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On Multiple Objective Optimization Involving Generalized Convexity

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***Abstract*—** The aim of the present work is to characterize weakly efficient solution of multiobjective programming problems under the assumptions of α-invexity, using the concepts of critical point and Kuhn-Tucker stationary point for multiobjective programming problems. In this paper, we also extend the above results to the nondifferentiable multiobjective programming problems. The use of α-invex functions weakens the convexity requirements and increases the domain of applicability of the multiobjective programming in physical sciences and economics.

***Keywords*—** α -invexity, KT- α -invex problems, Kuhn-Tucker optimality conditions.

1. INTRODUCTION

The field of multiple-objective optimization, also known as multiobjective programming has grown remarkably in different directions in the setting of optimality conditions and duality theory since necessary and sufficient optimality conditions for generalized minimax programming introduced. It has been enriched by the applications of various types of generalizations of convexity theory, with and without differentiability assumptions, and in the framework of continuous time programming, fractional programming, inverse vector optimization, saddle point theory, symmetric duality, variational problems and control problems. Parallel to the above development in multipleobjective optimization, there has been a very popular growth and application of invexity theory which was originated by Hanson [3]. The invex functions defined by Hanson [3] allow the use of Kuhn-Tucker conditions for optimality in constrained optimization problems. In 1983, Clarke [2] introduced the concept of subdifferential functions. Jeyakumar [4] discussed a class of nonsmooth nonconvex problems in which functions are locally Lipschitz and are satisfying some invex type conditions (see [9] also). Later Martin [5] proved that invexity hypotheses are not only sufficient but also necessary when using the Kuhn-Tucker optimality conditions for unconstrained scalar programming problems.

Osuna et al [12] generalized the results of Martin [5] making them applicable to vectorial optimization problems. Sach et al [15] considered a generalized Kuhn-Tucker point of a vector optimization problem involving locally Lipschitz functions, weakly efficient solutions of the problem and KT-pseudoinvexity of the problem, and shown that the generalized Kuhn-Tucker point of the problem is a weakly efficient solution if and only if the problem is KT-pseudoinvex. In 2004 Noor [10] introduced the concept of α-invex functions, which is a more general class to invex functions. Mishra and Noor [6] and Noor and Noor [11] introduced some classes of α-invex functions by relaxing the definition of an invex function. Mishra, Pant and Rautela [7] introduced the concept of strict pseudo α-invex and quasi α-invex functions (see [8] and [13] also).

In the present work we characterize weakly efficient solution of multiple criterion nonlinear programming problems under the assumptions of α-invexity, using the concepts of critical point and Kuhn-Tucker stationary point for multiobjective programming problems. Subsequently we extend these results to nonsmooth multiobjective optimization problems to increases the domain of applicability in physical sciences and economics.

2. α -INVEXITY AND MULTIOBJECTIVE PROGRAMMING

In the present section, we study the problem of finding generalized classes of functions which ensure that the vector Kuhn-Tucker conditions are sufficient and necessary for weak efficiency in differentiable vector optimization problems.

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Let S be a nonempty subset of R^n, $\eta : S \times S \to R^n$ be an n-dimensional vector valued function and $\alpha : S \times S \to R_+ \setminus \{0\}$ be a bifunction. First of all, we recall some known results and concepts.

Definition 2.1 (Noor [10]). A subset S is said to be an α -invex set, if there exist $\eta : S \times S \to R^n$ and $\alpha : S \times S \to R_+ \setminus \{0\}$ such that

$$u + \lambda\alpha(x,u)\eta(x,u) \in S, \ \ \forall\ x,u \in S,\ \lambda \in [0,1].$$

From now onward we assume that the set S is a nonempty α -invex set with respect to $\alpha(x,u)$ and $\eta(x,u)$, unless otherwise specified.

In general, the vectorial optimization problem is represented as follows

(VOP) Minimize $f(x) = [f_1(x), f_2(x), ..., f_p(x)]$,

subject to $x \in S$, an α -invex set.

Often, the feasible set can be represented by functional inequalities:

(CVOP) Minimize $f(x) = [f_1(x), f_2(x), ..., f_p(x)]$,

subject to $g(x) \leqq 0$, $x \in S$, an α -invex set,

where $f : R^n \to R^p$ and $g : R^n \to R^m$ are differentiable function on the α -invex set $S \subseteq R^n$.

