International Journal of Operations Research

International Journal of Operations Research Vol. 5, No. 4, 213−224 (2008)

Analysis of Single Server Retrial Queue with Batch Arrivals, Two Phases of Heterogeneous Service and Multiple Vacations with N–Policy

M. Senthil Kumar[1,*] **and R. Arumuganathan**[2,Ψ]

[1,2]Department of Mathematics and Computer Applications PSG College of Technology Coimbatore,

Tamil Nadu India - 641004

Received August 2007; *Revised October 2008*; *Accepted November 2008*

Abstract—We consider a single server retrial queue with batch arrivals, two phases of heterogeneous service and multiple vacations with N- policy. The primary arrivals find the server busy or doing secondary job (vacation) will join orbit (group of repeated calls). If the number of repeated calls in orbit is less than N, the server does the secondary job repeatedly until the retrial group size reaches N. At the secondary job completion epoch, if the orbit size is at least N, then server remains in the system to render service either for primary calls or for repeated calls. For the proposed model, we carry out steady state system size distribution of number of customers in retrial group. We discuss its application of the proposed model to the analysis of a communication protocol like SMTP (Simple Mail Transfer Protocol), TCP/IP (Transmission Control Protocol/Internet Protocol) and etc.

Keywords—N-Policy multiple vacations, Retrial queue, TCP/IP protocol, SMTP protocol.

1. INTRODUCTION

Queueing systems arise in modeling of many practical applications related to computer science, communication engineering, production, human computer interactions, and so on. A new class of queueing systems, systems with repeated calls (or retrial queues, queues with returning customers, repeated orders, etc.) is characterized by the following feature: *an arriving customer sees when all servers accessible for him are busy leaves the service area but after some random time repeats his demand.* This feature plays a vital role in several computer and communication networks as well. Other applications include stacked aircraft waiting to land, and queues of retail shoppers who may leave a long waiting hoping to return later when the line may be shorter.

The detailed overviews of the related references with retrial queues can be found in the recent book of Falin and Templeton (1997) and survey papers of Artalejo (1999a, 1999b). The first batch arrival retrial queueing model was introduced by Falin (1976) who assumed the following rule: "*if the server is busy at the arrival epoch, then the whole batch joins the retrial group, whereas the server is free, then one of the arriving units starts its service and the rest join the retrial group*". The single server retrial queue with priority calls have been studied by Choi et al. (1990, 1995, 1999) for many applications in telecommunication and mobile communication. The distribution of number of customers served in an M/G/1 retrial queue have been analysed by Lopez–Herrero (2002) for finding the probability that at most k customers were served during the busy period of an M/G/1 retrial queue. Krishna Kumar et.al. (2002b) analysed the M/G/1 retrial queue with feedback and starting failures using supplementary variable technique.

The overwhelming literature contributions consider queueing system with two phases of service. Madan (2000) considered the classical M/G/1 queueing system in which the server provides the first essential service (FES) to all the arriving customers whereas some of them receive second optional service. The FES follows general distribution and second optional service follows exponential distribution. Medhi (2002) generalized the model by considering that the second optional service is also governed by a general distribution. Madan et al. (2005) considered the queueing model with two phases of heterogeneous service under Bernoulli schedule and a general vacation time.

Krishna Kumar et al. (2002a) considered an M/G/1 retrial queue with additional phase of service. While at the first phase of service, the server may push–out the customer who is receiving such a service, to start the service of another priority arriving customer. The interrupted customers join a retrial queue and the customer at the head of this queue is allowed to conduct a repeated attempt in order to start again his first phase of service after some random time. The motivation for this model comes from some computer and communication networks where messages are processed in two stages by a single server. Artalejo et al. (2004) analyze the steady state analysis of M/G/1 queueing system with repeated attempts and two phases of service using Embedded Markov chain method. Recently Senthil Kumar and Arumuganathan (2008) have analysed the batch arrival single

* Corresponding author's email: ms_kumar_in@yahoo.com

Ψ Corresponding author's email: ran_psgtech@yahoo.co.in

server retrial queue in which the server provides two phases of heterogeneous service and receives general vacation time under Bernoulli schedule. *Our paper focuses on the two phases of heterogeneous service with N-policy multiple vacations.*

For a detailed survey on queueing system with server vacations, one can refer to refs. (Lee et al. 1994, Krisha Reddy et al. 1998 & Arumuganathan et al. 2005). Lee et al. (1994) analysed an $M^x/G/1$ queue with N–policy and multiple vacations. They have discussed the paper with bulk arrival and single service. Bulk queue with N–policy multiple vacations and setup times is analyzed by Krisha Reddy et al. (1998) in which the arrivals occur in bulk and service process is done in bulk. Recently, Arumuganathan et al. (2005) analyzed a bulk queue with multiple vacations, setup times with N–policy and closedown times. Lee et al. (1994) Krishna Reddy et al. (1998) and Arumuganathan et al. (2005) discussed N–policy in classical queueing model. *But this paper discusses the N–policy multiple vacations in retrial queueing model.*

In this paper, we consider a single server retrial queue with batch arrivals, two phases of heterogeneous service and multiple vacations with N–policy. Analytical treatment of this model is obtained using supplementary variable technique. We obtain the probability generating function of number of customers in the retrial group. Our main motivation is coming from some applications to Local Area Networks (LAN), client–server communication, telephone systems, electronic mail servers on Internet. In all these applications, the service is done in two phases. The detailed information regarding LAN and TCP/IP can be obtained from Behrouz Forouzan and Sophia Chung Fegan (2003).

1.1 Practical justification of the model

A possible application of bulk arrival retrial queueing system with two phases of heterogeneous service and multiple vacations with N–policy is as follows:

The following example is based on the work of Jau–Chuan Ke and Fu–Min Chang (2008). Mail system uses Simple Main Transfer Protocol (SMTP) to deliver messages between mail servers. When a mail transfer program contacts a server on a remote machine, it forms a TCP connection over which it communicates. Once the connection is established, the two programs follow SMTP that allows the sender to identify it, specify a recipient, and transfer an e–mail message. By having sender deposits the e–mail in his/her, own mail server, the mail server an repeatedly try to send the contact message to target server until the target server becomes operational. Typically, contacting messages arrive at the mail server following the Poisson stream. One message comprises collection of finite number of packets. (i.e. Threshold policy N). If all packets of a message are arrived to the mail server, the server starts to do service. When all the messages arrive at the mail server, one packet is selected to serve and the rest of the packets will join to the buffer (i.e., retrial group). In the buffer, each packet waits a certain amount of time and retries the service again. There is a daemon program implemented at mail server to manage the service requests from buffer. In addition, to keep the mail server functioning well, some maintenance activities (i.e., multiple vacations) are needed. For example, virus scan and spam filtering etc., are an important maintenance activity for the mail server. It can be performed when the mail server is idle and be programmed to perform on a regular basis. In this scenario, the buffer, mail server service mechanism and maintenance activities are corresponding to orbit, two phases of heterogeneous service, and multiple vacations in queueing terminology.

