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Multivariate Design for Mass Customization of Consumer Products

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Abstract—In this paper, we propose a model and a solution procedure to minimize the cost associated with mass customizing consumer products in the presence of multivariate constraints. Three algorithms based on enumeration, steepest descent and Lagrange relaxation are used to solve the non-linear optimization problem and their performance is evaluated with numerical experiments. The sufficient and necessary conditions under which the optimal solution can be achieved are presented as well.

Keywords—Mathematical model, Cost minimization, Nonlinear programming, Multivariate percentile, Anthropometric Design.

1. INTRODUCTION

Customer-centric enterprise is an important aspect for any business (Cox, 1998). As a result, manufacturers attempt to satisfy the different needs and preferences through a wide variety of products (Ishii, 1995). With the exponential growth of product variety today and the significant opportunities in e-commerce, the old paradigm of mass production has become extremely sluggish (Anderson, 2002), especially, with the change of business paradigm from producer-centered productivity to consumer-centered customization known today as mass customization (Anderson, 1997). Mass customization is an attempt to satisfy the varied individual customer needs with near mass production efficiency (Jiao, 1998). By breaking down the product features into components and offering those components to the consumer as choices, customization of the whole or part-product is possible (Kahn, 1998). However, the complexity of this process increases with the increasing number of variables. Consumer product design such as footwear and garments rely on attempting to satisfy one or two dimensions even though more than this number is necessary to get a customized fit (Choua, 2005, Kuo, 2005).

Historically, there has been a trend to increase product variety to cater to varying consumer tastes and styles (Cox, 1998, Lancaster, 1998). For example, from 1970 to 1988, the number of running shoe models increased from 5 to over 285 (167 men and 118 women) (Cox, 1998). In order to keep pace with ever changing customer tastes, thousands of new products are made annually and with each variation, manufacturers attempt to bring products closer to what the customer needs. Even though variety matters to consumers, each product "size" may have a differing performance function to different consumers (Lancaster, 1998). Finding the required item among a large assortment of sizes can be quite frustrating for any consumer. Hence, allowing a customer to choose one product from a "shelf" can be wasteful and can also constrain a customer's ultimate satisfaction even though a store shelf may have great marketing appeal (Du, 2000). The difficulty of selecting the right pair of shoes in a shoe store is a classic example. In this case, product variety can be a hindrance rather than a benefit. Hence, the need satisfaction process should be attained not purely through more variety, but through the manufacture of the "right" products. There are two primary reasons for considering only one or two sizing variables to satisfy the needs in consumer products. The first being the size of inventory that a retailer may have to carry and the second being the increased cost associated with satisfying the many variables in the production of the item. Getting the right product that consumer's need at the right cost is a necessary condition with the diminishing profit margins in a competitive business environment. This paper will focus on minimizing the cost with multivariate design parameters.

The general multivariate design problems can be quite "challenging" especially when considering anthropometry. One approximate method is to treat the multivariate design as a series of univariate problems, ignoring the correlation of those variables. Since the univariate problem is analytically manageable (Pheasant, 1996), such a method is easy to implement even

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though the actual percentage to which the product fits would be considerably smaller than expected, implying that such an approximation is too crude. A more accurate and alternative method is to generate smoothed percentile curves using a large sample of data (Chung, 2001, Ward, 2001). However, multivariate data collection can be rather expensive and thus the application of such a method is quite limited. Instead, the correlation coefficient and the summary statistics of the variables can be used to generate the percentile curves assuming that the multivariate variables are adequately represented as a multivariate normal distribution (Moroney, 1972). Polansky (2001) provides another method to generate computer-generated bivariate charts for anthropometric design with a further assumption that the target population dimensional requirements are met using identical percentile cutoffs and/or a complementary percentile cutoffs when setting design limits. Kreifeldt (1993) and Nah (1996) have taken the issues a step further by investigating the cost-benefit issues considering material, inventory, shipping and safety costs, and building them into an objective function while emphasizing the need to find a trade-off to minimize the total cost. However, they have not provided a suitable technique to solve the optimization problem in order to minimize the total cost. Even though mathematical models are common in many different applications (Choi, 2002, Grunwald, 1988, Weymann, 1995), they have had limited coverage in the anthropometric design arena (Karim, 2004). Hence this paper is an attempt to:

1. Formulate a mathematical model for the cost minimization of the multivariate anthropometric design problem considering the optimality conditions.
2. Provide an algorithm to obtain the optimal solution.
3. Devise an efficient heuristic which can produce a solution faster without significantly affecting the solution quality.
4. Provide an algorithm for the special case where the design variables are independent.
5. Validate the algorithms through numerical experiments.

Our methodology brings the researchers a new interpretation on the anthropometric design problem. A number of practical problems faced by the industrial participants can be directly formulated by our model and solved by our algorithms. Certainly, the researchers can also find a lot of extensions from the basic model. They deserve further investigation.

Those algorithms are evaluated by numerical experiments in various settings. It has been shown that they have pretty nice performance in very practical settings. Moreover, we design an approach to implement those algorithms for practical cases in which the distributions are not multivariate normal distributed but can be characterized by given raw data-base. We test that approach on a real case from (Paquette, 1997). Again, the numerical study shows our algorithms' superiority.

The rest of this paper is organized as follows. In section 2, we give the definition of "multivariate percentile" and the model. In section 3, we analyze this problem and provide necessary and sufficient optimality conditions. In section 4, we show the numerical approaches of generating the design solution. In section 5 numerical experiments are conducted. Section 6 concludes this paper.

