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Minimax Fractional Programming Involving Type I and Related Functions

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Abstract—The convexity assumptions for a minimax fractional programming problem of variational type are relaxed to those of a type I and related functions. Sufficient optimality conditions are established under the aforesaid assumptions. Several duality theorems are obtained for Wolfe type and Mond-Weir type duals and the original problem.

Keyword—Minimax programming; Fractional programming; Sufficient optimality conditions; Duality; Generalized convexity

1. INTRODUCTION

The problem considered in this paper consists of minimizing a maximum of several time-dependent ratio involving integral expressions. It is well known that fractional programming problems has benefited from advances in generalized convexity and vice-versa (see, Frenk and Schaible (2005) and references therein and Stancu-Minasian (1997)). Many authors have studied variational problems (see, Bector et al. (1992), Bector and Husain (1992), Craven (1993), Gregory and Lin (1996), Hanson (1964), Kim and Kim (2002), Kim and Lee (1998), Lai and Liu (2003), Liu (1994), Mishra (1996), Mishra and Mukherjee (1994a, 1994b, 1995), Mond and Hanson (1967), Mond et al. (1988), Mond and Husain (1989), Mond and Smart (1989), Mukherjee and Mishra (1994, 1995), Nahak and Nanda (1996, 1997), Valentine (1937), Ye and Zheng (1991) and Zalmai (1990)).

Consider a minimax problem with a fractional objective in the form:

$$\text{(P)} \qquad v^* = \min_{x} \max_{1\le i\le p} \frac{\int_a^b f_i\left(t,x(t),\dot{x}(t)\right)dt}{\int_a^b g_i\left(t,x(t),\dot{x}(t)\right)dt}$$

subject to $x \in PS\left(T,R^n\right),\ x(a)=\alpha,\ x(b)=\beta,$

$$\int_a^b h_j\left(t,x(t),\dot{x}(t)\right)dt \le 0, \quad j\in \underline{m} \equiv \{1,\dots,m\},\ t\in T=[a,b],$$

where the functions $f_i, g_i,\ i\in \underline{p}$ and $h_j,\ j\in\underline{m}$ are continuous in t, x and x and have continuous partial derivatives with respect to x and $\dot{x}$ and where $PS\left(T,R^n\right)$ is the space of all piecewise smooth state functions x defined on the compact time set T in R. The norm of $x\in PS\left(T,R^n\right)$ is defined by $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where D is the differential operator on $PS\left(T,R^n\right)$ defined by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_a^t u(s)\ ds,$$

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which α is a given boundary value. Therefore, $D = \frac{d}{dt}$ except at discontinuities.

Throughout the paper we assume that

$\int_a^b g_i\left(t, x(t), \dot{x}(t)\right)dt > 0$, $\int_a^b f_i\left(t, x(t), \dot{x}(t)\right)dt \geq 0$ for each $i \in \underline{p}$ and any $x \in \Im_P$, the set of all feasible solutions of (P). For simplicity, we write $x(t) = x$ and $\dot{x}(t) = \dot{x}$.

In order to obtain necessary and sufficient optimality conditions for problem (P), Bector *et al.* (1992) considered an equivalent parametric problem for $v \in R$ in the following form:

(EP_v) minimize q

subject to $x \in PS\left(T, R^n\right)$, $x(a) = \alpha$, $x(b) = \beta$,

$$\int_a^b \left[f_i\left(t, x, \dot{x}\right) - v g_i\left(t, x, \dot{x}\right)\right]dt \leq q,$$

$$\int_a^b h_j\left(t, x, \dot{x}\right)dt \leq 0, \text{ for } i \in \underline{p} \text{ and } j \in \underline{m},$$

and they established the following useful result:

Lemma 1. [Bector et al. (1992)]. The function x^* is an optimal solution for (P) with optimal value v^* if and only if the triplet $\left(x^*, v^*, q^*\right)$ is an optimal solution of (EP_v) with optimal value $q^* = 0$. That is,

$$v^* = \frac{\int_a^b f_i\left(t, x^*, \dot{x}^*\right)dt}{\int_a^b g_i\left(t, x^*, \dot{x}^*\right)dt}.$$

Recently, Lai and Liu (2003) considered the problem (P) and established the optimality results for the problem (P) and some duality results for (P) and its three different duals under the invexity (see, Hanson (1981)) and generalized invexity assumptions on the functionals involved in the problem.