2. MATHEMATICAL MODEL

We consider the single server retrial queueing system with batch arrival. The primary calls arrive in bulk according to Poisson process with rate λ. In the batch arrival retrial queue it is assumed that at every arrival epoch a batch of *k* primary calls arrives with probability g_k. If the server is busy at the arrival epoch, then all these calls join the orbit, whereas if the server is free, then one of the arriving calls begins its service and others form sources of repeated calls. The server provides preliminary FES and second essential service (SES) to all arriving calls. Primary calls finding the server free upon arrival automatically start their FES. However, if a primary call finds the server busy (attending FES or SES), then it joins the orbit in order to seek service again until it finds the server free. The time between two successive repeated attempts of each call in orbit is assumed to be exponentially distributed with rate '*ν*'. If the number of repeated calls (arrivals) in orbit is less than N, the server does the secondary job repeatedly until the retrial group size reaches N. At the secondary job completion epoch, if the orbit size is at least N, then server remains in the system to render service either for primary calls or for repeated calls. Let $S_1(.)$, $S_2(.)$ and $V(.)$ are cumulative distribution functions of FES, SES and vacation time, respectively. s_1 (x), s_2 (x) and v(x) denote probability density functions of FES, SES and vacation time, respectively. Laplace Stieltjes transform (LST) of FES, SES and vacation time are $\tilde{S}_1(\theta)$, $\tilde{S}_2(\theta)$, and $\tilde{V}(\theta)$, respectively. $S_1^0(t)$, $S_2^0(t)$ and $V^0(t)$ denote remaining service time of FES, SES and Vacation time at time t, respectively. N(t) denotes the number of customers in the orbit at time *t*.

$X(z) = \sum_{k=1}^{\infty} g_k z^k$ is denoted as the generating function of the batch size distribution. The server state is denoted as follows

$$C(t) = \begin{cases} 0, & \textit{if the server is idle} \\ 1, & \textit{if the server is doing FES} \\ 2, & \textit{if the server is doing SES} \\ 3, & \textit{if the server is on vacation} \end{cases}$$

Y(t) = j if the server is on the j^{th} vacation starting from the idle period.

Now we define,

$P_{0n}(t)dt = \Pr\{N(t) = n, C(t) = 0\} \quad n \geq N$ and

$P_{in}(x,t)dt = \Pr\{N(t) = n, C(t) = i,\ x \leq S_i^0(t) \leq x + dt\}$; $n \geq 0$; $i = 1,2$

$V_{jn}(x,t)dt = \Pr\{N(t) = n, Y(t) = j,\ C(t) = 3, x \leq V^0(t) \leq x + dt\}$; $n \geq 0$; $j \geq 1$

3. SYSTEM SIZE DISTRIBUTION

Now, the following equations are obtained for the queueing system, using supplementary variable technique:

$$P_{0,j}(t+\Delta t) = P_{0,j}(t)\left(1-\lambda\Delta t - j\nu\Delta t\right) + P_{2,j}(0,t)\Delta t + \sum_{l=1}^{\infty} V_{l,j}(0,t)\Delta t \qquad j \geq N$$

$$P_{1,j}(x-\Delta t, t+\Delta t) = P_{1,j}(x,t)(1-\lambda\Delta t) + \lambda\sum_{k=1}^{j} P_{1,j-k}(x,t)g_k\Delta t \qquad 0 \leq j < N$$

$$P_{1,j}(x-\Delta t, t+\Delta t) = P_{1,j}(x,t)(1-\lambda\Delta t) + \lambda\sum_{k=1}^{j+1} g_k P_{0,j-k+1}(t)s_1(x) + (j+1)\nu P_{0,j+1}(t)\Delta t\, s_1(x) + \sum_{k=1}^{j} P_{1,j-k}(x,t)\lambda g_k\Delta t \qquad j \geq N$$

$$P_{2,j}(x-\Delta t, t+\Delta t) = P_{2,j}(x,t)(1-\lambda\Delta t) + \lambda\sum_{k=1}^{j} g_k P_{2,j-k}(x,t)\Delta t + P_{1,j}(0,t)s_2(x)\Delta t \qquad j \geq 0$$

$$V_{1,0}(x-\Delta t, t+\Delta t) = V_{1,0}(x,t)(1-\lambda\Delta t) + P_{2,0}(0,t)v(x)\Delta t$$

$$V_{1,j}(x-\Delta t, t+\Delta t) = V_{1,j}(x,t)(1-\lambda\Delta t) + P_{2,j}(0,t)v(x)\Delta t + \sum_{k=1}^{j} V_{1,j-k}(x,t)\lambda g_k\Delta t \qquad j = 1,2,..N-1$$

$$V_{1,j}(x-\Delta t, t+\Delta t) = V_{1,j}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} V_{1,j-k}(x,t)\lambda g_k\Delta t \qquad j \geq N$$

$$V_{l,0}(x-\Delta t, t+\Delta t) = V_{l,0}(x,t)(1-\lambda\Delta t) + V_{l-1,0}(0,t)v(x)\Delta t$$

$$V_{l,j}(x-\Delta t, t+\Delta t) = V_{l,j}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} V_{l,j-k}(x,t)\lambda g_k\Delta t + V_{l-1,j}(0,t)v(x)\Delta t\,, \qquad j = 1,2...N-1$$

$$V_{l,j}(x-\Delta t, t+\Delta t) = V_{l,j}(x,t)(1-\lambda\Delta t) + \sum_{k=1}^{j} V_{l,j-k}(x,t)\lambda g_k\Delta t \qquad j \geq N$$

