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Optimality and Duality for Multiple-Objective Optimization with Generalized α-Univex Functions

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Abstract—The aim of the present paper is to obtain a number of Kuhn-Tucker type sufficient optimality conditions for a feasible solution to be an efficient solution under the assumptions of the new notions of weak strictly pseudo quasi α-univex, strong pseudo quasi α-univex, and weak strictly pseudo α-univex vector valued functions. We also derive the duality theorems for Mond-Weir and general Mond-Weir type duality under the aforesaid assumptions.

Keywords—Multiobjective programming, Duality, α-Univexity, Generalized convexity, Efficient solution.

1. INTRODUCTION

In optimization theory, convexity plays a vital role in many aspects of nonlinear programming (see Mangasarian (1969)) including sufficient optimality conditions and duality theorems. In order to study the optimization problems in a wider context various useful generalizations of the notion of convexity have been introduced. Hanson (1981) introduced the class of invex functions. Later Hanson and Mond (1987) defined two new classes of functions called type-I and type-II functions, and sufficient optimality conditions were established by using these concepts. This concept was extended by Rueda and Hanson (1988) to pseudo-type-I and quasi-type-I functions. Kaul et al. (1994) have considered a multiobjective nonlinear programming problem involving type-I functions to obtain some duality results. In the sequel of development of convexity theory Marusciac (1982) introduced constrained qualifications. Weir (1987) introduced the concept of converse duality theorem in multiple objective programming. Giorgi and Guerraggio (1998) have generalized the notion of invexity to vector-valued functions and they provided some duality results. Bector et al. (1992) introduced the concept of univex functions. Aghezzaf and Hachimi (1998) introduced the concept strong pseudo convex function. Later Aghezzaf and Hachimi (2000) introduced new class of generalized type-I vector valued functions and derived various duality results for a nonlinear multiobjective-programming problem. Mishra et al. (2004) extended the concept of type-I functions to the setting of Banach Spaces. Mishra et al. (2004b, 2005a) employed the new class of generalized *d*-type-I and generalized *d*-univex type-I functions and applied the notion of generalized convexity to complex minimax programming (see Mishra et al. (2004a)). Mishra et al. (2005b) extended the concept of generalized type-I vector valued functions to generalized univex type-I vector valued functions and used the notion of type-I preinvex functions to multiple objective fractional programming (see Mishra et al. (2005c)). Noor (2004) and Mishra and Noor (2005) have studied some properties of the α-preinvex functions and their differentials. Recently Mishra et al. (2007), Pant and Rautela (2006) introduced the various classes of α-invex functions.

Motivated by the work of Noor (2004), in the present paper, we consider a multiobjective programming problem and establish some sufficient optimality results. We also derive duality theorems for Mond-Weir and general Mond-Weir type duality under the generalized α-univex assumptions.

To compare vectors along the lines of Mangasarian (1969), we will distinguish between $\leq$ and $\leqq$ or between $\geq$ and $\geqq$. Specifically,

$x \in R^n$, $y \in R^n$, $x \leqq y \Leftrightarrow x_i \leqq y_i$, $i = 1, \ldots, n$.

$x \in R^n$, $y \in R^n$, $x \leq y \Leftrightarrow x_i \leqq y_i$, $i = 1, \ldots, n$ and $x \neq y$.

2. PRELIMINARIES

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In this section we introduce the notions of α-univex, weak strictly pseudo quasi α-univex, strong pseudo quasi α-univex, weak quasi strictly-pseudo α-univex and weak strictly pseudo α-univex functions by unifying the notion of α-invex and univex functions for (MOP).

We consider the following multiobjective programming problem:

(MOP) Minimize $f(x)$

Subject to $g(x) \leqq 0, \quad x \in X(\subseteq R^n)$,

X an α-invex set.

where $f : X \to R^p$ and $g : X \to R^m$ are differentiable functions on a set $X \subseteq R^n$ and minimization means obtaining efficient solutions for the problem (MOP). Let $A = \{x \in X: g(x) \leqq 0\}$ be the set of all the feasible solutions for (MOP) and denote $P = \{1, ..., p\}$, $M = \{1, ..., m\}$ and $I = \{j : g_j(y) = 0\}$.