In order to relax the convexity assumptions we impose the following definitions.

Let $f : S \subseteq R^n \to R^p$ be a differentiable vector function on the α -invex set S. To make things easier, we introduce the matrix $\nabla f(\bar{x})$ of dimensions $p \times n$, whose are gradient vectors of each component function valued at the point x .

Definition 2.2 (Noor [10]). The function $f : S \subseteq R^n \to R^p$ on the α -invex set is said to be an α -preinvex function, if there exist $\eta : S \times S \to R^n$ and $\alpha : S \times S \to R_+ \setminus \{0\}$ such that

$$f(u + \lambda\alpha(x,u)\eta(x,u)) \le (1-\lambda) f(u) + \lambda f(x), \ \ \forall\ x,u \in S,\ \lambda \in [0,1].$$

Definition 2.3 (Mishra and Noor [6]). Let $f : S \subseteq R^n \to R^p$ be a differentiable function on the α-invex set S, then f is said to be α-invex on S if, for all $x_1, x_2 \in S$, there exist α and η, such that

$$f(x_1) - f(x_2) \geqq \left\langle \alpha(x_1,x_2)\nabla f(x_2),\ \eta(x_1,x_2) \right\rangle. \tag{1}$$

Definition 2.4 (Mishra and Noor [6]). Let $f : S \subseteq R^n \to R^p$ be a differentiable function on the α -invex set S, then f is said to be pseudo α -invex on S if, for all $x_1, x_2 \in S$, there exist α and η, such that

$$f(x_1) - f(x_2) < 0 \Rightarrow \left\langle \alpha(x_1,x_2)\nabla f(x_2),\ \eta(x_1,x_2) \right\rangle < 0. \tag{2}$$

Relationships between the above notions of α -invexity are given by

Proposition 2.1.

(i) For any problem (VOP) and any point $x_2 \in S$,

α -invexity on S at $x_2 \Rightarrow$ pseudo α -invexity on S at x_2.

(ii) For any problem (VOP) with $p = 1$ and any point $x_2 \in S$,

α -invexity on S at $x_2 \Leftrightarrow$ pseudo α -invexity on S at x_2.

Proof. The first part of Proposition 2.1 is obvious from the definitions. To prove the second one it is enough to show that for the case $p = 1$ the following implication is true:

pseudo α -invexity on S at $x_2 \Rightarrow \alpha$ -invexity on S at x_2.

Indeed, let $x_1 \in S$. If $f(x_1) \geqq f(x_2)$ then Eq.(1) is satisfied, with $\eta(x_1,x_2) = 0$. If $f(x_1) < f(x_2)$ then by assumption there is a point $\eta(x_1,x_2)$ satisfying Eq.(2). Since $\kappa\left\langle \alpha(x_1,x_2)\nabla f(x_2),\ \eta(x_1,x_2) \right\rangle \to -\infty$ as $\kappa \to +\infty$, we can take $\kappa > 0$ such that

$$f(x_1) - f(x_2) \geqq \kappa\left\langle \alpha(x_1,x_2)\nabla f(x_2),\ \eta(x_1,x_2) \right\rangle = \left\langle \alpha(x_1,x_2)\nabla f(x_2),\ \kappa\eta(x_1,x_2) \right\rangle.$$

Therefore, Eq.(1) is satisfied, with $\kappa\eta(x_1,x_2)$ instead of $\eta(x_1,x_2)$. Implication is thus established.

In multiobjective optimization problems, multiple objectives are usually non commensurable and can not be combined into a single objective. Moreover, often the objectives conflict with each other. Consequently, the concept of optimality for single objective optimization problems can not be applied directly to (VOP). The concept of Pareto optimality, characterizing an efficient solution, has been introduced for (VOP). Mathematically, a slightly different notion of Pareto optimality is defined as:

Definition 2.5. A feasible point x is said to be an efficient point or efficient solution if and only if there does not exist another $x \in S$ such that $f(x) \le f(\bar{x})$.

Definition 2.6. A feasible point x is said to be a weakly efficient point or weakly efficient solution (WEP) if and only if there does not exist another $x \in S$ such that $f(x) < f(\bar{x})$.

It is clear that every efficient solution is also a weakly efficient solution. Ruiz and Rufian [14] characterized the weakly efficient solutions of a multiobjective programming problem in which the functions are not differentiable.