2. MULTIVARIATE PERCENTILE DEFINITION

The definition of univariate percentile has been well recognized and widely used in the literature (Eastman Kodak Company, 1983). The p^{th} percentile of a set is a value such that p% data have a value less than that under consideration. If the data follows a normal distribution $N(\mu,\sigma)$, where μ is the mean and σ is the standard deviation, the 25th percentile is $\mu-0.3849\sigma$, the 50th percentile is μ, 90th percentile is $\mu+1.2817\sigma$, 95th percentile is $\mu+1.6452\sigma$, and in general, the p^{th} percentile is $\mu+\Phi^{-1}(p\%)\sigma$, where the $\Phi^{-1}(x)$ is the Cumulative Density Function(CDF) of standard normal distribution (Bagby, 1995).

While the definition of univariate percentile is unambiguous, the definition of multivariate percentile is still unclear. We have adopted the following definitions for a multivariate percentile.

Definition 1. Given n variables $X_1,X_2,\ldots,X_n$ which follow a joint distribution of $F(X_1,X_2,\ldots,X_n)$, we define the equation $\Pr(X_1\le x_1,X_2\le x_2,\ldots,X_n\le x_n)=\alpha\%$ as the **"α-Percentage Constraint"**. We call any $x_1,x_2,\ldots,x_n$ as **"design level"**. Specifically x_i is called as the design level of variable X_i.

Definition 2. Given the targeted percentage α, let us define the **percentile** of $X_1,X_2,\ldots,X_n$ as the set of design levels $\{(x_1,x_2,\ldots,x_n)\}$ which satisfy the corresponding percentage constraint.

For the univariate case we have only one variable X and hence the solution of $\Pr(X\le x)=\alpha\%$ is unique. Thus the "percentile" can be regarded as a point. However, in the multivariate case, the solution of $\Pr(X_1\le x_1,\ldots,X_n\le x_n)$ may not be unique because it is possible that different design levels have the same cumulative density probability. Therefore, the set $\{(x_1,x_2,\ldots,x_n):\Pr(X_1\le x_1,\ldots,X_n\le x_n)=\alpha\%\}$ actually forms a hypersurface. Figure 2 shows the CDF function of multivariate normal distribution of two random variables X_1 and X_2. Figure 2 shows the projected 70-percentile of these two variables.

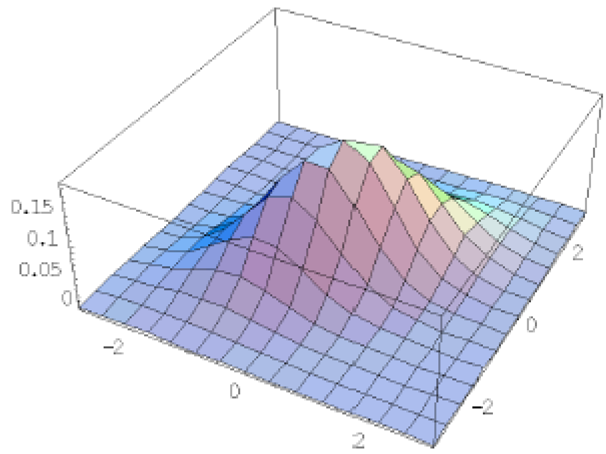

Figure 1. An example of multivariate normal distribution.

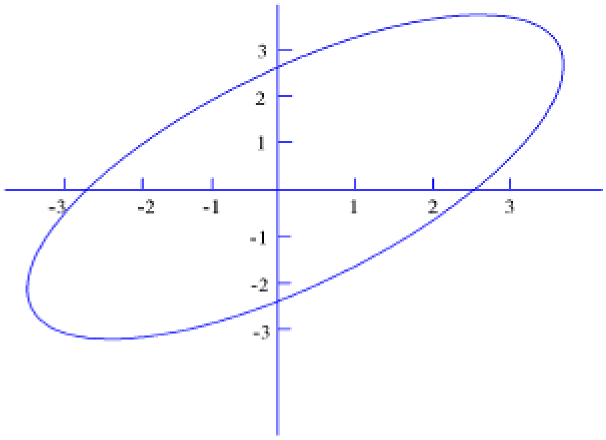

Figure 2. The 70-percentile of random variables characterized by Figure 2.

As Figure 2 shows, the percentile of a multivariate distribution corresponds to a set of points rather than a single point. If the problem is one of minimizing cost, then the optimal point has to chosen from the feasible set.

3. MATHEMATICAL MODELING

This design problem can be formulated as given below.

3.1 Problem formulation

Assume that the total design cost is a general separable function of the design level $z = [z_1, z_2, \ldots, z_n]$. The authors of (Kreifeldt, 1993) assume that the total cost is linear function of the design level, which is a special case of the cost function proposed in this paper. Let $C(z)$ be the total cost:

$$C(z) = c_1(z_1) + c_2(z_2) + \ldots + c_n(z_n) \quad (1)$$

In many cases, it is not the cost alone, but the quality of the product, or its performance, that form part of the objective. Therefore, the "cost" here is of a general sense, and to incorporate different measurements into one single objective, one can associate each measurement with a properly set weight.

To meet a targeted α -percentage population, the design level must satisfy the α^{th} percentage constraint. With both the objective function and the constraints, the problem can be formulated as follows:

Problem (P).

Minimize $C(z)=c_1(z_1)+c_2(z_2)+\ldots+c_n(z_n)$,

Subject to $F(z)=p$.

where $p=\alpha\%$ is a constant and $0<p<1$.

We assume that the decision variables, z, are continous. This assumption allows us to derive the optimality conditions and design more analytical approaches. In some applications, the decision variables can be integers. For instance, the color of a shoe. Those approaches might not be able to provide exact solutions for the integer cases, but they may provide a good starting point.