In the following definitions $\eta : X \times X \to R^n$ is an n-dimensional vector valued function and $\alpha(x, y) : X \times X \to R_+ \setminus \{0\}$ be a bifunction. Assume that $\phi_0 : R^p \to R^p$, $\phi_1 : R^m \to R^m$ satisfy $u \le 0 \Rightarrow \phi_0(u) \le 0$ and $u \leqq 0 \Rightarrow \phi_1(u) \leqq 0$, $b_0, b_1 : X \times X \to R_+$.

Definition 2.1. *(f, g)* is said to be α-univex at $y \in X$ if there exist functions b_0, b_1, ϕ_0, ϕ_1, α and η such that

$$b_0(x, y)\phi_0\left(f(x) - f(y)\right) \ge \left(\alpha(x, y)\nabla f(y)\right)\eta(x, y),$$
$$-b_1(x, y)\phi_1\left(g(y)\right) \ge \left(\alpha(x, y)\nabla g(y)\right)\eta(x, y).$$

Remark 2.1. Note that any α-invex pair *(f, g)* is α-univex if we define $\phi : R \to R$ with $\phi(V) = V$ and $b(x, a) = 1$. But the converse does not necessarily hold. It can be seen from the following example.

Example 2.1. Let $f, g : R \to R$ are defined by $f(x) = x^3$, $g(x) = x^3 + 5$ where $\alpha(x, a) = 1$,

$$b(x, a) = \begin{cases} a^2/(x - a), & x > a, \\ 0, & x \le a, \end{cases}$$

and

$$\eta(x, a) = \begin{cases} x^2 + a^2 + xa, & x > a, \\ x - a, & x \le a. \end{cases}$$

Let $\phi : R \to R$ be defined by $\phi(V) = 3V$. The function f is α-univex but not α-invex, because for

$$x = -3, \quad a = 1, \quad f(x) - f(a) < \langle \alpha(x, a)\nabla f(a), \eta(x, a)\rangle$$

and

$$-g(a) < \langle \alpha(x, a)\nabla g(a), \eta(x, a)\rangle.$$

Definition 2.2. *(f, g)* is said to be weak strictly pseudo quasi α-univex at $y \in X$ if there exist functions b_0, b_1, ϕ_0, ϕ_1, α and η such that

$$b_0(x, y)\phi_0\left(f(x) - f(y)\right) \le 0$$
$$\Rightarrow \left(\alpha(x, y)\nabla f(y)\right)\eta(x, y) < 0,$$
$$-b_1(x, y)\phi_1\left(g(y)\right) \leqq 0$$
$$\Rightarrow \left(\alpha(x, y)\nabla g(y)\right)\eta(x, y) \leqq 0.$$

If (MOP) is weak strictly pseudo quasi α-univex at each $y \in X$, (MOP) is said to be weak strictly pseudo quasi α-univex on X.

Definition 2.3. *(f, g)* is said to be strong pseudo quasi α-univex at $y \in X$ if there exist functions b_0, b_1, ϕ_0, ϕ_1, α and η such that

$$b_0(x, y)\phi_0\left(f(x) - f(y)\right) \le 0$$
$$\Rightarrow \left(\alpha(x, y)\nabla f(y)\right)\eta(x, y) \le 0,$$
$$-b_1(x, y)\phi_1\left(g(y)\right) \leqq 0$$
$$\Rightarrow \left(\alpha(x, y)\nabla g(y)\right)\eta(x, y) \leqq 0.$$

If (MOP) is strong pseudo quasi α-univex at each $y \in X$ (MOP) is said to be strong pseudo quasi α-univex on X. Instead of the class of weak strictly pseudo quasi α-univex, the class of strong pseudo quasi α-univex functions does contain the class of α-univex.

The following examples show that weak strictly pseudo quasi α-univex and strong pseudo quasi α-univex functions exist. Weak strictly pseudo quasi α-univex functions need not to be strictly pseudo quasi α-univex for the same α, b_0, b_1, ϕ_0, ϕ_1 and η as can be seen from the following example.

Example 2.2. The functions $f : R^2 \to R^2$ defined by $f(x) = (x_1\exp(\sin x_2),\ x_2(x_2 - 1)\exp(\cos x_1))$ and $g : R^2 \to R$ defined by $g(x) = (2x_1 + x_2 - 2)$ are weak strictly pseudo quasi α-univex with respect to $\alpha(x, y) = 1$, $b_0 = b_1 = 1$, ϕ_0 and ϕ_1 are the identity functions on R and $\eta(x, y) = (x_1 + x_2 - 1,\ x_2 - x_1)$ at $y = (0, 0)$.