In this paper, we give new characterizations of these solutions for constrained problems as well as the unconstrained problems, based on the Kuhn-Tucker optimality conditions applied to vectorial programming problems under the α -invex assumptions. For this purpose, we use the concept of critical point and Kuhn-Tucker stationary point, given in Osuna et al [12].

Definition 2.7. A feasible point $\bar{x} \in S$ is said to be a vector critical point (VCP) for problem (VOP) if there exists a vector $\lambda \in R^p$, with $\lambda \geq 0$, , such that $\lambda^T \nabla f(\bar{x}) = 0$.

Definition 2.8. A feasible point $\bar{x} \in S$ is said to be a vector Kuhn-Tucker point (VKTP) for problem (CVOP) if there exists a vector $(\bar{\lambda}, \bar{\mu}) \in R^{p+m}$, with $(\bar{\lambda}, \bar{\mu}) \geq 0$ and $\bar{\lambda} \neq 0$, such that

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0, \tag{3a}$$

$$\bar{\mu}^T g(\bar{x}) = 0. \tag{3b}$$

Craven [1] established the following theorem for problem (VOP).

Theorem 2.1. Let $\bar{x}$ be a weakly efficient solution for (VOP). Then, there exists $\bar{\lambda} \geq 0$, such that $\bar{\lambda}^T \nabla f(\bar{x}) = 0$.

Thus, every weakly efficient solution is a vector critical point. Now, we prove the converse of Theorem 2.1 using the concept of pseudo α -invexity.

Lemma 2.1. Let $\bar{x}$ be a vector critical point for (VOP) and let f be a pseudo α-invex function at $\bar{x}$ with respect to α and η. Then, $\bar{x}$ is a weakly efficient solution.

Proof. Let $\bar{x}$ be a vector critical point; i.e., there exists $\lambda \geq 0$ such that

$$\lambda^T \nabla f(\bar{x}) = 0.$$

If there exists another $x \in S$, such that

$$f(x) < f(\bar{x})$$

$$\text{i.e. } f(x) - f(\bar{x}) < 0.$$

By the pseudo α-invexity of f , the above inequality gives

$$\langle \alpha(x,\bar{x}) \nabla f(\bar{x}),\ \eta(x,\bar{x}) \rangle < 0.$$

By the positivity of $\alpha(x,\bar{x})$ the above inequality reduces to

$$\langle \nabla f(\bar{x}),\ \eta(x,\bar{x}) \rangle < 0.$$

But then, the system

$$\lambda^T \nabla f(\bar{x}) = 0,$$

$$\lambda \geq 0,\ \lambda \in R^p,$$

has no solution for λ . This completes the proof.

Thus, for multiobjective programming problems, weakly efficient points are those for which (and only those for which) the gradient vectors of the component functions of the objective functions, valued at that point are linearly independent.

Now we prove an even stronger result, which is true if and only if the objective function is pseudo α -invex.

Theorem 2.2. A vectorial function f is pseudo α -invex on S if and only if every vector critical point of f is a weakly efficient solution on S .

Proof. The sufficient condition has been proved already in Lemma 2.1. We must just prove that, if every vector critical point is a weakly efficient point, then the vectorial function f fulfills the pseudo α -invexity condition. Let $\bar{x}$ be a weakly efficient point. Then, the system

$$f_i(x) - f_i(\bar{x}) < 0,\quad i = 1, 2, ..., p, \tag{4}$$

has no solution in $x \in S$.

On the other hand, if $\bar{x}$ is a vector critical point, then there exists λ such that $\lambda^T \nabla f(\bar{x}) = 0$. Applying the Gordan theorem, the system below has no solution at $u \in R^n$,

$$\nabla f_i(\bar{x})^T u < 0, \quad i = 1, 2, ..., p. \tag{5}$$

So the system Eq.(4) and Eq.(5) are equivalent. If there exists $x \in S$ solution of Eq.(4), i.e.

$$f(x) - f(\bar{x}) < 0,$$

then there exists $\eta(x, \bar{x}) \in R^n$ solution of Eq.(5); therefore

$$\langle \nabla f(\bar{x}), \ \eta(x, \bar{x}) \rangle < 0.$$

Since $\alpha(x, \bar{x})$ is positive, the above inequality gives

$$\langle \alpha(x, \bar{x}) \nabla f(\bar{x}), \ \eta(x, \bar{x}) \rangle < 0.$$

This is precisely the pseudo α -invexity condition for f. This completes the proof.