In this paper, we will establish optimality conditions and duality theorems under the assumptions of type I and related functions in the problem (P). This will extend some results of Lai and Liu (2003) to a wider class of functions. Moreover, several results established in the literature are a particular case of the results obtained in this paper.

2. NOTATIONS AND PRELIMINARIES

For $x \in PS\left(T, R^n\right)$, we let $F_i(x) = \int_a^b f_i\left(t, x, \dot{x}\right)dt$, $G_i(x) = \int_a^b g_i\left(t, x, \dot{x}\right)dt$ and $H_j(x) = \int_a^b h_j\left(t, x, \dot{x}\right)dt$, for $i \in \underline{p}$ and $j \in \underline{m}$. Since the functions f_i, g_i, $i \in \underline{p}$ and h_j, $j \in \underline{m}$ are continuous in t, x and $\dot{x}$ and have continuous partial derivatives with respect to x and $\dot{x}$ then the functionals $F = \left(F_1, ..., F_p\right)$, $G = \left(G_1, ..., G_p\right)$ and $H = \left(H_1, ..., H_m\right)$ are (Frechet) differentiable on $PS\left(T, R^n\right)$. It follows that the problem (P) may be written in the form:

$$\text{(P)} \quad \min_{x \in PS\left(T, R^n\right)} \max_{i \in \underline{p}} \left(\frac{F_i(x)}{G_i(x)}\right)$$

subject to $x(a) = \alpha$, $x(b) = \beta$ and $H(x) \leq 0$.

Here α and β are fixed vectors in R^n. The equivalent parametric minimization problem (EP_v) is then given by

(EP_v) minimize q

subject to $F_i(x) - vG_i(x) \leq q$, $i \in \underline{p}$, $H_j(x) \leq 0$, $j \in \underline{m}$

$x \in PS\left(T, R^n\right)$ with fixed boundary conditions $x(a) = \alpha$ and $x(b) = \beta$.

As in [11], One can show that if $x \in \Im_P$, a feasible solution for (P), then

$$\varphi(x) = \max_{i \in \underline{p}} \frac{F_i(x)}{G_i(x)} = \max_{\substack{\langle y, e\rangle = 1 \\ y \in R_+^p}} \frac{\langle y, F(x)\rangle}{\langle y, G(x)\rangle}, \tag{1}$$

where e is a vector of ones in R_+^p and $\langle \cdot, \cdot \rangle$ denotes the inner product in R^p. For convenience, now on we set $I = \left\{ y \in R_+^p : \langle y, e \rangle = 1 \right\}$.

For simplicity, for $x \in PS\left(T, R^n\right)$, $y \in R_+^p$ and $z \in R_+^m$ we denote

$$\Phi(x, y) = \langle y, F(x) \rangle = \int_a^b \sum_{i=1}^p y_i f_i(t, x, \dot{x}) dt,$$

$$\Psi(x, y) = \langle y, G(x) \rangle = \int_a^b \sum_{i=1}^p y_i g_i(t, x, \dot{x}) dt,$$

$$\Omega(x, z) = \langle z, H(x) \rangle = \int_a^b \sum_{j=1}^m z_j h_j(t, x, \dot{x}) dt.$$

Clearly, $\Phi(x, \cdot)$, $\Psi(x, \cdot)$ and $\Omega(x, \cdot)$ are linear functionals.

From Eq.(1), if x^* is an optimal solution for (P), then

$$\begin{aligned} \varphi\left(x^*\right) = \max_{i \in \underline{p}} \frac{F_i\left(x^*\right)}{G_i\left(x^*\right)} = \max_{y \in I} \frac{\Phi\left(x^*, y\right)}{\Psi\left(x^*, y\right)} = \frac{\Phi\left(x^*, \hat{y}\left(x^*\right)\right)}{\Psi\left(x^*, \hat{y}\left(x^*\right)\right)} \\ = \min_x \max_{y \in I} \frac{\Phi(x, y)}{\Psi(x, y)} = \frac{\Phi\left(x^*, y^*\right)}{\Psi\left(x^*, y^*\right)}, \end{aligned} \tag{2}$$

where $\hat{y}\left(x^*\right) = y^*$.