3.2 Optimality conditions

Problem (P) is a classical nonlinear optimization problem that has been studied extensively and can be solved with Lagrangian relaxation (Jornsten, 1986, Guignard, 1987, Rangarajan, 1996, Monte, 2001).

The necessary optimality condition can be stated as follows:

Necessary optimality condition

For the local minimum of problem (P), z^*, there exists a scalar λ, called Lagrange multiplier, such that

$$\frac{\partial C(z^*)}{\partial z_i}+\lambda\frac{\partial(F(z^*))}{\partial z_i}=0 \tag{2}$$

The above equation can be interpreted as the "gradient" of the Lagrange function:

$$L(z,\lambda)=\sum_{i=1}^{n}c_i z_i+\lambda(F(z)-p)$$

The above conditions are only the necessary conditions. In other words, even with a feasible z satisfying Eq.(2), there is no guarantee that z is the local minimum. Optimality of z^* requires $\nabla^2 L(z^*,\lambda^*)$ to be positive definite (Bertsekas, 2000).

Sufficient optimality condition

The feasible solution z^* is optimal if there exists a scalar λ such that

$$\frac{\partial C(z^*)}{\partial z_i}+\lambda^*\frac{\partial(F(z^*))}{\partial z_i}=0$$

and, $\nabla^2 L(z^*,\lambda^*)$ is positive definite.

Methods such as Lagrangian relaxation can be applied to solve this nonlinear optimization problem. However, it is difficult to directly use the optimality conditions with multivariate distributions as $F(z)$, the cumulative density function cannot be evaluated in closed form. Hence the following numerical approaches are proposed.

4. NUMERICAL APPROACHES

Three algorithms are proposed with the first being enumerative wherein all candidates in the solution space (the hyperplane) are evaluated and the one with minimum cost is selected. The second algorithm is based on the method of steepest descent (Kirk, 1970) and the third is based on the Lagrange relaxation method for the special case where the design variables are independent of each other. A proof is also provided to show that the solution is exact with the Lagrange relaxation method.

4.1 Approximation of multi-variant normal probabilities

Our approaches rely on the efficiency of evaluating multivariant normal distribution functions. In general,

4.2 Enumerative search algorithm

A simplified approach to solve a multivariate design problem is to ignore the correlations among the different design variables and simply solve a series of univariate problems. This method is commonly adopted in anthropometric design. Even though such an approach is simple and used in practice, the actual percentage fitted by such a design is considerably smaller than the target values. Such issues can be overcome with a modified univariate approach as shown below.

To meet a given percentile, p, the designer can choose a percentile from a range for each design variable as follows

$$p_i\in\left[(1-\beta^L)p,(1+\beta^U)p\right], i=1,2,\ldots,n$$

where β^L and β^U are two adjusting parameters between (0, 1).

Let $z_i(p_i)$ denote the design level for design variable i given the targeted percentile p_i. Thus, for each combination $(p_1, p_2, \ldots, p_n)$, we solve n univariate problems, obtaining the solutions $z = (z_1, z_2, \ldots, z_n)$. We can then evaluate the corresponding multivariate percentile, namely $F(z)$, which might be less than the targeted level. After enumerating all possible combinations, the feasible solution with the minimal cost can be identified. A feasible combination may not exist in the given region if β^L and (or) β^U are too small. To achieve the optimal solution, one may have to set β^L and β^U relatively large, thereby increasing the number of possible combinations. This approach might not work well for applications with many design variables. However, this method will be suitable for applications with a relatively smaller number of design variables. This algorithm can be stated as follows:

Algorithm 1: Enumerative approach

Step 1: *Initialization*

Initialize parameters: step size Δ, maximum steps K, and solution qualify tolerance ε, β^L, β^U. Set $p_i^0 = (1-\beta^L)p$ and compute $z_i(p_i^0)$ for all $i \in \{0,1,2,\ldots,n\}$. (Note that $K\Delta = (\beta^L + \beta^U)p$.)

Step 2: *Solve univariate problems*

Solve the n univariate problems for $(p_1, p_2, \ldots, p_n) = (p_1^0 + k_1\Delta, p_2^0 + k_2\Delta, \ldots, p_n^0 + k_n\Delta)$ for all integers $k_1, k_2, \ldots, k_n$, where $k_i \in \{0,1,2,\ldots,K\}$.

Step 3: *Evaluate* $F(z)$ *and* $C(z)$

Compute $C(z)$ and $F(z)$ for all $(z_1, z_2, \ldots, z_n) = (z_1(p_1^0 + k_1\Delta), z_2(p_2^0 + k_2\Delta)), \ldots, z_n(p_n^0 + k_n\Delta)$

Step 4: *Search the minimum*

Find z^* such that $C(z^*)$ is the minimum and such that $|F(z^*) - p| < \varepsilon$.

To obtain the "optimal solution", the step size should be small so that the method approximates an exhaustive method, thereby examining almost all possible contenders to achieve the target percentile with minimum cost within a known tolerance.

4.3 An efficient heuristic to solve Problem (P)

The steepest descent method can be used to solve the above problem as well (Bertsekas, 2000, Kirk, 1970). An initial solution $z = [z_1, z_2, \ldots, z_n]$ which may violate the percentage constraint is first used. Thereafter, the design level of one variable is adjusted along the direction of steepest descent in cost. If the approximate coordinate direction is:

$$d_i = \frac{F(z_1, \ldots, z_i, \ldots, z_n) - F(z_1, \ldots, z_i + \Delta, \ldots, z_n)}{c_i(z_i + \Delta) - c_i(z_i)} \tag{3}$$

In each iteration, we can move along the direction which is $\hat{d} = \max\{d_i : i = 1,2,\ldots,n\}$.

The key steps of the heuristic are summarized below.