Now, let us assume that weakly efficient solution and vector Kuhn-Tucker points for a constrained multiobjective programming problem are equivalent even under α -invex assumptions. So we define KT- α -invexity, which is a weaker condition than (f and g) α -invexity.

Definition 2.9. Problem (CVOP) is said to be KT- α -invex on the feasible set if there exists $\alpha : S \times S \to R_+$ and $\eta : S \times S \to R^n$ such that, $\forall \ x_1, x_2 \in S$, with $g(x_1) \leqq 0$ and $g(x_2) \leqq 0$

$$f(x_1) - f(x_2) \geqq \langle \alpha(x_1, x_2) \nabla f(x_2), \ \eta(x_1, x_2) \rangle,$$

$$-g_j(x_2) \geqq \langle \alpha(x_1, x_2) \nabla g_j(x_2), \ \eta(x_1, x_2) \rangle, \ \forall \ j \in I(x_2),$$

where $I(x_2) = \{ j : j = 1, 2, ..., m \text{ such that } g_j(x_2) = 0 \}$.

Definition 2.10. Problem (CVOP) is said to be KT-pseudo α -invex on the feasible set if there exists $\alpha : S \times S \to R_+$ and $\eta : S \times S \to R^n$ such that, $\forall \ x_1, x_2 \in S$, with $g(x_1) \leqq 0$ and $g(x_2) \leqq 0$

$$f(x_1) - f(x_2) < 0 \Rightarrow \langle \alpha(x_1, x_2) \nabla f(x_2), \ \eta(x_1, x_2) \rangle < 0, \tag{6a}$$

$$-\langle \alpha(x_1, x_2) \nabla g_j(x_2), \ \eta(x_1, x_2) \rangle \geq 0, \ \forall \ j \in I(x_2), \tag{6b}$$

where $I(x_2) = \{ j : j = 1, 2, ..., m \text{ such that } g_j(x_2) = 0 \}$.

Relationships between the above notions of α -invexity are given by the following proposition.(The proof of the following proposition is similar to the proof of Proposition 2.1, so we sate the result but omit the proof)

Proposition 2.2.

(i) For any problem (CVOP) and any point $x_2 \in S$,

KT- α -invexity on S at $x_2 \Rightarrow$ KT-pseudo α -invexity on S at x_2.

(ii) For any problem (CVOP) with $p = 1$ and any point $x_2 \in S$,

KT- α -invexity on S at $x_2 \Leftrightarrow$ KT-pseudo α -invexity on S at x_2.

Theorem 2.3. Every vector Kuhn-Tucker point is a weakly efficient solution if and only if problem (CVOP) is KT-pseudo α -invex.

Proof. Let $\bar{x}$ be a vector Kuhn-Tucker point for (CVOP), and let us assume that problem (CVOP) is KT-pseudo α -invex. Since $\bar{x}$ was assumed to be a (VKTP), we have

$$\langle \nabla f(\bar{x}), \ \eta(x, \bar{x}) \rangle + \sum_{j \in I(\bar{x})} \bar{\mu}_j \langle \nabla g_j(\bar{x}), \ \eta(x, \bar{x}) \rangle = 0. \tag{7}$$

We see that $\bar{x}$ is a weakly efficient solution for (CVOP). If there exists a feasible point x such that

$$f(x) < f(\bar{x})$$

i.e. $f(x) - f(\bar{x}) < 0.$

By Eq.(6a), there exist α and $\eta(x, \bar{x}) \in R^n$ such that

$$\langle \alpha(x, \bar{x}) \nabla f(\bar{x}), \ \eta(x, \bar{x}) \rangle < 0.$$

By the positivity of α the above inequality gives

$$\left\langle \nabla f(\bar{x}),\ \eta(x,\bar{x}) \right\rangle < 0. \tag{8}$$

It follows from Eq.(7) and Eq.(8) that

$$\sum_{j \in I(\bar{x})} \bar{\mu}_j \left\langle \nabla g_j(\bar{x}),\ \eta(x,\bar{x}) \right\rangle > 0.$$

Again by the positivity of α, we get

$$\sum_{j \in I(\bar{x})} \bar{\mu}_j \left\langle \alpha(x,\bar{x}) \nabla g_j(\bar{x}),\ \eta(x,\bar{x}) \right\rangle > 0. \tag{9}$$