From the above equations, the steady state system size equations are obtained as follows:

$$(\lambda + j\nu)P_{0,j} = P_{2,j}(0) + \sum_{l=1}^{\infty} V_{l,j}(0) \qquad j \geq N \tag{1}$$

$$-P_{1,j}{}'(x) = -\lambda P_{1,j}(x) + \sum_{k=1}^{j} P_{1,j-k}(x)\lambda g_k \qquad 0 \leq j < N \tag{2}$$

$$-P_{1,j}{}'(x) = -\lambda P_{1,j}(x) + \lambda\sum_{k=1}^{j+1} P_{0,j-k+1}g_k s_1(x) + (j+1)\nu P_{0,j+1}s_1(x) + \sum_{k=1}^{j} P_{1,j-k}(x)\lambda g_k \qquad j \geq N \tag{3}$$

$$-V_{1,0}{}'(x) = -\lambda V_{1,0}(x) + P_{2,0}(0)v(x) \tag{4}$$

$$-V_{1,j}{}'(x) = -\lambda V_{1,j}(x) + P_{2,0}(0)v(x) + \sum_{k=1}^{j} V_{1,j-k}(x)\lambda g_k \qquad j = 1,2,....N-1 \tag{5}$$

$$-V_{l,0}{}'(x) = -\lambda V_{l,0}(x) + V_{l-1}(0)v(x) \qquad l \geq 2 \tag{6}$$

$$-V_{l,j}{}'(x) = -\lambda V_{l,j}(x) + V_{l-1,j}(0)v(x) + \sum_{k=1}^{j} V_{l,j-k}(x)\lambda g_k \qquad 0 \leq j < N \tag{7}$$

$$-V_{l,j}{}'(x) = -\lambda V_{l,j}(x) + \sum_{k=1}^{j} V_{1,j-k}(x)\lambda g_k \qquad j \geq N \tag{8}$$

Taking LST on both sides of the Eq.(2)–(8), we have,

$$(\lambda + jv)P_{0,j} = P_{2,j}(0) + \sum_{l=1}^{\infty} V_{l,j}(0) \qquad j \geq N \tag{9}$$

$$\theta \tilde{P}_{1,j}(\theta) - P_{1,j}(0) = \lambda \tilde{P}_{1,j}(\theta) - \sum_{k=1}^{j} \tilde{P}_{1,j-k}(\theta)\lambda g_k \qquad 0 \leq j < N \tag{10}$$

$$\theta \tilde{P}_{1,j}(\theta) - P_{1,j}(0) = \lambda \tilde{P}_{1,j}(\theta) - \sum_{k=1}^{j} \tilde{P}_{1,j-k}(\theta)\lambda g_k - \lambda \sum_{k=1}^{j+1} P_{0,j-k+1}\, g_k \tilde{S}_1(\theta) - (j+1)vP_{0,j+1}\tilde{S}_1(\theta) \qquad j \geq N \tag{11}$$

$$\theta \tilde{P}_{2,j}(\theta) - P_{2,j}(0) = \lambda \tilde{P}_{2,j}(\theta) - \sum_{k=1}^{j} \tilde{P}_{2,j-k}(\theta)\lambda g_k - P_{1,j}(0)\tilde{S}_2(\theta) \qquad j \geq 0 \tag{12}$$

$$\theta\, \tilde{V}_{1,0}(\theta) - V_{1,0}(0) = \lambda \tilde{V}_{1,0}(\theta) - P_{2,0}(0)\tilde{V}(\theta) \tag{13}$$

$$\theta \tilde{V}_{1,j}(\theta) - V_{1,j}(0) = \lambda \tilde{V}_{1,j}(\theta) - P_{2,j}(0)\tilde{V}(\theta) - \sum_{k=1}^{j} \tilde{V}_{1,j-k}(\theta)\lambda g_k \qquad j = 1, 2, \ldots N-1 \tag{14}$$

$$\theta \tilde{V}_{1,j}(\theta) - V_{1,j}(0) = \lambda \tilde{V}_{1,j}(\theta) - \sum_{k=1}^{j} \tilde{V}_{1,j-k}(\theta)\lambda g_k \qquad j \geq N \tag{15}$$

$$\theta \tilde{V}_{l,0}(\theta) - V_{l,0}(0) = \lambda \tilde{V}_{l,0}(\theta) - \tilde{V}_{l-1,0}(0)\tilde{V}(\theta) \qquad l \geq 2 \tag{16}$$

$$\theta \tilde{V}_{l,j}(\theta) - V_{l,j}(0) = \lambda \tilde{V}_{l,j}(\theta) - \tilde{V}_{l-1,j}(0)\tilde{V}(\theta) - \sum_{k=1}^{j} \tilde{V}_{l,j-k}(\theta)\lambda g_k \qquad 0 \leq j < N \tag{17}$$

$$\theta \tilde{V}_{l,j}(\theta) - V_{l,j}(0) = \lambda \tilde{V}_{l,j}(\theta) - \sum_{k=1}^{j} \tilde{V}_{l,j-k}(\theta)\lambda g_k \qquad j \geq N \tag{18}$$

Now, we define the following probability generating functions (PGF)

$$\begin{aligned} &P_0(z) = \sum_{j=N}^{\infty} P_{0,j} z^j \;; \quad \tilde{P}_1(z,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{1,j}(\theta) z^j \;; \quad P_1(z,0) = \sum_{j=0}^{\infty} P_{1,j}(0) z^j \\ &\tilde{P}_2(z,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{2,j}(\theta) z^j \;; \quad P_2(z,0) = \sum_{j=0}^{\infty} P_{2,j}(0) z^j \\ &\tilde{V}_l(z,\theta) = \sum_{j=0}^{\infty} \tilde{V}_{l,j}(\theta) z^j \;; \quad V_l(z,0) = \sum_{j=0}^{\infty} V_{l,j}(0) z^j \end{aligned} \tag{19}$$

Using PGF, the Eq.(9)–(18) can be written as follows

$$\lambda P_0(z) + vzP_0{}'(z) = P_2(z) - \sum_{j=0}^{N-1} P_{2,j}(0) z^j + \sum_{l=1}^{\infty} \left(V_l(z,0) - \sum_{j=0}^{N-1} V_{l,j}(0) z^j \right) \tag{20}$$

$$\left(\theta - \lambda + \lambda X(z)\right) \tilde{P}_1(z,\theta) = P_1(z,0) - \lambda \frac{X(z)}{z} P_0(z)\tilde{S}_1(\theta) - v\tilde{S}_1(\theta)P_0{}'(z) \tag{21}$$

$$\left(\theta - \lambda + \lambda X(z)\right) \tilde{P}_2(z,\theta) = P_2(z,0) - P_1(z,0)\tilde{S}_2(\theta) \tag{22}$$

$$\left(\theta-\lambda+\lambda X(z)\right)\tilde{V}_1(z,\theta)=V_1(z,0)-\sum_{j=0}^{N-1}P_{2,j}(0)\tilde{V}(\theta)z^j \tag{23}$$

$$\left(\theta-\lambda+\lambda X(z)\right)\tilde{V}_l(z,\theta)=V_l(z,0)-\tilde{V}(\theta)\sum_{j=0}^{N-1}V_{l-1,j}(0)z^j \tag{24}$$