Algorithm 2: Steepest descent Search

Step 1: *Initialization*

Input ε and step size Δ. Let $k = 0$. Find a starting point $z^k = [z_1^k, z_2^k, \ldots, z_n^k]$. One good choice is $z_i^k = u_i$.

Step 2: *Solve univariate problems*

Set $k = k+1$. Evaluate d_i for $i = 1,2,\ldots,n$. Choose the largest value of d_i. Let the variable is m, update z_m^k with $z_m^k = z_m^{k-1} + \Delta$.

Step 3: *Evaluate* $F(z^k)$ *and* $C(z^k)$

Update z^k, evaluate $F(z^k)$. If $|F(z^k) - p| < \varepsilon$. Stop. Otherwise, go to step 2.

The efficiency of this heuristic will be examined with numerical experiments later.

4.4 The special case when design variables are independent.

In the special case where the decision variables are independent of each other and the cost structure is linear, the evaluation of $F(z)$ is relatively easier.

$$F(z) = \Pr(Z_1 \le z_1, \ldots, Z_n \le z_n) = \prod_{i=1}^{n} \Pr(Z_i \le z_i) \tag{4}$$

If $F_i(z_i)$ denotes the function $\Pr(Z_i \le z_i)$, then for the special case we have the following properties.

Property 1 *If a design problem requires its percentile is greater than 50%, i.e.,* $p > 0.5$ *, then each variable must be greater than its mean value.*

Proof. We prove it by contradiction. Suppose that there exists a k for a design level z^* such that $z_k^* < \bar{z}_k$. By Eq.(4), we have

$$F(z^*) = \prod_{i=1}^{n} \Pr(Z_i \le z_i^*) = \Pr(Z_k \le z_k^*) \prod_{i=1, i\ne k}^{n} \Pr(Z_i \le z_i^*) < \Pr(Z_k \le \bar{z}_k) \times 1 < 0.5 .$$

On the other hand, the design requires that $F(z^*) > 0.5$. There exists a contradiction. Therefore, each variable must be greater than its mean value. QED.

Let us recall the sufficient optimality condition. The first order condition implies that

$$\frac{\partial(F_i(z^*))/\partial z_i}{F(z^*)c_i} = -\lambda^* p .$$

Note that the probability density function (PDF) for each variable is a monotonically decreasing function in the region $[\mu, +\infty)$ while $F(z)$ is monotonically increasing. We can conclude that $\frac{\partial(F_i(z_i))/\partial z_i}{F(z)c_i}$ is monotonically decreasing.

Since

$$\frac{\partial L(z, L)}{\partial z_i} = c_i + \lambda p \frac{\partial F_i(z_i)/\partial z_i}{F_i(z_i)}$$

and $\lambda < 0$, it follows that

Property 2 $\frac{\partial L(z,\lambda)}{\partial z_i}$ *is monotonically increasing with* z_i.

Property 3 *For any solution* z^*, λ^* *satisfying the first order condition,* $\nabla^2 L(z^*, \lambda^*)$ *is positive definite.*

According to property 3, $\frac{\partial^2 L(z^*, \lambda^*)}{\partial^2 z_i} \ge 0$, for any give i. Because for any $i \ne j$, Z is independent with Z_j, $\frac{\partial L(z^*, \lambda^*)}{\partial z_i \partial z_j} = 0$. So the matrix $\nabla^2 L(z^*, \lambda^*)$ is a dialogue matrix where the elements in the dialogue are positive, which implies $\nabla^2 L(z^*, \lambda^*)$ is positive definite.

With the optimality conditions and Property 3, it can be seen that the first order optimality condition is sufficient to guarantee optimality. Hence the optimal solution can be found by searching the optimal multiplier, λ^* for which the first-order condition holds. The algorithm can be described as follows:

Algorithm 3: Lagrange relaxation method for the special case

Step 1: *Initialization*

Set λ_0 and $\Delta\lambda$.

Step 2: *Solve the equations*

Set $n = n + 1$.

Let $\lambda = \lambda_0$. Solve the equations $\frac{\partial(F_i(z_i))/\partial z_i}{F(z^*)c_i} = -\lambda p$ for all $i = 1, 2, \ldots, n$.

Step 3: *Evaluate the percentile*

Evaluate $F(z)$. If $|F(z) - p| < \varepsilon$. Stop. Otherwise, go to step 4.

Step 4: *Test for termination*

a) If $F(z) > p + \varepsilon$, reduce $\lambda = \lambda - \Delta\lambda$.

b) If $F(z) < p - \varepsilon$, increase $\lambda = \lambda + \Delta\lambda$, Goto step 2.

The critical aspect of Algorithm 3 is the equations in step 2, which is given in the appendix. The starting value for the lagrange multiplier is set according to the following rule:

$$\lambda_0 = \frac{\Pr(Z_1 = \phi^{-1}(p))}{c_1 p}$$

5. NUMERICAL EXPERIMENTS

The efficiency of the algorithms is checked using a series of numerical experiments with the univariate approach (Algorithm 4) as the benchmark for evaluation. A high dimensional problem is then presented to validate the performance of the algorithms and thereafter an application of the suitability of the algorithms is provided for the case of bi-modal distributions.

5.1 Benchmark approaches

The simple univariate approach (Algorithm 4) is used as the benchmark for evaluation.

Algorithm 4: Univariate based algorithm

Step 1: *Initialization*

Calculate $p_i = \sqrt[n]{p}$, for all $i \in \{1,2,\ldots,n\}$.

Step 2: *Evaluate univariate percentile*

Obtain $z_i(p_i)$, that is, the z_i , such that $F_i(z_i) = p_i$, for all $i \in \{0,1,2,\ldots,n\}$.