Since, problem (CVOP) is KT-pseudo α-invex. Then by Eq.(6b),

$$-\left\langle \alpha(x,\bar{x}) \nabla g_j(\bar{x}),\ \eta(x,\bar{x}) \right\rangle \geq 0, \quad \forall\, j \in I(\bar{x}),$$

and with $\bar{\mu}_j \geq 0$, ,

$$-\bar{\mu}_j \left\langle \alpha(x,\bar{x}) \nabla g_j(\bar{x}),\ \eta(x,\bar{x}) \right\rangle \geq 0, \quad \forall\, j \in I(\bar{x}),$$

which is equivalent to

$$\bar{\mu}_j \left\langle \alpha(x,\bar{x}) \nabla g_j(\bar{x}),\ \eta(x,\bar{x}) \right\rangle < 0, \quad \forall\, j \in I(\bar{x}).$$

This contradicts Eq.(9).

Let us prove the converse. We suppose that every vector Kuhn-Tucker point is a weakly efficient solution. If $\bar{x}$ is a vector Kuhn-Tucker point, the following system does not have any solution:

$$\nabla f_i(\bar{x})^T u < 0, \quad i = 1, 2, \ldots, p, \tag{10a}$$

$$\nabla g_j(\bar{x})^T u < 0, \quad j \in I(\bar{x}). \tag{10b}$$

If $\bar{x}$ is a weakly efficient solution, the system

$$f_i(x) - f_i(\bar{x}) < 0, \quad i = 1, 2, \ldots, p, \tag{11a}$$

$$g(x) \leqq 0 \tag{11b}$$

does not have any solution.

Then, Eq.(10) and Eq.(11) are equivalent at $\bar{x}$. So problem (CVOP) is KT-pseudo α-invex on the feasible set. This completes the proof.

3. α-INVEXITY AND NONSMOOTH OPTIMIZATION

In this section we further generalize our results by assuming that the functions need not be differentiable. The recent growth of nonsmooth analysis has generated the interest in the field of α-invex functions and their applications. Clarke [1] introduced generalized directional derivative and generalized subdifferentials for locally Lipschitz functions. Therefore it was natural to extend these results to α-invex functions. We begin our analysis with main concepts and definitions in this area.

Definition 3.1. A function $f: S \subseteq R^n \to R$ is said to be Lipschitz near $x \in S$ if for some $K > 0$,

$$\left| f(y) - f(z) \right| \leq K \| y - z \|, \ \forall\, y, z \text{ within a neighborhood of } x.$$

We say that $f: S \subseteq R^n \to R$ is locally Lipschitz on S if it is Lipschitz near any point of S.

Definition 3.2. If $f: S \subseteq R^n \to R$ is Lipschitz at $x \in S$, the generalized derivative (in the sense of Clarke) of f at $x \in S$ in the direction $v \in R^n$, denoted by $f^0(x; v)$, is given by

$$f^0(x; v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

We shall say that a locally Lipschitz function at x is Clarke-differentiable at x, with directional derivative given by $f^0(x;v)$. By the Lipschitz condition it follows that $\left|f^0(x;v)\right| \leqq K\|v\|$, so $f^0(x;v)$ is a well-defined finite quantity. Moreover, $f^0(x;v)$ is a sublinear function in the direction v and for any $v \in R^n$

$$f^0(x;v) = \max\left\{\xi^T v : \xi \in \partial f(x)\right\},$$

where $\partial f(x)$ is a convex and compact set of R^n, called the Clarke subdifferential of f at $x \in S$ or Clarke generalized gradient of f at x. This is given by

$$\partial f(x) = \left\{\xi \in R^n : f^0(x;v) \geqq \xi^T v \quad \text{for all } v \in R^n\right\}.$$

The fundamental results concerning $\partial f(x)$ are given below:

(a) If f is continuously differentiable at $x \in S$, then $\partial f(x) = \{\nabla f(x)\}$.

(b) $\partial_C(-f)(x) = -\partial_C f(x)$; if $g : S \to R$ is locally Lipschitz at $x \in S$, then $\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x)$.

(c) Let D_f be the set of points in S at which f is not differentiable (By Rademacher's theorem D_f has Lebesgue measure zero) and let X be any other set of measure zero in R^n. Then $\partial f(x) = conv\left\{\lim_{k\to\infty} \nabla f(x^k) : x^k \to x,\ x^k \notin X \bigcup D_k\right\}$;

That is, $\partial f(x)$ is the convex hull of all points of the form $\lim \nabla f(x^k)$, where $\{x^k\}$ is any sequence which converges to $x \notin X \bigcup D_f$. The term *conv* stands for convex hull.