Substituting $\theta=\lambda-\lambda X(z)$ in Eq.(21)–(24), we get

$$P_1(z,0)=\lambda\frac{X(z)}{z}P_0(z)\tilde{S}_1(\lambda-\lambda X(z))+v\,\tilde{S}_1(\lambda-\lambda X(z))P_0\,'(z) \tag{25}$$

$$P_2(z,0)=P_1(z,0)\tilde{S}_2(\lambda-\lambda X(z)) \tag{26}$$

$$V_1(z,0)=\sum_{j=0}^{N-1}P_{2,j}(0)\,\tilde{V}(\lambda-\lambda X(z))z^j \tag{27}$$

$$V_l(z,0)=\tilde{V}(\lambda-\lambda X(z))\sum_{j=0}^{N-1}V_{l-1,j}(0)z^j \tag{28}$$

Substituting Eq.(25)–(28) in the Eq.(20), we have,

$$P_0\,'(z,0)=\frac{1}{v\left(z-\tilde{S}_1(\lambda-\lambda X(z))\tilde{S}_2(\lambda-\lambda X(z))\right)}\left[f(z)-P_0(z)\lambda\left(1-\tilde{S}_1(\lambda-\lambda X(z))\tilde{S}_2(\lambda-\lambda X(z))\frac{X(z)}{z}\right)\right]$$

where $f(z)=\sum_{l=1}^{\infty}\left(V_l(z,0)-\sum_{j=0}^{N-1}V_{l,j}(0)z^j\right)-\sum_{j=0}^{N-1}P_{2,j}(0)z^j$

Substituting Eq.(25)–(28) in the Eq.(21)–(24), we get the partial PGF of FES, SES and Vacation time

$$\left(\theta-\lambda+\lambda X(z)\right)\tilde{P}_1(z,\theta)=\left(\tilde{S}_1(\lambda-\lambda X(z)-\tilde{S}_1(\theta)\right)\left[\lambda\frac{X(z)}{z}P_0(z)-vP_0\,'(z)\right] \tag{29}$$

$$\left(\theta-\lambda+\lambda X(z)\right)\tilde{P}_2(z,\theta)=P_1(z,0)\left(\tilde{S}_2(\lambda-\lambda X(z))-\tilde{S}_2(\theta)\right) \tag{30}$$

$$\left(\theta-\lambda+\lambda X(z)\right)\tilde{V}_1(z,\theta)=\left(\tilde{V}(\lambda-\lambda X(z))-\tilde{V}(\theta)\right)\sum_{j=0}^{N-1}P_{2,j}(0)z^j \tag{31}$$

$$\left(\theta-\lambda+\lambda X(z)\right)\tilde{V}_l(z,\theta)=\left(\tilde{V}(\lambda-\lambda X(z))-\tilde{V}(\theta)\right)\sum_{j=0}^{N-1}V_{l-1,j}(0)z^j \tag{32}$$

Let P(z) be the PGF of the orbit size at an arbitrary time epoch. Then,

$$\begin{aligned} P(z) &= P_0(z) + \tilde{P}_1(z,0) + \tilde{P}_2(z,0) + \sum_{l=1}^{\infty} \tilde{V}_l(z,0) \\ &= P_0(z) + \frac{\left(\tilde{S}_1(\lambda - \lambda X(z) - 1\right)\left(\lambda \frac{X(z)}{z} P_0(z,0) + vP_0{}'(z,0)\right)}{\left(-\lambda + \lambda X(z)\right)} \\ &+ \frac{\left(\tilde{S}_2(\lambda - \lambda X(z)) - 1\right)\tilde{S}_1(\lambda - \lambda X(z)\left(\lambda \frac{X(z)}{z} P_0(z,0) + vP_0{}'(z,0)\right)}{\left(-\lambda + \lambda X(z)\right)} \\ &+ \frac{\left(\tilde{V}(\lambda - \lambda X(z)) - 1\right)}{\left(-\lambda + \lambda X(z)\right)}\left[\sum_{j=0}^{N-1} P_{2,j}(0) z^j + \sum_{l=2}^{\infty}\sum_{j=0}^{N-1} V_{l-1,j}(0) z^j\right] \end{aligned} \tag{33}$$

Using the Eq.(20), Eq.(29)–(32), the PGF of number of customers in orbit at an arbitrary epoch can be expressed as follows.

$$P(z) = \frac{(z-1)\left\{\left(\lambda X(z) - \lambda\right) P_0(z) + \left(\tilde{V}(\lambda - \lambda X(z)) - 1\right)\left[\sum_{j=0}^{N-1} P_{2,j}(0) z^j + \sum_{l=1}^{\infty}\sum_{j=0}^{N-1} V_{l,j}(0) z^j\right]\right\}}{\left[z - \tilde{S}_1(\lambda - \lambda X(z))\tilde{S}_2(\lambda - \lambda X(z))\right]\left(\lambda X(z) - \lambda\right)} \tag{34}$$

where

$$P_0(z) = k(z)P_0(1) + k(z)\int_1^z \frac{f(t)}{k(t)v(t - \tilde{S}_1(\lambda - \lambda X(t))\tilde{S}_2(\lambda - \lambda X(t)))} dt \,,$$

$$k(z) = \exp\left(\frac{-\lambda}{v}\int_1^z \left(\frac{1 - \tilde{S}_1(\lambda - \lambda X(u))\tilde{S}_2(\lambda - \lambda X(u))\frac{X(u)}{u}}{u - \tilde{S}_1(\lambda - \lambda X(u))\tilde{S}_2(\lambda - \lambda X(u))}\right) du\right)$$

and

$$f(z) = \left(\tilde{V}(\lambda - \lambda X(z)) - 1\right)\left[\sum_{j=0}^{N-1} P_{2,j}(0) z^j + \sum_{l=1}^{\infty}\sum_{j=0}^{N-1} V_{l,j}(0) z^j\right]$$

From the Eq.(34), we have $\lim_{z \to 1} P(z) = 1$. It immediately follows that the steady state condition is

$\rho = \lambda E(X)\left(E(S_1) + E(S_2)\right) < 1$. Let $\sum_{l=1}^{\infty} V_{lj}(0) = q_j$ and $P_{2,j}(0) = p_j$.