Example 2.3. The functions $f : R^2 \to R^2$ defined by $f(x) = (x_1(x_1 - 1)^2,\ x_2(x_2 - 1)^2(x_2^2 + 2))$ and $g : R^2 \to R$ defined by $g(x) = (x_1^2 + x_2^2 - 9)$ are strong pseudo quasi α-univex with respect to $\alpha(x, y) = 1$, $b_0 = b_1 = 1$, ϕ_0 and ϕ_1 are the identity functions on R

and $\eta(x, y) = (x_1 - 1,\ x_2 - 1)$ at $y = (0,0)$ but (*f*, *g*) is not weak strictly pseudo quasi α-univex with respect to same $\alpha(x, y)$, b_0, b_1, ϕ_0, ϕ_1 and $\eta(x, y)$ at y as can be seen by taking $x = (1, -1)$.

Definition 2.4. (*f*, *g*) is said to be weak strictly pseudo α-univex at $y \in X$ if there exist functions b_0, b_1, ϕ_0, ϕ_1, α and η such that

$$b_0(x, y)\phi_0\left(f(x) - f(y)\right) \leq 0$$
$$\Rightarrow \left(\alpha(x, y)\nabla f(y)\right)\eta(x, y) < 0,$$
$$-b_1(x, y)\phi_1(g(y)) \leqq 0$$
$$\Rightarrow \left(\alpha(x, y)\nabla g(y)\right)\eta(x, y) < 0.$$

If (MOP) is weak strictly pseudo α-univex at each $y \in X$, (MOP) is said to be weak strictly pseudo α-univex on X.

Remark 2.2. In the above definitions if we take $\phi : \mathrm{R} \to \mathrm{R}$ with $\phi(V) = V$ and $b(x, a) = 1$, $\alpha(x, a) = 1$, the above definitions reduce to the definitions given in Aghezzaf and Hachimi (2000).

Remark 2.3. If we take $\alpha(x, a) = 1$, the above definitions reduces to the definitions given in Mishra et al. (2005a).

Remark 2.4. If we take $\phi : \mathrm{R} \to \mathrm{R}$ with $\phi(V) = V$ and $\alpha(x, a) = 1$, the above definitions reduce to the case of B-vex functions discussed in Bector et al. (1992).

Remark 2.5. If we take $\phi : \mathrm{R} \to \mathrm{R}$ with $\phi(V) = V$, $\eta(x, a) = x - a$ and $\alpha(x, a) = 1$, the above definitions reduce to the well known classes of generalized convexity.

Definition 2.5. A point $y \in A$ is an efficient solution for (MOP) if and only if there exist no $x \in A$ such that

$$f(x) \leq f(y).$$

Definition 2.6. A point $y \in A$ is a weak efficient solution for (MOP) if and only if there exist no $x \in A$ such that

$$f(x) < f(y).$$

Definition 2.7. (*f*, *g*) is said to satisfy the Maeda's constraint qualification at $y \in A$ such that

$$\nabla f(y)h \leqq 0 \text{ and } \nabla g_j(y)h \leqq 0,\ j \in I,$$

where $h \in \mathrm{R}^n$.

3. OPTIMALITY CONDITIONS

In this section, we establish Karush-Kuhn-Tucker type sufficient optimality conditions for $y \in X$ to be an efficient solution of problem (MOP) under various generalized α-univex functions defined in the previous section.

Theorem 3.1. (Sufficiency). Suppose that

(i) $y \in X$;

(ii) there exist $\tau^0 \in \mathrm{R}^p$, $\tau^0 > 0$, $\lambda^0 \in \mathrm{R}^m$ and $\lambda^0 \geqq 0$ such that

  (a) $\tau^0 \nabla f(y) + \lambda^0 \nabla g(y) = 0,$

  (b) $\lambda^0 g(y) = 0,$

  (c) $\tau^0 e = 1,$ where $e = (1,\ 1,\ \ldots, 1)^t \in \mathrm{R}^p$;

(iii) problem (MOP) is strong pseudo quasi α-univex at $y \in X$ with respect to some α, b_0, b_1, ϕ_0, ϕ_1 and η for all feasible x.

Then y is an efficient solution to (MOP).