Problem (P) is equivalent to

$$\min_x \max_y \frac{\Phi(x, y)}{\Psi(x, y)} \quad \text{subject to} \quad H(x) \le 0, \; x \in R^n \text{ and } y \in I.$$

Theorem 1. [Craven (1988)] (Necessary conditions). If x^* is an optimal solution of (P), then there exist $y^* \in I$ and multipliers $v^* \in R$ and $z^* \in R_+^m$ such that $\left(x^*, v^*, y^*, z^*\right)$ satisfies

$$\Phi'\left(x^*, y^*\right) - v^* \Psi'\left(x^*, y^*\right) + \Omega'\left(x^*, z^*\right) = 0, \tag{3}$$

$$\Phi\left(x^*, y^*\right) - v^* \Psi\left(x^*, y^*\right) = 0, \tag{4}$$

$$\Omega\left(x^*, z^*\right) = 0, \tag{5}$$

where Φ' and Ψ' are the gradients of Φ and Ψ at $\left(x^*, y^*\right)$ respectively and $\Omega'\left(x^*, z^*\right) = \left\langle z^*, \nabla H(x) \right\rangle$.

We can replace v^* by $\dfrac{\Phi\left(x^*, y^*\right)}{\Psi\left(x^*, y^*\right)}$ using (4) and then restate Theorem 1 as follows:

Theorem 2. (Necessary Condition). If x^* is an optimal solution of (P), then there exist $y^* \in I$ and multipliers $v^* \in R$ and $z^* \in R_+^m$ such that $\left(x^*, y^*, z^*\right)$ satisfies

$$\Psi\left(x^*, y^*\right) \Phi'\left(x^*, y^*\right) - \Phi\left(x^*, y^*\right) \Psi'\left(x^*, y^*\right) + \Psi\left(x^*, y^*\right) \Omega'\left(x^*, z^*\right) = 0, \tag{6}$$

$$\Omega\left(x^*, z^*\right) = 0, \tag{7}$$

and obtain the optimal value by

$$\varphi\left(x^*\right)=\frac{\Phi\left(x^*,y^*\right)}{\Psi\left(x^*,y^*\right)}=\max_{y\in I}\frac{\Phi\left(x^*,y\right)}{\Psi\left(x^*,y\right)}\left(=\min_{x}\max_{1\le i\le p}\frac{F_i(x)}{G_i(x)}\right). \tag{8}$$

3. TYPE I AND RELATED FUNCTIONS

Let $x\in PS\left(T,R^n\right)$ and $f\in C^1\left(T\times R^n\times R^n\right)$ define a functional $J:PS\left(T,R^n\right)\to R$ by $J(x)=\int_a^b f\left(t,x,\dot{x}\right)dt$.

We assume that the boundary points $x(a)$ and $x(b)$ are fixed. Consider the admissible vectors $x+w$ with admissible variations $w\in C\left(\left[a,b\right],R^n\right)$ vanishing at the boundary points. Then the differential of J is a linear functional on $C\left(\left[a,b\right],R^n\right)$,

$$J'(x)=\frac{d}{d\alpha}\int_a^b f\left(t,x+\alpha w,\dot{x}+\alpha\dot{w}\right)dt\Big|_{\alpha=0}$$

$$=\int_a^b\left[f_x\left(t,x,\dot{x}\right)w(t)+f_{\dot{x}}\left(t,x,\dot{x}\right)\dot{w}(t)\right]dt$$

$$=\int_a^b\left[f_x\left(t,x,\dot{x}\right)-Df_{\dot{x}}\left(t,x,\dot{x}\right)\right]w(t)\,dt+f_{\dot{x}}\left(t,x,\dot{x}\right)w(t)\Big|_a^b$$

$$=\int_a^b\left[f_x\left(t,x,\dot{x}\right)-Df_{\dot{x}}\left(t,x,\dot{x}\right)\right]w(t)\,dt.$$

That is,

$$J'(x)w=\int_a^b\left[f_x\left(t,x,\dot{x}\right)-Df_{\dot{x}}\left(t,x,\dot{x}\right)\right]w(t)\,dt,$$

for all $w\in C\left(\left[a,b\right],R^n\right)$, $w(a)=0=w(b)$, where $D=d/dt$.

Define a function $\eta:PS\left(T,R^n\right)\times PS\left(T,R^n\right)\to C\left(T,R^n\right)$ with condition $\eta(x,u)=0$ if $x=u$.