(d) For D_f and X as in (c), $f^0(x, v) = \limsup_{y\to x^0}\left\{\left(\nabla f(y)\right)^T v : y \notin X \bigcup D_f\right\}$.

The following theorem (Clark [4]) assuming easy convergence of property (b), provides a necessary condition for a local minimum of f.

Theorem 3.1. Let $f : S \to R$ be nondifferentiable on the open set $S \subseteq R^n$ and let $x \in S$ be a point of local minimum of f over S; then $0 \in \partial f(x)$.

To make things easier, we introduce the notion of α -invexity and pseudo-α -invexity for the locally Lipschitz function f.

Definition 3.3. Let f be locally Lipschitz on the α -invex set $S \subseteq R^n$; then f is said to be α -invex on S if, for all $\forall\ x_1, x_2 \in S$, there exist α and η, $\forall\ i = 1,2,..., p$, such that

$$f_i(x_1) - f_i(x_2) \geqq \left\langle \alpha(x_1,x_2)\xi_i,\ \eta(x_1,x_2)\right\rangle,\ \forall\ \xi_i \in \partial f_i(x_2). \tag{12}$$

Definition 3.4. A locally Lipschitz function f on the α -invex set $S \subseteq R^n$ is said to be pseudo-α -invex on S if, for all $\forall\ x_1, x_2 \in S$, there exist α and η, $\forall\ i = 1,2,..., p$, such

$$f_i(x_1) - f_i(x_2) < 0 \Rightarrow \left\langle \alpha(x_1,x_2)\xi_i,\ \eta(x_1,x_2)\right\rangle < 0,\ \forall\ \xi_i \in \partial f_i(x_2). \tag{13}$$

Let S be a nonempty subset of R^n, $\eta : S \times S \to R^n$ be an n-dimensional vector valued function, $\alpha : S \times S \to R^n \setminus \{0\}$ be a bifunction. First, we recall some known results and concepts.

In general, the nonsmooth vector optimization problem is represented as follows

(NVOP) Minimize $f(x) = [f_1(x), f_2(x), ..., f_p(x)]$,

subject to $x \in S \subseteq R^n$,

where $f_i : S \subseteq R^n \to R,\ i = 1,2,..., p$ is a locally Lipschitz functions on the α -invex set $S \subseteq R^n$.

Often, the feasible set can be represented by functional inequalities:

(CNVOP) Minimize $f(x) = [f_1(x), f_2(x), ..., f_p(x)]$,

subject to $g_j(x) \leqq 0, j = 1,2,..., m$,

$x \in S \subseteq R^n$,

where $f_i : S \subseteq R^n \to R,\ i = 1,2,..., p$ and $g_j : R^n \to R, j = 1,2,..., m$ are locally Lipschitz functions on the α -invex set $S \subseteq R^n$.

Now, we define the concept of critical point and Kuhn-Tucker stationary point for nonsmooth multiobjective programming along the lines of Osuna et al [12].

Definition 3.5. A feasible point $\bar{x} \in S$ is said to be a vector critical point for problem (NVOP) if there exists a vector $\lambda \in R^p$, with $\lambda \in R^p$, such that $0 \in \lambda^T \partial f(\bar{x})$.

Definition 3.6. A feasible point x $\bar{x} \in S$ is said to be a vector Kuhn-Tucker point (VKTP) for problem (CNVOP) if there exists a vector $(\bar{\lambda}, \bar{\mu}) \in R^{p+m}$, with $(\bar{\lambda}, \bar{\mu}) \geq 0$ and $\bar{\lambda} \neq 0,$, such that

$$0 \in \bar{\lambda}\partial f(\bar{x}) + \bar{\mu}\partial g(\bar{x}), \tag{14a}$$

$$\bar{\mu}^T g(\bar{x}) = 0. \tag{14b}$$

The following Craven [1] type of result will be needed in the sequel of the paper.

Lemma 3.1. Let $\bar{x}$ be a weakly efficient solution for (NVOP). Then, there exists $\bar{\lambda} \geq 0$, such that $0 \in \bar{\lambda}^{\,T} \partial f(\bar{x})$.