Step 3: *Evaluate multivariate percentile*

Evaluate $F(z)$.

The above Algorithm is based on the assumption that the variables are independent and hence the multivariate percentile is the product of all percentiles among all dimensions. Therefore, $p_i = \sqrt[n]{p}$ is supposed to provide a good estimate of the percentile in each variate.

Moreover, Algorithm 1 is also used for comparison as it is expected to generate a nearoptimal solution. When the step size is small, the solution approaches the optimal one but, the computational time can be prohibitively large as the step size approaches 0.

5.2 A Numerical example

An example similar to the one in (Kreifeldt, 1993) is used to illustrate the efficiency of the algorithms, where a cover guard frame should be designed to prevent children's fingers from contacting the rotating fan blades. The design variables are the grill diameter (D) and the distance (L) to the blade. The objective is to choose a pair of (l,d) such that a specified percentile (e.g., 95%) of the child population is protected while the total cost of manufacture is minimized. The total cost consists of the engineering cost and a human factor cost. Thus,

Minimize $C_T(l,d) = C_1 l + C_2 d$

Subject to $\Pr(L \le l, D \le d) = p$

where the mean and covariance matrix of $X = (D,L)^T$ are

$$\mu = \begin{bmatrix} 0.8 \\ 25 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.01 & \rho 0.6 \\ \rho 0.6 & 36 \end{bmatrix}.$$

Note that standard deviations of D and L are 0.1 and 6, respectively, and the correlation coefficient between D and L is ρ .

In this paper, we examine the three scenarios of positive, negative and no correlations. The ρ 's in these three cases are $\frac{5}{6}$, 0, and $-\frac{5}{6}$, respectively. Let $C_1 = 10$ and $C_2 = 200$.

The results of Algorithm 4 (Univariate), Algorithm 1 (Enumerative search) and Algorithm 2 (Steepest Descent) are compared in terms of solution quality and computational time. In addition, for the case of independent variables where the correlation coefficient is 0, Algorithm 3 (Lagrange relaxation method) is presented as well. All the experiments were run on a PC with 2.5G CPU and 1G RAM. Algorithms 1, 2 and 4 were implemented in Mathematica 4 and Algorithm 3 was implemented in Matlab 7.0. The settings in the algorithms are as follows.

- **Algorithm 1:** $p_i^0 = 0.8$, $\forall i$, $\Delta = 0.001$, and $K = 180$.
- **Algorithm 2:** $\Delta = 0.001$, and $\varepsilon = 0.001$.
- **Algorithm 3:** $\varepsilon = 0.001$, and $\Delta\lambda = 0.00001$.

Tables 1 to 3 present the results of Algorithm 4, Algorithm 1 and Algorithm 2 in a positively correlated, non-correlated, and negatively correlated scenarios, respectively. In each table, the first column is the objective multivariate percentile, which ranges from 0.85 to 0.95. The second to tenth columns are separated into three blocks, corresponding to the results obtained when using Algorithm 4, Algorithm 3 and Algorithm 2, respectively. In each block, the first column is the percentile achieved, the second column is the objective cost of the output solution and the third column gives the computational time in seconds. The cost savings of Algorithm 2 over Algorithm 4 is given as a percentage in the last column of each table.

For the negatively correlated scenario, the computational time is much longer than the other two scenarios. This is due to the percentile evaluation process in Mathematica where the computation time significantly increases as the correlation approaches 1. As a result, the solution of Algorithm 1 (A1) cannot be obtained in 8 hours, implying the computational prohibitiveness of the enumerative search method for the negatively correlated situation. It can be seen that the computation time of Algorithm 4 is the shortest since it only evaluates one point, while Algorithm 1 (A1) takes the longest time as it enumerates many possible points while algorithm A2 takes sometime in-between the two.

The results show that Algorithm 2 produces a mean cost saving of 5.97%, 5.96%, and 5.96% over Algorithm 4 in the positively correlated, non-correlated, negatively correlated scenarios, respectively. A closer look tells us that in the positively correlated scenario, Algorithm 4 results in a percentile that is over the target values, while in the negatively correlated scenario, Algorithm 4 produces under-qualified solutions whose corresponding percentiles are smaller than the target values implying that the percentage constraint is violated. Surprisingly, Algorithm 2 produces solutions that are slightly better, but similar to Algorithm 1, which is expected to generate a near optimal solution. Thus, Algorithm 2 appears to generate near optimal solutions as well. As stated before, Algorithm 1 produces an optimal solution when the step size approaches 0 given an unlimited amount of time.

Table 1. Multivariate percentile solutions in positively correlated scenario.

TP	A4			A1			A2			Savings of A2
	P	Cost	Time	P	Cost	Time	P	Cost	Time	over A4
0.85	0.8889	6711	0.03	0.8500	6259	812.66	0.8500	6255	144.44	6.81%
0.86	0.8961	6757	0.02	0.8600	6312	811.81	0.8601	6308	149.89	6.65%
0.87	0.9033	6805	0.02	0.8701	6368	813.19	0.8700	6363	156.97	6.50%
0.88	0.9105	6856	0.02	0.8801	6426	811.31	0.8801	6422	163.42	6.34%
0.89	0.9178	6910	0.02	0.8900	6488	811.53	0.8902	6484	169.98	6.17%
0.90	0.9250	6968	0.03	0.9001	6555	813.00	0.9001	6550	177.97	6.00%
0.91	0.9323	7031	0.03	0.9101	6627	813.09	0.9101	6621	186.83	5.83%
0.92	0.9396	7099	0.02	0.9201	6705	810.45	0.9201	6698	196.63	5.65%
0.93	0.9469	7174	0.02	0.9302	6799	809.39	0.9301	6783	206.56	5.45%
0.94	0.9542	7259	0.03	0.9400	6896	812.11	0.9400	6877	217.70	5.26%
0.95	0.9616	7355	0.02	0.9502	7008	811.05	0.9500	6985	230.74	5.03%

Table 2. Multivariate percentile solutions in non-correlated scenario.