Proof. Suppose contrary to the result that y is not an efficient solution to (MOP). Then there exist a feasible solution x to (MOP) such that

$$f(x) \leq f(y).$$

Since $b_0(x, y) > 0$ and $(.) \leq 0 \Rightarrow \phi_0(.) \leq 0$, from the above inequality, we get

$$b_0(x, y)\phi_0\left[f(x) - f(y)\right] \leq 0. \tag{1}$$

By the feasibility of y, we have

$$-\lambda^0 g(y) \leqq 0.$$

Since $b_1(x, y) \geqq 0$ and $(.) \leqq 0 \Rightarrow \phi_0(.) \leqq 0$, from the above inequality, we get

$$-b_1(x, y)\phi_1\left[\lambda^0 g(y)\right] \leqq 0 \tag{2}$$

By inequalities (1), (2) and condition (iii), we have

$$\alpha(x, y)\left[\nabla f(y)\right]\eta(x, y) \leq 0 \text{ and}$$
$$\alpha(x, y)\left[\lambda^0 \nabla g(y)\right]\eta(x, y) \leqq 0.$$

By the positivity of α, the above inequalities reduce to

$$\left[\nabla f(y)\right]\eta(x, y) \leq 0 \quad \text{and} \quad \left[\lambda^0 \nabla g(y)\right]\eta(x, y) \leqq 0.$$

Since $\tau^0 > 0$, the above inequalities give

$$\left[\tau^0 \nabla f(y) + \lambda^0 \nabla g(y)\right]\eta(x, y) < 0, \tag{3}$$

which contradicts (iii). This completes the proof.

Example 3.1. Consider function $f = (f_1, f_2)$ defined on

$X = R$, by $f_1(x) = x^2$, $f_2(x) = x^3$ and function g defined on $X = R$, by $g = (g_1, g_2)$ defined on $X = R$, by $g_1(x) = -2x^2$, $g_2(x) = -x^3$.

Clearly the feasible region is nonempty. Let $\alpha(x, \bar{x}) = 1$, $b(x, \bar{x}) = 1$, $\eta(x, \bar{x}) = (x - \bar{x})/2$ and $\bar{x} = 0$.

(i) $-g(\bar{x}) = 0$, implies that $\alpha(x, \bar{x}) \nabla g(\bar{x}) \eta(x, \bar{x}) = 0$.

(ii) $b(x, \bar{x})[f(x) - f(\bar{x})] \leqq 0$
$\Rightarrow \alpha(x, \bar{x}) \nabla f(x, \bar{x}) \eta(x, \bar{x}) = 0$, for all x.

Thus (f, g) is strong pseudo-quasi α-univex at $x = 0$. But (f, g) is not α-univex functions at $x = 0$ with respect to $\alpha(x, \bar{x}) = 1$, $b(x, \bar{x}) = 1$ and $\eta(x, \bar{x}) = (x - \bar{x})/2$. Then, by Theorem 3.1, $\bar{x}$ is a weak Pareto efficient solution for the given multiobjective programming problem.

The proofs of the following theorems follow along the lines of Theorem 3.1; therefore we state the theorems but omit the proofs.

Theorem 3.2. (Sufficiency). Suppose that

(i) $y \in X$;

(ii) there exist $\tau^0 \in R^p$, $\tau^0 \geq 0$, $\lambda^0 \in R^m$ and $\lambda^0 \geq 0$ such that

(a) $\tau^0 \nabla f(y) + \lambda^0 \nabla g(y) = 0$,

(b) $\lambda^0 g(y) = 0$,

(c) $\tau^0 e = 1$, where $e = (1, 1, \ldots, 1)^t \in R^p$;

(iii) problem (MOP) is weak strictly pseudo quasi α-univex at $y \in X$ with respect to some α and η for all feasible x.

Then y is an efficient solution to (MOP).

Theorem 3.3. (Sufficiency). Suppose that

(i) $y \in X$;

(ii) there exist $\tau^0 \in R^p$, $\tau^0 \geqq 0$, $\lambda^0 \in R^m$ and $\lambda^0 \geqq 0$ such that

(a) $\tau^0 \nabla f(y) + \lambda^0 \nabla g(y) = 0$,

(b) $\lambda^0 g(y) = 0$,

(c) $\tau^0 e = 1$, where $e = (1, 1, \ldots, 1)^t \in R^p$;

(iii) problem (MOP) is weak strictly pseudo α-univex at $y \in X$ with respect to some α and η for all feasible x.