Thus, every weakly efficient solution is a vector critical point. Now, we prove the converse of Lemma 3.1 using the concept of pseudo-α-invexity for nonsmooth functions.

Theorem 3.2. Let $\bar{x}$ be a vector critical point for (NVOP) and let f be a pseudo-α-invex function at $\bar{x}$ with respect to α and η. Then, $\bar{x}$ is a weakly efficient solution.

Proof. Let $\bar{x}$ be a vector critical point; i.e., there exists $\lambda \geq 0$ such that

$$0 \in \lambda^T \nabla f(\bar{x}) = 0.$$

If there exists another $x \in S$, such that

$$f_i(x) < f_i(\bar{x}),\ \forall\ i = 1,2,\ldots,p$$

$$\text{i.e. } f_i(x) - f_i(\bar{x}) < 0,\ \forall\ i = 1,2,\ldots,p.$$

By the pseudo α-invexity of f, the above inequality gives

$$\langle \alpha(x,\bar{x})\ \xi_i,\ \eta(x,\bar{x}) \rangle < 0,\ \forall\ \xi_i \in \partial f_i(\bar{x}).$$

By the positivity of $\alpha(x,\bar{x})$ the above inequality reduces to

$$\langle\ \xi_i,\ \eta(x,\bar{x}) \rangle < 0,\ \forall\ \xi_i \in \partial f_i(\bar{x}).$$

But then, the system

$$0 \in \lambda^T \partial f(\bar{x}),$$

$$\lambda \geq 0,\ \lambda \in R^p,$$

has no solution for λ. This completes the proof.

Thus, for nonsmooth multiobjective programming problems, weakly efficient points are those for which (and only those for which) the generalized derivatives (in the sense of Clarke [2]) of the component functions of the objective functions, valued at that point are linearly independent.

Now we prove an even stronger result, which is true if and only if the objective function is pseudo α-invex.

Theorem 3.3. A locally Lipschitz function f is pseudo α-invex on S if and only if every vector critical point of f is a weakly efficient solution on S.

Proof. The sufficient condition has been proved already in Theorem 3.2. We must just prove that, if every vector critical point is a weakly efficient point, then the locally Lipschitz function f fulfills the pseudo-α-invexity condition. Let $\bar{x}$ be a weakly efficient point. Then, the system

$$f_i(x) - f_i(\bar{x}) < 0,\quad i = 1,2,\ldots,p, \tag{15}$$

has no solution in $x \in S$.

On the other hand, if $\bar{x}$ is a vector critical point, then there exists λ such that $0 \in \lambda^T \partial f(\bar{x})$. Applying the Gordan theorem, the system below has no solution at $u \in R^n$,

$$\langle \xi_i, u \rangle < 0,\ \forall\ \xi_i \in \partial f_i(\bar{x}),\quad i = 1,2,\ldots,p. \tag{16}$$

So the system Eq.(15) and Eq.(16) are equivalent. If there exists $x \in S$ solution of Eq.(15), i.e.

$$f_i(x) - f_i(\bar{x}) < 0,$$

then there exists $\eta(x,\bar{x}) \in R^n$ solution of Eq.(16); therefore

$$\langle \xi_i,\ \eta(x,\bar{x}) \rangle < 0,\ \forall\ \xi_i \in \partial f_i(\bar{x}).$$

Since $\alpha > 0$, the above inequality gives

$$\langle \alpha(x,\bar{x})\xi_i,\ \eta(x,\bar{x})\rangle < 0,\ \forall\ \xi_i \in \partial f_i(\bar{x}).$$

This is precisely the pseudo α -invexity condition for f. This completes the proof.

Now, let us assume that weakly efficient solution and vector Kuhn-Tucker points for a constrained nonsmooth multiobjective programming problem are equivalent even under α -invex assumptions. So we define KT-pseudo- α -invexity, which is a weaker condition then (f and g) α -invexity.

Definition 3.7. Problem (CNVOP) is said to be KT-pseudo α -invex on the feasible set if there exists α and η such that, $\forall\ x_1, x_2 \in S$, with $g(x_1) \leqq 0$ and $g(x_2) \leqq 0$

$$f_i(x_1) - f_i(x_2) < 0 \Rightarrow \langle \alpha(x_1,x_2)\xi_i,\ \eta(x_1,x_2)\rangle < 0,\ \forall\ \xi_i \in \partial f_i(x_2), \tag{17a}$$

$$-\langle \alpha(x_1,x_2)\zeta_j,\ \eta(x_1,x_2)\rangle \geq 0,\ \forall\ \zeta_j \in \partial g_j(x_2),\ j \in I(x_2), \tag{17b}$$

where $I(x_2) = \{ j : j = 1,2,...,m \text{ such that } g_j(x_2) = 0 \}$.