Then $$P(z) = \frac{(z-1)\left\{(\lambda X(z) - \lambda)P_0(z) + (\tilde{V}(\lambda - \lambda X(z)) - 1)\left[\sum_{j=0}^{N-1} (p_j + q_j) z^j\right]\right\}}{\left[z - \tilde{S}_1(\lambda - \lambda X(z))\tilde{S}_2(\lambda - \lambda X(z))\right](\lambda X(z) - \lambda)}$$

Theorem 1.

If α_n is the probability of n customers arrive during a vacation,

$q_n = \sum_{i=0}^{n} k_i p_{n-i}$, $n = 0, 1 \ldots N-1$ where $k_0 = \alpha_0 / (1 - \alpha_0)$ and $k_n = \left(\alpha_n + \sum_{i=1}^{n} \alpha_i k_{n-i}\right) / (1 - \alpha_0)$

Proof.

Using $\sum_{l=1}^{\infty} V_{lj}(0) = q_j$, $P_{2,j}(0) = p_j$, Eq.(31)–(32), $\sum_{l=1}^{\infty} V_l(z,0)$ simplifies to

$$\begin{aligned}\sum_{n=0}^{\infty} q_n z^n &= \tilde{V}(\lambda - \lambda X(z)) \sum_{j=0}^{N-1} (p_j + q_j) z^j \\ &= \sum_{n=0}^{\infty} \alpha_n z^n \left(\sum_{j=0}^{N-1} (p_j + q_j) z^j \right) \\ &= \sum_{n=0}^{N-1} \left(\sum_{j=0}^{n} (p_j + q_j)\alpha_{n-j} \right) z^n + \sum_{n=N}^{\infty} \left(\sum_{j=0}^{N-1} (p_j + q_j)\alpha_{n-j} \right) z^n\end{aligned}$$

Equating the coefficient of z^n, n=0,1,2,…N-1 on both sides of the above equation, we have,

$$q_n = \left(\sum_{i=0}^{n} (p_i + q_i)\alpha_{n-i} \right) / (1 - \alpha_0)$$

Coefficient of p_n in q_n = $\alpha_0 / (1 - \alpha_0)$ = k_0
Coefficient of p_{n-1} in q_n = $(\alpha_1 + \alpha_1 k_0)/(1 - \alpha_0)$ = k_1.

4. PERFORMANCE MEASURES

Some useful results of our model are listed below.

a) *The mean number of customers in the orbit*

$$L_Q = E[N(t)] = \underset{z \to 1}{Lt} \frac{d}{dz} P(z)$$

$$= \frac{(\lambda E(X^2) P_0(1) + \lambda E(X) P_0'(1) + 2\lambda E(X) E(V) \sum_{j=0}^{N-1} j(p_j + q_j)}{2(1-p)\lambda E(X)} \quad (35)$$
$$\frac{+ V_2 \sum_{j=0}^{N-1} (p_j + q_j) + \lambda E(X)(S_{12} + 2S_{11}S_{21} + S_{22}) - \lambda E(X^2)(1-p))}{2(1-p)\lambda E(X)}$$

where

$$S_{11} = \lambda E(X)E(S_1) \; ; \; S_{21} = \lambda E(X)E(S_2)$$

$$S_{12} = \lambda E(X^2)E(S_1) + \lambda^2 E(X)^2 E(S_1^2) \; ; \; S_{22} = \lambda E(X^2)E(S_2) + \lambda^2 E(X)^2 E(S_2{}^2)$$

$$V_2 = \lambda E(X^2)E(V) + \lambda^2 E(X)^2 E(V^2)$$

(36)

$$P_0(1) = 1 - \rho - \lambda E(X)E(V) \sum_{i=0}^{N-1} (p_i + q_i)$$

$$P_0'(1) = \frac{\lambda E(X)E(V) \sum_{i=0}^{N-1} (p_i + q_i) - \lambda(1 - \rho - E(X))}{\nu(1-\rho)}$$

b) *Mean waiting time in retrial queue*

We have the mean waiting time in the retrial queue (W) as follows,

$$W_Q = \frac{L}{\lambda E(X)} \quad (37)$$

5. PARTICULAR CASES

Case I: If there is no N-policy multiple vacations ($\tilde{V}$ (0) =1) and no second phase of essential service ($\tilde{S}_2(0)=1$), Eq.(34) reduces to the following form

$$P(z)=\frac{\left(z-1\right)P_0(z)}{z-\tilde{S}_1(\lambda-\lambda X(z))} \tag{38}$$

where $P_0(z)=(1-\lambda E(X)E(S_1))\exp\left(\dfrac{-\lambda}{v}\displaystyle\int_1^z\frac{1-\tilde{S}_1(\lambda-\lambda X(u))\dfrac{X(u)}{u}}{z-\tilde{S}_1(\lambda-\lambda X(u))}du\right)$

The Eq.(38) coincides the result of orbit size distribution of $M^x/G/1$ retrial queueing system in Falin and Templeton (1997).

Case II: *Single Server Batch Arrival Retrial Queue with two phase of Service times (k_1 –Erlang for FES, k_2 –Erlang for SES) and N- policy multiple vacations.*

It is assumed that k-Erlang with probability density function, $s(x)=\dfrac{\left(ku\right)^k x^{k-1}e^{-kux}}{\left(k-1\right)!}$, k>0 ; where u is the parameter. Hence the PGF of the retrial queue size distribution is given by,

$$\tilde{S}_i(\lambda-\lambda X(z))=\left(\frac{u_ik_i}{u_ik_i+\lambda(1-X(z))}\right)^{k_i}; \qquad i=1,2.$$

$$P(z)=\frac{(z-1)\left\{(\lambda X(z)-\lambda)P_0(z)+(\tilde{V}(\lambda-\lambda X(z))-1)\left[\displaystyle\sum_{j=0}^{N-1}(p_j+q_j)z^j\right]\right\}}{\left[z-\left(\dfrac{u_1k_1}{u_1k_1+\lambda(1-X(z))}\right)^{k_1}\left(\dfrac{u_2k_2}{u_2k_2+\lambda(1-X(z))}\right)^{k_2}\right](\lambda X(z)-\lambda)} \tag{39}$$

