the state of the server and thus have to verify the server's state from time to time. The unreliability of the server and the impatience behavior are also included while investigating the classical and constant retrial policy. The batch input incorporated in retrial queue is quite common in a number of real situations, such as in the transmission in the computer communication systems, wherein the messages are often being transmitted in a random number of packets, the local area network (LAN) system, etc. In previous studies reported in literature, the authors have not included the concept of batch arrivals for both the policies along with server breakdowns and Bernoulli schedule into consideration for retrial queueing model. This also motivated us to develop a realistic model to deal with more versatile situations for retrial queueing system along with the batch arrivals, unreliable server, Bernoulli schedule and discouragement.

The steady-state distributions of the orbit size and the system size when the server is idle failed or busy have been established in terms of probability generating functions which are further employed to obtain explicit formulae for various performance indices of interest. The system performances are measured numerically for both the policies and it is noticed that the constant retrial policy proves to be better than the classical one due to shorter queue length. The analytical results derived in explicit form are not only beneficial to the queue theorists but may also be helpful to the system designers and practitioners to implement them in future to design the efficient service systems at the reasonable cost. The concept of the set up time, heterogeneous arrival rates and working vacations can be further incorporated to extend the present study.

ACKNOWLEDGEMENTS

The authors would like to thank the anonymous referees for their critical comments and valuable suggestions, which have helped to upgrade the quality and the clarity of the paper.

REFERENCES

1. Artalejo, J.R., Atnecia, I., and Moreno, P. (2005). A discrete time $\text{Geo}^X/G/1$ retrial queue with control of admission. *Applied Mathematical Modelling*, 29: 1100-1120.
2. Artalejo, J.R. and Falin, G.I. (1994). Stochastic decompositions for retrial queues. *Top*, 2: 329-342.
3. Atenica, I. and Moreno, P. (2006). *Geo/G/*1 Retrial queue with 2nd optional service. *International Journal of Operational Research*, 1(4): 340-362.
4. Atenica, I., Fortes, I.Y., and Moreno, P. (2003). A retrial queue with starting failures, feedback and linear retrial policy. *Proceedings of the 6th Hellenic European Conference on Computer Mathematics and Its Applications*, Athens, Greece, pp. 669-678.
5. Atnecia, I., Moreno, P., Fortes, I., and Sanchez, S. (2006). An $M/G/1$ retrial queue with active breakdowns and Bernoulli schedule in the server. *Information Management and Sciences*, 17(1): 1-17.
6. Chang, W. (2006). The optimal batch size of M/DN /1 queue. *International Journal of Operational Research*, 1(4): 396-409.
7. Choi, B.D., Shin, Y.W., and Ahn, W.C. (1992). Retrial queues with collision arising from unslotted *CSMA/CD* protocol. *Queueing Systems*, 11: 335-356.
8. Choudhary, G. and Madan, K.C. (2007). A batch arrival Bernoulli vacation queue with a random setup time under restricted admissibility policy. *International Journal of Operational Research*, 2(1): 81-97.
9. Choudhary, G., Tadj, L., and Paul, M. (2007). Steady state analysis of an $M^X/G/1$ queue with two-phase service and Bernoulli vacation schedule under multiple vacation policy. *Applied Mathematical Modelling*, 31(6): 1079-1091.
10. Choudhary, M.L. and Templeton, J.G.C. (Eds.) (1983). *A First Course in Bulk Queues*, John Wiley and Sons, New York.
11. Dudin, A.N., Krishnamoorthy, A., Joshua, V.C., and Tsanankov, G.V. (2004). Analysis of the $BMAP/G/1$ retrial system with search of customers from the orbit. *European Journal of Operational Research*, 157(1): 169-179.
12. Falin, G.I. and Templeton, J.G.C. (Eds.) (1997). *Retrial Queues*, Chapman and Hall, London.
13. Falin, G.I. (1990). A survey of retrial queues. *Queueing Systems*, 7: 127-167.
14. Fayolle, G. (1986). A simple telephone exchange with delayed feedbacks. In: O.J. Boxma, J.W. Cohen, and H.C. Tijmis (Eds.), *Teletraffic Analysis and Computer Performance Evaluation*, Elesvier, Amsterdam, pp. 245-253.
15. Fuhrmann, S.W. and Cooper, R.B. (1985). Stochastic decompositions in the $M/G/1$ queue with generalized vacations. *Operations Research*, 33: 1117-1129.
16. Ke, J.C. (2006). On $M/G/1$ system under NT policies with breakdowns, startup and close down. *Applied Mathematical Modeling*, 30: 49-66.
17. Kumar, B.K. and Madheswari, S.P. (2003). $M^X/G/1$ retrial queue with multiple vacations and starting failures. *Opsearch*, 40(2): 115-135.
18. Liu, W.Y., Xu, X.L., and Tian, N.S. (2007). Stochastic decompositions in the $M/M/1$ queue with working vacations. *Operations Research Letters*, 35: 595-600.
19. Natalia, V.D. (2006). On the single server retrial queue. *Yugoslav Journal of Operations Research*, 16(1): 45-53.
20. Takagi, H. (1991). *Queueing Analysis: A Foundation of Performance Evaluation, Vol. 1: Vacation and Priority Systems, Part I*, Elsevier Science Publishers B.V., Amsterdam.
21. Takagi, H. (1996). A note on the response time in $M/G/1$ queue with service in random order and Bernoulli feedback. *Journal of the Operational Research Society of Japan*, 39(4): 486-500.
22. Wang, J., Cao, J., and Li, Q. (2001). Reliability analysis of the retrial queue with server breakdowns and repairs. *Queueing Systems*, 38: 363-380.