Theorem 3.4. Every vector Kuhn-Tucker point is a weakly efficient solution if and only if problem (CNVOP) is KT-pseudo α -invex.

Proof. Let $\bar{x}$ be a vector Kuhn-Tucker point for (CNVOP), and let us assume that problem (CNVOP) is KT-pseudo α -invex. Since $\bar{x}$ was assumed to be a (VKTP), we have

$$\langle \xi_i,\ \eta(x,\bar{x})\rangle + \sum_{j\in I(\bar{x})} \bar{\mu}_j \langle \zeta_j,\ \eta(x,\bar{x})\rangle = 0,\ \forall\ \xi_i \in \partial f(\bar{x}),\ \zeta_j \in \partial g_j(\bar{x}). \tag{18}$$

We see that $\bar{x}$ is a weakly efficient solution for (CNVOP). If there exists a feasible point x such that

$$f_i(x) < f_i(\bar{x}),\ \forall\ i = 1,2,..., p$$

$$\text{i.e. } f_i(x) - f_i(\bar{x}) < 0,\ \forall\ i = 1,2,..., p.$$

By Eq.(17a), there exist $\alpha(x,\bar{x})$ and $\eta(x,\bar{x}) \in R^n$ such that

$$\langle \alpha(x,\bar{x})\xi_i,\ \eta(x,\bar{x})\rangle < 0,\ \forall\ \xi_i \in \partial f_i(\bar{x}).$$

Since $\alpha(x,\bar{x}) \geq 0$, we get

$$\langle \xi_i,\ \eta(x,\bar{x})\rangle < 0,\ \forall\ \xi_i \in \partial f(\bar{x}). \tag{19}$$

It follows from Eq.(18) and Eq.(19) that

$$\sum_{j\in I(\bar{x})} \bar{\mu}_j \langle \zeta_j,\ \eta(x,\bar{x})\rangle > 0\ \forall\ \zeta_j \in \partial g(\bar{x}). \tag{20}$$

Since, problem (CNVOP) is KT-pseudo α -invex. Then by Eq.(17b),

$$-\langle \alpha(x,\bar{x})\zeta_j,\ \eta(x,\bar{x})\rangle \geq 0,\ \forall\ \zeta_j \in \partial g_j(\bar{x}),\ j \in I(\bar{x}),$$

and with $\bar{\mu}_j \geq 0$, ,

$$-\bar{\mu}_j \langle \alpha(x,\bar{x})\zeta_j,\ \eta(x,\bar{x})\rangle \geq 0,\ \forall\ \zeta_j \in \partial g_j(\bar{x}),\ j \in I(\bar{x}),$$

which is equivalent to

$$\sum_{j\in I(\bar{x})} \bar{\mu}_j \langle \zeta_j,\ \eta(x,\bar{x})\rangle \leq 0,\ \forall\ \zeta_j \in \partial g_j(\bar{x}),\ j \in I(\bar{x}),\ \text{(by the positivity of } \alpha(x,\bar{x})\text{)}.$$

This contradicts Eq.(20).

Let us prove the converse. We suppose that every vector Kuhn-Tucker point is a weakly efficient solution. If $\bar{x}$ is a vector Kuhn-Tucker point, the following system does not have any solution:

$$\langle \xi_i, u\rangle < 0,\ \forall \xi_i \in \partial f_i(\bar{x}),\ i = 1,2,..., p, \tag{21a}$$

$$\langle \zeta_j, u\rangle < 0,\ \forall \zeta_j \in \partial g_j(\bar{x}),\ j \in I(\bar{x}). \tag{21b}$$

If $\bar{x}$ is a weakly efficient solution, the system

$$f_i(x) - f_i(\bar{x}) < 0, \quad i = 1, 2, ..., p, \tag{22a}$$

$$g(x) \leqq 0 \tag{22b}$$

does not have any solution.

Then, Eq.(21) and Eq.(22) are equivalent at $\bar{x}$. So problem (CNVOP) is KT-pseudo α -invex on the feasible set. This completes the proof.

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