We now give the following definition:

Definition 1. The problem (P) is said to be *type I* with respect to η at $x^*\in\Im_P$ if for all $x\in\Im_P$, we have

$$\Phi\left(x,y^*\right)-\Phi\left(x^*,y^*\right)\ge\Phi_I'\left(x^*,y^*\right)\eta\left(x,x^*\right),$$

$$-\left(\Psi\left(x,y^*\right)-\Psi\left(x^*,y^*\right)\right)\ge-\Psi_I'\left(x^*,y^*\right)\eta\left(x,x^*\right),$$

$$-\Omega\left(x^*,z^*\right)\ge\Omega_I'\left(x^*,z^*\right)\eta\left(x,x^*\right),$$

where for each y^* and z^*, Φ_I', Ψ_I' and Ω_I' are partial Frechet derivatives of $\Phi\left(\cdot,y^*\right)$, $-\Psi\left(\cdot,y^*\right)$ and $\Omega\left(\cdot,z^*\right)$ at y^*, respectively.

Definition 2. The problem (P) is said to be *pseudo-quasi type I* with respect to η at $x^*\in\Im_P$ if for all $x\in\Im_P$, we have

$$\Psi\left(x^*,y^*\right)\Phi\left(x,y^*\right)-\Psi\left(x,y^*\right)\Phi\left(x^*,y^*\right)<0=\Psi\left(x^*,y^*\right)\Phi\left(x^*,y^*\right)-\Psi\left(x^*,y^*\right)\Phi\left(x^*,y^*\right)$$

$$\Rightarrow\left(\Psi\left(x^*,y^*\right)\Phi_I'\left(x,y^*\right)-\Phi\left(x^*,y^*\right)\Psi_I'\left(x^*,y^*\right)\right)\eta\left(x,x^*\right)<0,$$

$$-\Omega\left(x^*,z^*\right)\le 0\Rightarrow\Omega_I'\left(x^*,z^*\right)\eta\left(x,x^*\right)\le 0,$$

where for each y^* and z^*, Φ_I', Ψ_I' and Ω_I' are partial Frechet derivatives of $\Phi\left(\cdot,y^*\right)$, $-\Psi\left(\cdot,y^*\right)$ and

$\Omega\left(\cdot\,,\, z^*\right)$ at y^*, respectively.

Definition 3. The problem (P) is said to be *quasi-strictly-pseudo type I* with respect to η at $x^* \in \Im_P$ if for all $x \in \Im_P$, we have

$$\Psi\left(x^*, y^*\right)\Phi\left(x, y^*\right) - \Psi\left(x, y^*\right)\Phi\left(x^*, y^*\right) \le 0 = \Psi\left(x^*, y^*\right)\Phi\left(x^*, y^*\right) - \Psi\left(x^*, y^*\right)\Phi\left(x^*, y^*\right)$$

$$\Rightarrow \left(\Psi\left(x^*, y^*\right)\Phi'_I\left(x, y^*\right) - \Phi\left(x^*, y^*\right)\Psi'_I\left(x^*, y^*\right)\right)\eta\left(x, x^*\right) \le 0\,,$$

$$\Omega'_I\left(x^*, z^*\right)\eta\left(x, x^*\right) > 0 \Rightarrow -\Omega\left(x^*, z^*\right) > 0,$$

where for each y^* and z^*, Φ'_I, Ψ'_I and Ω'_I are partial Frechet derivatives of $\Phi\left(\cdot\,, y^*\right)$, $-\Psi\left(\cdot\,, y^*\right)$ and $\Omega\left(\cdot\,,\, z^*\right)$ at x^*, respectively.

Now we establish sufficient optimality conditions for (P) under certain assumptions on the problem (P).

Theorem 3. (Sufficient optimality conditions). Let $x^* \in \Im_P, y^* \in I, z^* \in R_+^m$ and $\left(x^*, y^*, z^*\right)$ satisfy Eq.(6)-Eq.(8). If, for the given y^* and z^* the problem (P) is type I with respect to η. Then x^* is an optimal solution for (P).

Proof. If x^* is not an optimal solution for (P), then there is a feasible solution $u \in \Im_P$ such that

$$\varphi\left(x^*\right) > \varphi\left(u\right). \tag{9}$$

By Eq.(1) and Eq.(2), we have

$$\frac{\Phi\left(x^*, y^*\right)}{\Psi\left(x^*, y^*\right)} = \varphi\left(x^*\right) > \varphi\left(u\right) = \max_{y \in I}\frac{\Phi\left(u, y\right)}{\Psi\left(u, y\right)} \ge \frac{\Phi\left(u, y^*\right)}{\Psi\left(u, y^*\right)},\quad y^* \in I.$$