where $P_0(z)=k(z)P_0(1)+k(z)\displaystyle\int_1^z\frac{f(t)}{k(t)v\left(t-\left(\dfrac{u_1k_1}{u_1k_1+\lambda(1-X(t))}\right)^{k_1}\left(\dfrac{u_2k_2}{u_2k_2+\lambda(1-X(t))}\right)^{k_2}\right)}dt$,

$$k(z)=\exp\left(\frac{-\lambda}{v}\int_1^z\left(\frac{1-\left(\dfrac{u_1k_1}{u_1k_1+\lambda(1-X(u))}\right)^{k_1}\left(\dfrac{u_2k_2}{u_2k_2+\lambda(1-X(u))}\right)^{k_2}\dfrac{X(u)}{u}}{u-\left(\dfrac{u_1k_1}{u_1k_1+\lambda(1-X(u))}\right)^{k_1}\left(\dfrac{u_2k_2}{u_2k_2+\lambda(1-X(u))}\right)^{k_2}}\right)du\right)$$

$$k(z)=\exp\left(\frac{-\lambda}{v}\int_1^z\left(\frac{1-\left(\dfrac{u_1k_1}{u_1k_1+\lambda(1-X(u))}\right)^{k_1}\left(\dfrac{u_2k_2}{u_2k_2+\lambda(1-X(u))}\right)^{k_2}\dfrac{X(u)}{u}}{u-\left(\dfrac{u_1k_1}{u_1k_1+\lambda(1-X(u))}\right)^{k_1}\left(\dfrac{u_2k_2}{u_2k_2+\lambda(1-X(u))}\right)^{k_2}}\right)du\right)$$

and $f(z)=(\tilde{V}(\lambda-\lambda X(z))-1)\left[\sum_{j=0}^{N-1}(p_j+q_j)z^j\right]$

Case III: *Single Server Batch Arrival Retrial Queue with two phase of Service times (Hyper-Exponential Service times for both FES & SES) and N- policy multiple vacations.*

Considering the case of Hyper Exponential *Service times for both FES & SES,* the pdf of Hyper Exponential service times are as follows,

$$v_i(x)=cu_ie^{-u_ix}+(1-c)w_ie^{-w_ix}\qquad i=1,2$$

$$\tilde{S}_i(\lambda\alpha-\lambda z\alpha)=\left(\frac{u_ic}{u_i+\lambda\alpha(1-z)}\right)+\left(\frac{w_i(1-c)}{w_i+\lambda\alpha(1-z)}\right);\qquad i=1,2$$

$$P(z)=\frac{(z-1)\left\{(\lambda X(z)-\lambda)P_0(z)+(\tilde{V}(\lambda-\lambda X(z))-1)\left[\sum_{j=0}^{N-1}(p_j+q_j)z^j\right]\right\}}{\left[z-\left(\left(\frac{u_1c}{u_1+\lambda(1-X(z))}\right)+\left(\frac{w_1(1-c)}{w_1+\lambda(1-X(z))}\right)\right)\left(\left(\frac{u_2c}{u_2+\lambda(1-X(z))}\right)+\left(\frac{w_2(1-c)}{w_2+\lambda(1-X(z))}\right)\right)\right](\lambda X(z)-\lambda)}\tag{40}$$

Case IV: *Single Server Batch Arrival Retrial Queue with two phase of Service times (Exponential Service times for both FES & SES) and N- policy multiple vacations.*

$$P(z)=\frac{(z-1)\left\{(\lambda X(z)-\lambda)P_0(z)+(\tilde{V}(\lambda-\lambda X(z))-1)\left[\sum_{j=0}^{N-1}(p_j+q_j)z^j\right]\right\}}{\left[z-\tilde{S}_1(\lambda-\lambda X(z))\tilde{S}_2(\lambda-\lambda X(z))\right](\lambda X(z)-\lambda)}\tag{41}$$

where $\tilde{S}_i(\lambda-\lambda X(z))=\left(\frac{\mu_i}{\mu_i+\lambda(1-X(z))}\right)\ ;i=1,2$

6. NUMERICAL RESULTS

Mail system uses Simple Mail Transfer Protocol (SMTP) to deliver messages between mail servers. When a mail transfer program contacts a server on a remote machine, it forms a TCP connection over which it communicates. Once the connection is established, the two programs follow SMTP that allows the sender to identify it, specify a recipient, and transfer an e-mail message. By having sender deposits the e-mail in his/her, own mail server, the mail server can repeatedly try to send the contact message to target server until the target server becomes operational. Typically, contacting messages arrive at the mail server following the Poisson stream. One message comprises collection of finite number of packets (*i.e., Threshold policy N*). If all packets of a message are arrived to the mail server, the server starts to do service. When all messages arrive at the mail server, one packet is selected to serve and the rest of the packets will join to the buffer (*i.e., retrial group*). In the buffer, each packet waits a certain amount of time and retries the service again. There is a daemon program implemented at mail server to manage the service requests from buffer. In addition, to keep the mail server functioning well, some maintenance activities (*i.e., multiple vacations*) are needed. For example, virus scan and spam filtering etc., are an important maintenance activity for the mail server. It can be performed when the mail server is idle and be programmed to perform on a regular basis. In this scenario, the buffer, mail server service mechanism and maintenance activities are corresponding to orbit, two phases of heterogeneous service and multiple vacations in queueing terminology.

It is essential to verify the effect of the parameters N and retrial rate 'v' with mean number of packets in the buffer L_Q and with average waiting time of a packet in buffer. The proposed model can be modelled with following assumptions:

(i) Average arrival rate λ =0.1,

(ii) Vacation time is exponential with parameter 2 and

(iii) Batch arrival size distribution is geometric with mean 2.

Table 1 represents the effect of retrial rate 'v' with L_Q is observed. Considering the two phases of heterogeneous service time as exponential, Erlangian-2 and hyper-exponential, we observe that L_Q is decreasing when retrial rate increases. As threshold value N increases, L_Q increases. Figure 1(a) depicts the effect of L_Q with threshold value N and retrial rate v. Considering the two phases of heterogeneous service time as Erlangian-2, L_Q increases as N increases and L_Q decreases as 'v' increases. Figure 1(b) depicts the effect of L_Q with retrial rate 'v' and fixed N=4, considering the two phases of heterogeneous service time as Exponential, Erlangian-2 and Hyper–Exponential.