Then y is an efficient solution to (MOP).

4. MOND-WEIR TYPE DUALITY

In this section, to establish a connection between (MOP) and (DMOP) we present some weak and strong duality relations under various generalized α-univex functions defined in the previous section.

We consider the following Mond-Weir dual problem (DMOP) in Egudo (1989) type format for (MOP).

(DMOP) Maximize $f(y)$

Subject to $\tau \nabla f(y) + \lambda \nabla g(y) = 0$,
$\lambda g(y) \geqq 0$,
$\lambda \geqq 0$, $\tau \geqq 0$ and $\tau e = 1$,

where $e = (1, 1, \ldots, 1)^t \in R^p$.

We denote the set of all feasible solutions of problem (DMOP) by Y_0 i.e.

$$Y_0 = \{(y, \tau, \lambda) : \tau \nabla f(y) + \lambda \nabla g(y) = 0, \lambda g(y) \geq 0, \tau \in R^p, \lambda \in R^m, \lambda \geq 0\}$$

Theorem 4.1. (Weak Duality). Suppose that

(i) $x \in X$;

(ii) $(y, \tau, \lambda) \in Y_0$ and $\tau > 0$;

(iii) problem (DMOP) is strong pseudo quasi α-univex at y with respect to some α, b_0, b_1, ϕ_0, ϕ_1 and η.

Then

$f(x) > f(y)$.

Proof. By contradiction, suppose that

$f(x) \leq f(y)$.

Since $b_0(x, y) > 0$ and $(.) \leq 0 \Rightarrow \varphi_0(.) \leq 0$, from the above inequality, we get

$$b_0(x, y) \phi_0 [f(x) - f(y)] \leq 0. \tag{4}$$

Since (y, τ, λ) is feasible for (DMOP). It follows that $-\lambda g(y) \leqq 0$.

Since $b_1(x, y) \geqq 0$ and $(.) \leqq 0 \Rightarrow \phi_0(.) \leqq 0$, from the above inequality, we get

$$-b_1(x, y) \phi_1 [\lambda g(y)] \leqq 0. \tag{5}$$

By inequalities (4), (5) and condition (iii), we have

$\alpha(x, y)[\nabla f(y)] \eta(x, y) \leq 0$ and
$\alpha(x, y)[\lambda \nabla g(y)] \eta(x, y) \leqq 0$.

But $\alpha > 0$, so above inequalities reduce to

$$[\nabla f(y)] \eta(x, y) \leq 0, \tag{6}$$
$$[\lambda \nabla g(y)] \eta(x, y) \leqq 0. \tag{7}$$

But $\tau > 0$, the above two inequalities give

$$[\tau \nabla f(y) + \lambda \nabla g(y)] \eta(x, y) < 0,$$

which contradicts (iii). This completes the proof.

The proofs of the following theorems follow along the lines of Theorem 4.1; therefore we state the theorems but omit the proofs.

Theorem 4.2. (Weak Duality). Suppose that
(i) $x \in X$;
(ii) $(y,\tau,\lambda) \in Y_0$ and $\tau \geq 0$;
(iii) problem (DMOP) is weak strictly pseudo quasi α-univex at y with respect to some α, b_0, b_1, ϕ_0, ϕ_1 and η.
Then

$$f(x) > f(y).$$

Theorem 4.3. (Weak Duality). Suppose that
(i) $x \in X$;
(ii) $(y,\tau,\lambda) \in Y_0$ and $\tau \geqq 0$;
(iii) problem (DMOP) is weak strictly pseudo α-univex at y with respect to some α, b_0, b_1, ϕ_0, ϕ_1 and η.
Then

$$f(x) > f(y).$$

Theorem 4.4. (Strong Duality). Let $\bar{x}$ be an efficient solution for (MOP) and $\bar{x}$ satisfies a constraint qualification for (MOP) in Maeda (1994). Then, there exist $\bar{\tau} \in R^p$ and $\bar{\lambda} \in R^m$ such that $(\bar{x},\bar{\tau},\bar{\lambda})$ is feasible for (DMOP). If any of the weak duality in Theorems 4.1, 4.2 and 4.3 also holds, then $(\bar{x},\bar{\tau},\bar{\lambda})$ is efficient solution for (DMOP).