It follows that

$$\Phi\left(u, y^*\right)\Psi\left(x^*, y^*\right) - \Phi\left(x^*, y^*\right)\Psi\left(u, y^*\right) < 0. \tag{10}$$

For $y^* \in I, z^* \in R_+^m$ employing the type I condition of the problem, we get

$$\Phi\left(u, y^*\right) - \Phi\left(x^*, y^*\right) \ge \Phi'_I\left(x^*, y^*\right)\eta\left(u, x^*\right), \tag{11}$$

$$-\left(\Psi\left(u, y^*\right) - \Psi\left(x^*, y^*\right)\right) \ge -\Psi'_I\left(x^*, y^*\right)\eta\left(u, x^*\right), \tag{12}$$

$$-\Omega\left(x^*, z^*\right) \ge \Omega'_I\left(x^*, z^*\right)\eta\left(u, x^*\right). \tag{13}$$

Here for each y^* and z^*, Φ'_I, Ψ'_I and Ω'_I are partial Frechet derivatives of $\Phi\left(\cdot\,, y^*\right)$, $-\Psi\left(\cdot\,, y^*\right)$ and $\Omega\left(\cdot\,,\, z^*\right)$ at x^*, respectively. Since $\Psi\left(x^*, y^*\right) > 0$ and $\Phi\left(x^*, y^*\right) \ge 0,$ we multiply (11) by $\Psi\left(x^*, y^*\right)$, Eq.(12) by $\Phi\left(x^*, y^*\right)$. Adding up the resulting inequalities and using Eq.(10), we get

$$\left(\Psi\left(x^*, y^*\right)\Phi'_I\left(x^*, y^*\right) - \Phi\left(x^*, y^*\right)\Psi'_I\left(x^*, y^*\right)\right)\eta\left(u, x^*\right) < 0\,. \tag{14}$$

From Eq.(7), Eq.(13) and $\Psi\left(x^*, y^*\right) > 0$, we get

$$\Psi\left(x^*, y^*\right)\Omega'_I\left(x^*, z^*\right)\eta\left(u, z^*\right) \le 0. \tag{15}$$

From Eq.(14) and Eq.(15) we get a contradiction to Eq.(6). Hence Eq.(9) does not hold, and so x^* must be optimal for (P).□

Theorem 4. (Sufficient optimality conditions). Let $x^* \in \Im_P, y^* \in I, z^* \in R_+^m$ and $\left(x^*, y^*, z^*\right)$ satisfy Eq.(6)-Eq.(8). If, for the given y^* and z^* the problem (P) is pseudo-quasi type I with respect to η. Then x^* is an optimal solution for (P).

Proof. If x^* is not an optimal solution for (P), then there is a feasible solution $u \in \Im_P$ such that

$$\varphi\left(x^*\right) > \varphi\left(u\right). \tag{16}$$

By Eq.(1) and Eq.(2), we have

$$\frac{\Phi\left(x^*, y^*\right)}{\Psi\left(x^*, y^*\right)} = \varphi\left(x^*\right) > \varphi\left(u\right) = \max_{y \in I} \frac{\Phi\left(u, y\right)}{\Psi\left(u, y\right)} \ge \frac{\Phi\left(u, y^*\right)}{\Psi\left(u, y^*\right)}, \quad y^* \in I.$$

It follows that

$$\Phi\left(u, y^*\right)\Psi\left(x^*, y^*\right) - \Phi\left(x^*, y^*\right)\Psi\left(u, y^*\right) < 0. \tag{17}$$

Note that

$$\Phi\left(x^*, y^*\right)\Psi\left(x^*, y^*\right) - \Phi\left(x^*, y^*\right)\Psi\left(x^*, y^*\right) = 0. \tag{18}$$

For $y^* \in I, z^* \in R_+^m$ employing the pseudo-quasi type I condition of the problem and using Eq.(7), Eq.(17) and Eq.(18), we get

$$\left(\Psi\left(x^*, y^*\right)\Phi_I'\left(x, y^*\right) - \Phi\left(x^*, y^*\right)\Psi_I'\left(x^*, y^*\right)\right)\eta\left(x, x^*\right) < 0 , \tag{19}$$

$$\Omega_I'\left(x^*, z^*\right)\eta\left(x, x^*\right) \le 0, \tag{20}$$

Here for each y^* and z^*, Φ_I', Ψ_I' and Ω_I' are partial Frechet derivatives of $\Phi\left(\cdot, y^*\right)$, $-\Psi\left(\cdot, y^*\right)$ and $\Omega\left(\cdot, z^*\right)$ at x^*, respectively. Since $\Psi\left(x^*, y^*\right) > 0$, multiplying (20) by $\Psi\left(x^*, y^*\right)$ and adding up the resulting inequalities, we get a contradiction to (6). Hence (16) does not hold, and so x^* must be optimal for (P).□

Theorem 5. (Sufficient optimality conditions). Let $x^* \in \Im_P, y^* \in I, z^* \in R_+^m$ and $\left(x^*, y^*, z^*\right)$ satisfy Eq.(6)-Eq.(8).