Table 2 represents the effect of retrial rate 'v' with W_Q is observed. Considering the two phases of heterogeneous service time as exponential, Erlangian-2 and hyper-exponential, we observe that W_Q is decreasing when retrial rate increases. As threshold value N increases, W_Q increases. Figure 2(a) depicts the effect of W_Q with threshold value N and retrial rate v. Considering the two phases of heterogeneous service time as Erlangian-2, W_Q increases as N increases & W_Q decreases as retrial rate 'v' increases. Figure 2(b) depicts the effect of W_Q with retrial rate 'v' and fixed N=4 , considering the two phases of heterogeneous service time as Exponential, Erlangian-2 and Hyper–Exponential.

Table 1. Mean orbit size.

v	N=2			N=3			N=4			N=5		
	Exp	Erl-2	HyperExp	Exp	Erl-2	HyperExp	Exp	Erl-2	HyperExp	Exp	Erl-2	HyperExp
1	0.564241	0.64581	0.565572	0.626819	0.711375	0.628486	0.655305	0.74121	0.657124	0.669802	0.756389	0.671698
2	0.514635	0.590299	0.515316	0.576152	0.654703	0.577159	0.604277	0.684142	0.605431	0.618626	0.69916	0.619856
3	0.498099	0.571795	0.498564	0.559264	0.635813	0.56005	0.587268	0.66512	0.5882	0.601567	0.680083	0.602575
4	0.489831	0.562543	0.490188	0.550819	0.626368	0.551495	0.578763	0.655608	0.579585	0.593038	0.670545	0.593935
5	0.484871	0.556992	0.485162	0.545752	0.620701	0.546362	0.57366	0.649902	0.574416	0.587921	0.664822	0.588751
6	0.481564	0.553291	0.481812	0.542375	0.616922	0.54294	0.570258	0.646097	0.570969	0.584509	0.661007	0.585294
7	0.479202	0.550648	0.479418	0.539962	0.614224	0.540496	0.567828	0.64338	0.568508	0.582072	0.658282	0.582826
8	0.47743	0.548665	0.477624	0.538152	0.6122	0.538663	0.566006	0.641341	0.566662	0.580244	0.656238	0.580974
9	0.476052	0.547123	0.476228	0.536745	0.610626	0.537237	0.564588	0.639756	0.565226	0.578823	0.654648	0.579534
10	0.47495	0.54589	0.475111	0.535619	0.609366	0.536097	0.563454	0.638488	0.564077	0.577685	0.653377	0.578382

(FES and SES follow exponential, Erlangian-2 & Hyper Exponential $d(x)=ae^{-vx}+v^2e^{-v^2x}(1-c)$ with service rate of FES and SES $s_1 = 20$ & $s_2 = 10$),

Table 2. Mean waiting time.

v	N=2			N=3			N=4			N=5		
	Exp	Erl-2	HyperExp	Exp	Erl-2	HyperExp	Exp	Erl-2	HyperExp	Exp	Erl-2	HyperExp
1	2.821206	3.229052	2.827859	3.134094	3.556875	3.142429	3.276527	3.706049	3.285621	3.34901	3.781944	3.358488
2	2.573173	2.951494	2.576579	2.880762	3.273517	2.885793	3.021385	3.420711	3.027156	3.09313	3.495799	3.099278
3	2.490496	2.858975	2.492818	2.796318	3.179065	2.800248	2.936338	3.325598	2.941002	3.007837	3.400417	3.012875
4	2.449157	2.812715	2.450938	2.754096	3.131839	2.757475	2.893814	3.278042	2.897924	2.965191	3.352727	2.969674
5	2.424354	2.784959	2.42581	2.728762	3.103503	2.731811	2.8683	3.249508	2.872078	2.939603	3.324112	2.943753
6	2.407819	2.766455	2.409058	2.711874	3.084612	2.714702	2.85129	3.230485	2.854847	2.922544	3.305036	2.926472
7	2.396008	2.753238	2.397092	2.69981	3.071119	2.702481	2.839141	3.216898	2.842539	2.91036	3.29141	2.914129
8	2.387149	2.743326	2.388118	2.690762	3.060999	2.693316	2.830028	3.206707	2.833308	2.901221	3.28119	2.904872
9	2.38026	2.735616	2.381138	2.683725	3.053128	2.686187	2.822941	3.198781	2.826128	2.894113	3.273242	2.897671
10	2.374748	2.729448	2.375554	2.678096	3.046831	2.680484	2.817271	3.19244	2.820385	2.888427	3.266883	2.891911

(FES and SES follow exponential, Erlangian -2 & Hyper Exponential $d(x)=ae^{-vx}+v^2e^{-v^2x}(1-c)$ with service rate of FES and SES $s_1 = 20$ & $s_2 = 10$),

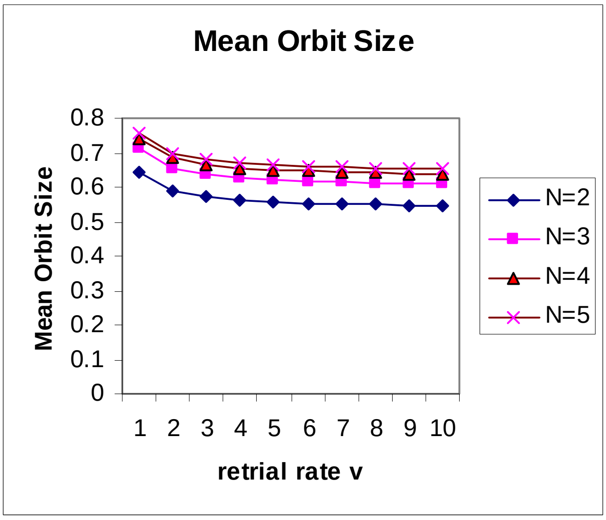

Figure 1(a).

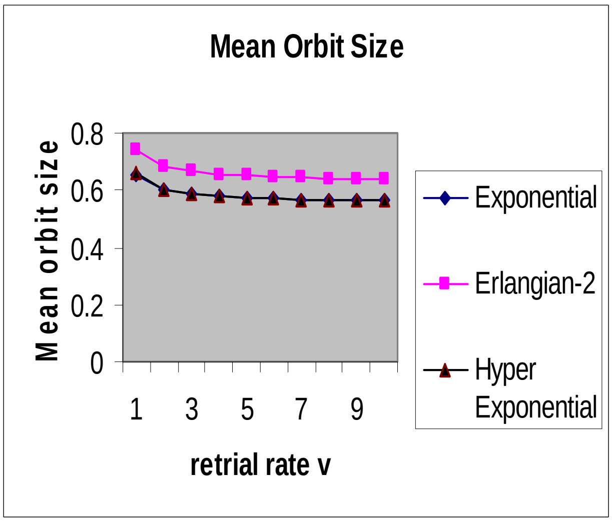

Figure 1(b).