Proof. Since $\bar{x}$ is efficient solution for (MOP) and satisfies a generalized constraint qualification (Maeda (1994)), by Kuhn-Tucker necessary conditions (Maeda (1994)) there exist $\bar{\tau} > 0$ and $\bar{\lambda} \geqq 0$ such that

$$\bar{\tau}\nabla f(\bar{x}) + \bar{\lambda}\nabla g(\bar{x}) = 0,$$
$$\bar{\lambda}_i g_i(\bar{x}) = 0.$$

The vector $\bar{\tau}$ may be normalized according to $\bar{\tau}e = 1$, $\bar{\tau} > 0$, which gives that the triplet $(\bar{x},\bar{\tau},\bar{\lambda})$ is feasible for (DMOP). The efficiency follows from the weak duality in Theorems 4.1, 4.2 and 4.3. This completes the proof.

5. GENERALIZED MOND-WEIR TYPE DUALITY

We shall continue our discussion on duality for (MOP) in the present section by considering a general Mond-Weir type dual problem (GMOP) and proving weak and strong duality theorem under some mild assumption of α-univexity introduced in section 2. The results given in this section may help to develop numerical algorithms as it provides suitable stopping rules for primal problem (MOP) and dual problem (GMOP).

We consider the following general Mond-Weir (1981) type dual to (MOP)

(GMOP) Maximize $f(y) + \lambda_{J_0} g_{J_0}(y)e$,

Subject to
$$\tau\nabla f(y) + \lambda\nabla g(y) = 0, \tag{8}$$
$$\lambda_{J_t} g_{J_t}(y) \geqq 0, \quad 1 \leqq t \leqq r, \tag{9}$$
$$\lambda \geqq 0, \tag{10}$$
$$\tau \geqq 0, \tag{11}$$
$$\tau e = 1, \tag{12}$$

where $e = (1, 1, ..., 1)^t \in R^p$ and J_t, $0 \leqq t \leqq r$ are partitions of set M.

Theorem 5.1. (Weak Duality). Let x and (y, τ, λ) be feasible solutions for (MOP) and (GMOP) respectively. Assume that one of the following conditions holds:

(a) $\tau > 0$ and $(f + \lambda_{J_0} g_{J_0}(.)e, \lambda_{J_t} g_{J_t}(.))$ is strong pseudo quasi α-univex at y with respect to some α, b_0, b_1, ϕ_0, ϕ_1 and η for any t, $1 \leqq t \leqq r$,
(b) $(f + \lambda_{J_0} g_{J_0}(.)e, \lambda_{J_t} g_{J_t}(.))$ is weak strictly pseudo quasi α-univex at y with respect to some α, b_0, b_1, ϕ_0, ϕ_1 and η for any t, $1 \leqq t \leqq r$,
(c) $(f + \lambda_{J_0} g_{J_0}(.)e, \lambda_{J_t} g_{J_t}(.))$ is weak strictly pseudo α-univex at y with respect to some α, b_0, b_1, ϕ_0, ϕ_1 and η for any t, $1 \leqq t \leqq r$.

Then the following condition can not hold:

$$f(x) \leq f(y) + \lambda_{J_0} g_{J_0}(y)e. \tag{13}$$

Proof. Suppose contrary to the result. Thus, we have

$$f(x) \leq f(y) + \lambda_{J_0} g_{J_0}(y)e.$$

Since x is feasible for (MOP) and $\lambda \geqq 0$, the above inequality implies that

$$f(x) + \lambda_{J_0} g_{J_0}(x)e \leq f(y) + \lambda_{J_0} g_{J_0}(y)e.$$

By the feasibility of (y,τ,λ) inequality (9) gives

$$-\lambda_{J_t} g_{J_t}(y) \leqq 0, \quad \text{for all } 1 \leqq t \leqq r.$$

Since ϕ_0 and ϕ_1 are increasing, the above two inequalities give

$$b_0(x,y)\phi_0\left[\left(f(x) + \lambda_{J_0} g_{J_0}(x)e\right) - \left(f(y) + \lambda_{J_0} g_{J_0}(y)e\right)\right] \leq 0, \tag{14}$$

$$-b_1(x,y)\phi_1\left[\lambda_{J_t} g_{J_t}(y)\right] \leqq 0 \tag{15}$$

By condition (a), from (14) and (15), we have

$$\alpha(x,y)\left(\nabla f(y)+\lambda_{Jo}\nabla g_{Jo}(y)e\right)\eta(x,y)\le 0$$

and

$$\alpha(x,y)\left(\lambda_{Jt}\nabla g_{Jt}(y)\right)\eta(x,y))\le 0,\quad \forall 1\leqq t\leqq r.$$