TP	A4			A1			A2			Savings of A2
	P	Cost	Time	P	Cost	Time	P	Cost	Time	over A4
0.85	0.8500	6711	0.00*	0.8504	6301	111.59	0.8500	6256	16.88	6.79%
0.86	0.8600	6757	0.02	0.8603	6356	111.61	0.8601	6309	17.50	6.63%
0.87	0.8700	6805	0.00	0.8702	6414	111.63	0.8700	6364	18.14	6.49%
0.88	0.8800	6856	0.02	0.8801	6476	111.74	0.8801	6423	18.83	6.32%
0.89	0.8900	6910	0.00	0.8900	6541	111.66	0.8901	6485	19.64	6.16%
0.90	0.9000	6968	0.00	0.9009	6619	112.03	0.9001	6551	20.39	5.99%
0.91	0.9100	7031	0.02	0.9108	6696	112.13	0.9101	6622	21.38	5.82%
0.92	0.9200	7099	0.00	0.9207	6781	112.08	0.9201	6699	22.50	5.64%
0.93	0.9300	7174	0.00	0.9306	6876	112.09	0.9301	6784	23.63	5.44%
0.94	0.9400	7259	0.00	0.9405	6984	111.74	0.9400	6878	24.91	5.24%
0.95	0.9500	7355	0.00	0.9504	7111	111.67	0.9501	6987	26.22	5.01%

*: "0.00" means the computational time is less than 0.01 second.

The non-correlated scenario was used to understand the performance of Algorithm 3 (Table 4). It is evident that a mean cost saving of 5.40% exists with Algorithm 3 when compared with Algorithm 4. Even though the cost is slightly higher than those output by Algorithm 2, in most case Algorithm 3 takes shorter time than Algorithm 2. The computational merit is more significant with more variables and when they are independent.

In summary, Algorithm 2 produces near optimal solutions with a cost savings of, on average, more than 5% over Algorithm 4. Moreover, when compared with Algorithm 1, Algorithm 2 produces similar results but takes much shorter computational time. Hence, considering both effectiveness and efficiency, Algorithm 2 is quite promising. When the variables are independent of each other, Algorithm 3 takes less time having a solution quality similar to Algorithm 2.

Table 3. Multivariate percentile solutions in negatively correlated scenario.

TP	A4			A1			A2			Savings of A2
	P	Cost	Time	P	Cost	Time	P	Cost	Time	over A4
0.85	0.8439	6711	2.48	-*	-	-	0.8500	6256	12783.50	6.79%
0.86	0.8547	6757	2.53	-	-	-	0.8601	6309	13277.70	6.63%
0.87	0.8655	6805	1.27	-	-	-	0.8700	6364	13792.60	6.49%
0.88	0.8762	6856	2.52	-	-	-	0.8801	6423	14282.90	6.32%
0.89	0.8868	6910	1.27	-	-	-	0.8901	6485	14830.30	6.16%
0.90	0.8974	6968	2.53	-	-	-	0.9001	6551	15263.80	5.99%
0.91	0.9079	7031	2.55	-	-	-	0.9101	6622	15920.30	5.82%
0.92	0.9183	7099	1.28	-	-	-	0.9201	6699	16640.80	5.64%
0.93	0.9287	7174	1.30	-	-	-	0.9301	6784	17437.90	5.44%
0.94	0.9391	7259	2.63	-	-	-	0.9400	6878	18319.50	5.24%
0.95	0.9494	7355	2.63	-	-	-	0.9501	6987	19241.70	5.01%

*: "-" implies that solutions can not be obtained in 8 hours.

Table 4. Solutions from Algorithm 3.

TP	A4			A3			Savings of A3
	P	Cost	Time	P	Cost	Time	over A4
0.85	0.8500	6711	0.00	0.8494	6343	20.12	5.50%
0.86	0.8600	6757	0.02	0.8601	6398	15.23	5.32%
0.87	0.8700	6805	0.00	0.8700	6454	15.34	5.16%
0.88	0.8800	6856	0.02	0.8801	6510	10.12	5.05%
0.89	0.8900	6910	0.00	0.8902	6562	4.34	5.05%
0.90	0.9000	6968	0.00	0.9001	6607	2.33	5.18%
0.91	0.9100	7031	0.02	0.9100	6649	2.67	5.43%
0.92	0.9200	7099	0.00	0.9201	6679	2.89	5.91%
0.93	0.9300	7174	0.00	0.9301	6706	2.23	6.53%
0.94	0.9400	7259	0.00	0.9400	6906	10.19	4.85%
0.95	0.9500	7355	0.00	0.9500	6954	11.35	5.46%

5.3 Random generated cases

We further create 10 random generated cases for a comprehensive comparison between these algorithms. All cases are of three dimentions and of multi-normal distributions. More specifically, each case is generated by

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}, \quad Q = \begin{bmatrix} \sigma_1^2 & \sigma_1 \cdot \sigma_2 \cdot \rho_1 \cdot \rho_2 & \sigma_1 \cdot \sigma_3 \cdot \rho_1 \cdot \rho_3 \\ \sigma_1 \cdot \sigma_2 \cdot \rho_1 \cdot \rho_2 & \sigma_2^2 & \sigma_2 \cdot \sigma_3 \cdot \rho_2 \cdot \rho_1 \\ \sigma_1 \cdot \sigma_3 \cdot \rho_1 \cdot \rho_3 & \sigma_2 \cdot \sigma_3 \cdot \rho_3 \cdot \rho_2 & \sigma_3^2 \end{bmatrix}.$$

The cost vector is

$$c = (c_1, c_2, c_3)^T,$$

where $(.)^T$ denotes the transpose.