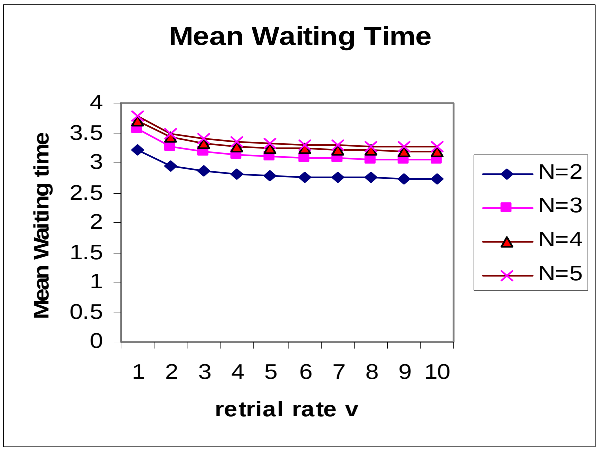

Figure 2(a).

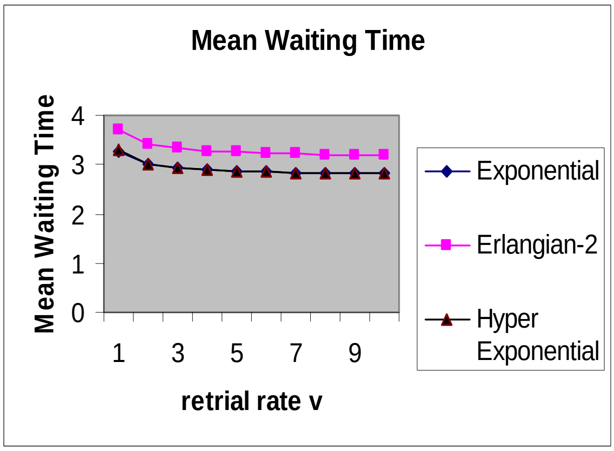

Figure 2(b).

7. CONCLUSION

The steady state analysis of a single server retrial queue with batch arrivals, two phases of heterogeneous service and multiple vacations with N- policy is obtained. The PGF of the number of customers in orbit and mean orbit size L_Q are obtained. Also we discussed special cases of PGF of orbit size. Further, numerical illustrations are also presented for the potential application.

8. ACKNOWLEDGEMENTS

The authors are very grateful to the anonymous referees for their valuable comments and suggestions to improve the paper in the present form.

REFERENCES

1. Artalejo, J.R. (1999a). Accessible bibliography on retrial queues. *Mathematical and Computer Modeling*, 30, 1-6.
2. Artalejo, J.R. (1999b). A classified bibliography of research on retrial queues: progress in 1990–1999, *Top* 7, 187–211.
3. Artalejo, J.R. (2004). Steady state analysis of an M/G/1 queue with repeated attempts and two phase service. *Quality Technology and Quantitative Management*, Vol.1, No. 2, 189–199.
4. Arumaganathan, R. and Jeyakumar, S. (2005). Steady state analysis of a bulk queue with multiple vacations, setup times with N-policy and closedown times. *Applied Mathematical Modeling* Vol. 29,10, 972–986.
5. Behrouz, A. Forouzan and Sophia Chung Fegan (2003). *TCP/IP Protocol Suite*. McGraw Hill, New York.
6. Choi,B.D. and Park, K.K.(1990), The M/G/1 retrial queue with Bernoulli schedule, *Queueing Systems* 7, 219–227.
7. Choi,B.D., Choi, K.B. and Lee, Y.W. (1995), M/G/1 retrial queueing system with two types of calls and finite capacity, *Queueing Systems*, 19,215–229.
8. Choi, B.D. and Chang, Y. (1999), Single server retrial queues with priority calls. *Mathematical and Computer Modeling*. 30, 7–32.
9. Choudhury, G. (2003). Some aspects of an M/ G /1 queueing system with optional second service. Top, 11, 141–150.
10. Falin, G.I. (1976). Aggregate arrival of customers in one line queue with repeated calls. *Ukrainian Math J.* 28, 437–440.

11. Falin, G.I. and Templeton, J.G.C. (1997) *Retrial Queues*. Chapman and Hall, London.
12. Jau-Chuan Ke and Fu-Min Chang (2008) $M^{[x]}$ /(G_1 , G_2)/1 retrial queue under Bernoulli vacation schedules, *Applied Mathematical Modeling, (*Article In press)
13. Krishnakumar, B., Vijayakumar, A. and Arivudainambi, D. (2002a). An M/G/1 retrial queueing system with two–phase service and preemptive resume. *Annals of Operations Research*, 113, 61–79.
14. Krishnakumar, B., Pavai Madheswari, S. and Vijayakumar, A. (2002b). The M/G/1 retrial queue with feedback and starting failures, *Applied Mathematical Modeling*. Vol. 26, 11, 1057–1076.
15. Krishna Reddy, G.V., Nadarajan, R., and Arumuganathan, R. (1998). Analysis of Bulk Queue with N-Policy Multiple Vacations and Setup Times, *Computers and Operations Research*. Vol. 25, No. 11. pp. 957-967.
16. Lee, H.W., Lee, S.S., Park, J.O. and Chae, K.C. (1994), Analysis of the M^x/G/1 queue with N- policy and multiple vacations *Journal of Applied Probability,* 31, 476-496.
17. Lopez–Herrero. M.J. (2002). Distribution of number of customers served in an M/G/1 retrial queue. *Journal of Applied Probability*, 39, 2, 407–413.
18. Madan, K.C. (2000). An M/G/1 queue with second optional service. *Queueing systems*, 34, 37 – 46.
19. Madan, K.C. and Gautam Choudhury (2005). A single Server Queue with Two phases of Heterogeneous Service under Bernoulli Schedule and a General Vacation time. *International Journal of Information and Management Sciences,* Vol.16, No.2, 1-16.
20. Medhi, J. (2002). A single server Poisson input queue with a second optional channel. *Queueing systems*, 42, 239 – 242.
21. Senthil Kumar, M. and Arumuganathan. R. (2008). On the single server Batch Arrival Retrial Queue with General vacation Time under Bernoulli schedule and two phases of Heterogeneous service, *Quality Technology and Quantitative Management,* Vol.5, 2, 145–160.