But $\alpha>0$, we get

$$\left(\nabla f(y)+\lambda_{Jo}\nabla g_{Jo}(y)\ e\right)\eta(x,y)\le 0$$

and

$$\left(\lambda_{Jt}\nabla g_{Jt}(y)\right)\eta(x,y))\leqq 0,\quad \forall 1\leqq t\leqq r.$$

Since $\tau>0$, the above two inequalities yield

$$\left[\tau\nabla f(y)+\sum_{t=0}^{r}\lambda_{Jt}\nabla g_{Jt}(y)\right]\eta(x,y)<0. \tag{16}$$

Since J_t, $0\leqq t\leqq r$, are partition of set M, (16) is equivalent to

$$\left[\tau\nabla f(y)+\lambda\nabla g(y)\right]\eta(x,y)<0. \tag{17}$$

This contradicts (8).

Using hypothesis (b), we see that (14) and (15) together give

$$\alpha(x,y)\left(\nabla f(y)+\lambda_{Jo}\nabla g_{Jo}(y)\ e\right)\eta(x,y)<0$$

and

$$\alpha(x,y)\left(\lambda_{Jt}\nabla g_{Jt}(y)\right)\eta(x,y)\leqq 0,\quad \forall 1\leqq t\leqq r.$$

But $\alpha>0$, we get

$$\left(\nabla f(y)+\lambda_{Jo}\nabla g_{Jo}(y)e\right)\eta(x,y)<0$$

and

$$\left(\lambda_{Jt}\nabla g_{Jt}(y)\right)\eta(x,y)\leqq 0,\quad \forall 1\leqq t\leqq r.$$

Since $\tau\ge 0$, the above two inequalities yield

$$\left[\tau\nabla f(y)+\sum_{t=0}^{r}\lambda_{Jt}\nabla g_{Jt}(y)\right]\eta(x,y)<0.$$

The above inequality leads to (5.10), which contradicts (8).

By condition (c), from (14) and (15), we have

$$\alpha(x,y)\left(\nabla f(y)+\lambda_{Jo}\nabla g_{Jo}(y)\ e\right)\eta(x,y)<0$$

and

$$\alpha(x,y)\left(\lambda_{Jt}\nabla g_{Jt}(y)\right)\eta(x,y))<0,\quad \forall 1\leqq t\leqq r.$$

But $\alpha>0$, we get

$$\left(\nabla f(y)+\lambda_{Jo}\nabla g_{Jo}(y)e\right)\eta(x,y)<0$$

and

$$\left(\lambda_{Jt}\nabla g_{Jt}(y)\right)\eta(x,y))<0,\quad \forall 1\leqq t\leqq r.$$

Since $\tau\ge 0$, the above two inequalities give (16), which leads to (17). This is a contradiction to (8). This completes the proof.

Theorem 5.2. (Strong Duality). Let $\bar{x}$ be an efficient solution for (MOP) at which the generalized constraint qualification is satisfied (Maeda (1994)). Then, there exist $\bar{\tau}\in R^p$ and $\bar{\lambda}\in R^m$ such that $(\bar{x},\bar{\tau},\bar{\lambda})$ is feasible for (GMOP). If also weak duality (Theorem 5.1) holds, then $(\bar{x},\bar{\tau},\bar{\lambda})$ is efficient solution for (GMOP).

Proof. Since $\bar{x}$ is efficient solution for (MOP) and satisfies a generalized constraint qualification (Maeda (1994)), by Kuhn-Tucker necessary conditions (Maeda (1994)) there exist $\bar{\tau}>0$ and $\bar{\lambda}\geqq 0$ such that

$$\bar{\tau}\nabla f(\bar{x})+\bar{\lambda}\nabla g(\bar{x})=0,$$
$$\bar{\lambda}_i g_i(\bar{x})=0,\ \forall 1\leqq t\leqq r.$$

The vector $\bar{\tau}$ may be normalized according to $\bar{\tau}e=1$, $\bar{\tau}>0$, which gives that the triplet $(\bar{x},\bar{\tau},\bar{\lambda})$ is feasible for (GMOP). The efficiency follows from the weak duality in Theorem 5.1.

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