The parameters are randomly generated in this way:

1. Let $c_3 = 1$, and the ratio $\log(c_1/c_2)$, $\log(c_2/c_3)$ is uniformly distributed in [−1, 1].
2. Let $\mu_3 = 1$, and the ratio $\log(\mu_1/\mu_2)$, $\log(\mu_2/\mu_3)$ is uniformly distributed in [−1, 1].
3. Let $\sigma_1 = \mu_1/10$, $\sigma_2 = \mu_2/10$ and $\sigma_3 = \mu_3/10$.
4. The ratio ρ_1, ρ_2 and ρ_3 are uniformly distributed in [−0.9, 0.9].

Table 5 illustrates the parameters of those random cases, and the results can be found in Table 6.

Table 5. Test Problems Characteristics.

Problem	c_1/c_2	c_2/c_3	μ_1/μ_2	μ_2/μ_3	ρ_1	ρ_2	ρ_3
P1	8.01	2.86	9.54	6.68	-0.89	0.92	-0.84
P2	0.26	4.69	9.83	0.57	0.37	-0.32	-0.47
P3	5.15	0.82	7.70	0.14	0.98	0.23	-0.78
P4	0.44	0.54	2.77	3.13	-0.56	0.70	-0.07
P5	0.24	4.26	5.82	2.80	-0.24	0.26	-0.22
P6	1.18	1.20	0.43	0.25	0.47	-0.40	-0.72
P7	1.96	0.90	3.27	4.13	0.61	-0.82	-0.02
P8	2.56	1.00	0.99	1.50	-0.33	0.84	-0.38
P9	0.16	0.18	0.50	0.21	-0.65	-0.61	0.69
P10	6.98	4.05	8.57	3.64	-0.33	-0.84	0.95

Table 6. Computational Results for Random Cases.

Problem	A4			A1			A2			Savings of A2
	P	Cost	Time	P	Cost	Time	P	Cost	Time	over A1
P1	0.5955	475.1	0.03	0.8500	-*	-*	0.8500	477.2	125.0	-*
P3	0.6173	9.2	0.02	0.8504	9.9	15049.8	0.8505	9.6	47.0	2.81%
P4	0.8500	21.6	0.02	0.8503	21.2	11524.7	0.8503	22.6	32.0	9.83%
P5	0.6000	3.6	0.02	0.8504	3.8	9776.2	0.8505	3.8	27.0	0.00%
P6	0.6141	18.7	0.02	0.8502	20.6	6901.1	0.8502	19.4	194	5.97%
P6	0.6075	1.9	0.02	0.8500	2.0	3648.8	0.8502	2.0	13.0	0.00%
P7	0.6044	15.7	0.03	0.8500	16.4	11528.3	0.8503	16.2	48.0	1.41%
P8	0.6337	6.3	0.03	0.8503	6.6	5761.5	0.8504	6.9	18.0	4.55%
P9	0.6331	1.2	0.02	0.8501	1.2	2382.4	0.8502	1.3	7.0	5.44%
P10	0.6448	357.5	0.02	0.8502	-*	-*	0.8502	343.2	72.1	-*

The results clearly show that the A1 cannot guarantee a feasible solution. Among 10 cases, there is only one feasible solution of all solutions generated by A1. Though A2 can guarantee a feasible solution, the computational times are extremely higher that A2 (There are two cases in which the solver cannot obtain satisfactory result in 8 hours). Meanwhile, in some cases, A4 can even get better solution than A2. In summary, A4 outperforms the other two algorithms.

5.4 Computational performance with high dimensions

To evaluate computational performance, a 5-variable example with the following distribution was used:

$$\mu = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 15 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.01 & 0.016 & 0.024 & 0.032 & 0.4 \\ 0.01 & 0.04 & 0.048 & 0.064 & 0.8 \\ 0.024 & 0.048 & 0.09 & 0.096 & 1.2 \\ 0.032 & 0.064 & 0.096 & 0.16 & 1.6 \\ 0.4 & 0.8 & 1.2 & 1.6 & 25 \end{bmatrix}.$$

The cost vector is

$$c = (10, 20, 30, 40, 150)^T,$$

where $(.)^T$ denotes the transpose.

When the targeted percentile is 0.9, the results of Algorithm 4 and Algorithm 2 are given in Table 7.

Table 7. Multivariate percentile solutions in 5-variate case.

TP	A4			A2			Savings of A2
	P	Cost	Time	P	Cost	Time	over A4
0.9	0.9458	4138	0.05	0.9001	3591	1093.19	13.22%

Algorithm 1 could not generate a solution within 8 hours. Table 5 shows that Algorithm 2 gives a cost saving of 13.22% over Algorithm 4. More importantly, Algorithm 2 in this case is usable with a computation time of less than 20 minutes whereas the time of Algorithm 1 is excessively prohibitive.

5.5 Experiments: using a database as an evaluation function

In some applications, it may be difficult to compute the distributions of the design variables because neither the exact functions nor the approximation functions may be available. In such cases, a database of a large sample set can be utilized to determine a solution. Suppose there are sufficient number of samples, given any $(x_1, x_2, \ldots, x_n)$, an approximation of $F(x_1, x_2, \ldots, x_n)$ can be obtained by probing the database.

Consider the situation of two design variables, "Heel Ankle Circumference" (HAC) and "Heel Breadth" (HB) each having a bi-modal distribution. The summary statistics data for the females and the males can be obtained from (Paquette, 1997), and are given in Table 8.

Table 8: Summary Statistics of the heel ankle circumference and heel breadth for US marine corps.

Male				Female			
HAC (mm)		HB (mm)		HAC (mm)		HB (mm)	
Mean	SD	Mean	SD	Mean	SD	Mean	SD
339.0	15.8	69.0	4.9	304.0	14.5	62.0	4.5

The correlation of the two variables for both male and female are assumed to be:

$$Q = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}.$$

If one product is designed to suit both males and females, one needs to consider a design problem where the distributions of HAC and HB are bi-modal. Suppose there N samples in the database, $F(x_1, x_2, \ldots, x_n)$ can be evaluated as given below

Evaluate $F(x_1, x_2, \ldots, x_n)$ with the database

Step 1: *Initialization*

Let the counter $i = 0$, $j = 0$.

Step 2: *Compare with one sample*

For the sample i, $(z_1, z_2, \ldots, z_n)$, examine whether $x_k > z_k$ for $k = 1, \ldots, n$. If yes, stop. Otherwise, $j = j + 1$ and repeat this step.

Step 3: *Evaluate the multivariate percentile*

$F(x_1, x_2, \ldots, x_n)$ can be computed as j/N.

In this way, the algorithm A2 can be applied. The cost vector is

$$c = (10,1)^T .$$

The result is shown in Table 9:

Even after using the approximation to the real distributions, the proposed algorithms provide a better solution than the benchmark.

Table 9. Results for a bi-modal example.

TP	A2			A4			Savings of A2
	P	Cost	Time	P	Cost	Time	over A4
0.9	0.9001	3612.0	0.1	0.9001	3624.5	0.0	0.3%

5.6 Implementation issues

Implementation of the proposed approach is rather straightforward if the multivariate distribution and the cost function is available. The algorithms are easy to code.

6. CONCLUSIONS AND DISCUSSIONS

This paper examined the cost minimization in the presence of many variables. Three algorithms were proposed to solve this problem with the first being an enumerative approach based on standard univariate methods. The strength of this algorithm is its simplicity of implementation and the guarantee of optimality. However, the time to attain a solution may be relatively high with many variables. The second algorithm is based on a standard search technique in nonlinear optimization theory. Numerical experiments showed that a solution can be achieved relatively fast without loss of solution quality. The third algorithm is solely for the case of independent design variables and is extremely efficient. The numerical experiments have shown the merits of the different algorithms and the value of using the approaches over the commonly used univariate method.

Several extensions of this problem deserve in-depth research. One extension is the cost minimization problem with constraints which represent the restrictions due to human factor concerns. For example, in the design of a workstation, the relationships among the design variables such as length, height etc, can be formulated as a group of linear constraints (Gupta, 2004). Using the mathematical models, state-of-art decision support tools that implement optimization techniques can be developed to support decision making.

Another extension is the design problem with more complex objective functions. In this paper, the objective is simply the "cost" minimization and the "cost" has a relatively simple structure. However, the cost structure could be quite complex (Zhang, 2007). On the other hand, besides costs, the social goals, such as the concern on the environment, are also important issues (Dula, 2004). It is interesting to model and analyze the impact of those issues with mathematical models.

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APPENDIX: THE METHOD OF SOLVING EQUATIONS IN STEP 2 IN ALGORITHM 2.

Without loss of generality, we assume that the distribution of Z_i is standard normal distribution. To solve the equations in step 2, we first obtain the relationships between z_i and z_1. Observe that,

$$\frac{\partial\left(F_i\left(z_i\right)\right)/\partial z_i}{F\left(z^*\right)c_i}=\frac{\partial\left(F_i\left(z_i\right)\right)/\partial z_1}{F\left(z^*\right)c_1}$$

We have

$$\frac{\partial\left(F_i\left(z_i\right)\right)/\partial z_i}{c_i}=\frac{\partial\left(F_i\left(z_1\right)\right)/\partial z_1}{c_1}$$

It follows that,

Shi, Xu, Goonetilleke, Fu and Song: *Multivariate Design for Mass Customization of Consumer Products*
IJOR Vol. 6, No. 1, 10−23 (2009)

$$\frac{\partial\left(F_i\left(z_i\right)\right)/\partial z_i}{\partial\left(F_i\left(z_1\right)\right)/\partial z_1}=\frac{c_i}{c_1}$$

Knowing that

$$\frac{\partial\left(F_i\left(z_i\right)\right)}{\partial z_i}=\frac{1}{\sqrt{2\pi}}e^{-z_i^2/2}$$

We can obtain the relationship between z_i and z_1 as,

$$-\frac{z_i^2}{2}+\frac{z_1^2}{2}=\ln\left(\frac{c_i}{c_1}\right) \quad \text{for all} \quad i\neq 1$$

We evaluate $F(z)$ using the approximation of $\Pr\left(Z_i\leq z_i\right)$ given in (Bagby, 1995):

Approximation 1.

$$\Pr\left(Z_i\leq z_i\right)\approx 1/2+1/2\sqrt{\left(1-\frac{1}{30}7e^{-z_i^2/2}+16e^{-z_i^2\left(2-\sqrt{2}\right)/2}+e^{-z_i^2/2}\left(7+0.25\pi\cdot z_i^2/2\right)\right)} \quad z_i\geq 0$$

Together with (5), we can get an approximation of $F(z)$ which is a function of z_1 only. Then, solving the equation (using a some standard solver),

$$\frac{\partial\left(F_1\left(z_1\right)\right)/\partial z_1}{F\left(z^*\right)c_i}=-\lambda p$$

We will get the value of z_1, then all the z_i as well as $F(z)$.