If, for the given y^* and z^* the problem (P) is quasi-strictly-pseudo type I with respect to η. Then x^* is an optimal solution for (P).

The proof of this Theorem can be given on the lines of the proof of Theorem 4.

4. WOLFE TYPE DUAL

Using Theorem 2, we will construct the following Wolfe type dual:

(WD) Maximise $\dfrac{\Phi\left(u, y\right) + \Omega\left(u, z\right)}{\Psi\left(u, y\right)}$

subject to $\left(u, z\right) \in PS\left(T, R^n\right) \times R_+^m$ and $y \in I \subset R_+^m$,

$$u\left(a\right) = \alpha, \quad u\left(b\right) = \beta,$$

$$\Psi\left(u, y\right)\Phi_I'\left(u, y\right) - \left(\Phi\left(u, y\right) + \Omega\left(u, z\right)\right)\Psi_I'\left(u, y\right) + \Psi\left(u, y\right)\Omega_I'\left(u, z\right) = 0. \tag{21}$$

Denote by K_{WD} the set of all feasible solutions of problem (WD). We assume throughout this section that

$$\Phi\left(u, y\right) + \Omega\left(u, z\right) \ge 0 \quad \text{and} \quad \Psi\left(u, y\right) > 0 \quad \text{for all} \quad \left(u, y, z\right) \in K_{WD}. \tag{22}$$

Theorem 6. (Weak duality). Let $x \in \Im_P$, $\left(u, y, z\right) \in K_{WD}$. If, for each y and z, the problem (P) is type I with respect to η (see Definition 1). Then

$$\varphi\left(x\right) \ge \frac{\Phi\left(u, y\right) + \Omega\left(u, y\right)}{\Psi\left(u, y\right)}, \tag{23}$$

where $\varphi\left(x\right)$ is defined by Eq.(1).

Proof. If Eq.(23) does not hold, then

$$\varphi(x) < \frac{\Phi(u, y) + \Omega(u, y)}{\Psi(u, y)}. \tag{24}$$

It follows from Eq.(2) that for any $y \in I$,

$$\frac{\Phi(x, y)}{\Psi(x, y)} \le \max_{\beta \in I} \frac{\Phi(x, \beta)}{\Psi(x, \beta)} = \varphi(x) < \frac{\Phi(u, y) + \Omega(u, z)}{\Psi(u, y)},$$

or

$$\Phi(x, y)\Psi(u, y) - \left(\Phi(u, y) + \Omega(u, z)\right)\Psi(x, y) < 0. \tag{25}$$

Since the problem (P) is Type I with respect to η, for each $y \in I$, $z \in R_+^m$, we get

$$\Phi(x,y) - \Phi(u,y) \ge \Phi_I'(u,y)\eta(x,u), \tag{26}$$

$$-\left(\Psi(x,y) - \Psi(u,y)\right) \ge -\Psi_I'(u,y)\eta(x,u), \tag{27}$$

$$-\Omega(u,z) \ge \Omega_I'(u,z)\eta(x,u). \tag{28}$$

Multiplying Eq.(26) by $\Psi(u, y) > 0$, Eq.(27) by $\Phi(u, y) + \Omega(u, z) \ge 0$ and Eq.(28) by $\Psi(u, y) > 0$ and adding up the resulting inequalities and using Eq.(25), we get

$$\left\{\Psi(u, y)\Phi_I'(u, y) - \left(\Phi(u, y) + \Omega(u, z)\right)\Psi_I'(u, y) + \Psi(u, y)\Omega_I'(u, z)\right\}\eta(x,u) < 0,$$

which contradicts Eq.(21). Hence Eq.(24) does not hold, and Eq.(23) does hold.□

Theorem 7. (Strong duality). If x^* is an optimal solution for (P) satisfying the conditions of Theorem 6, then there exist $y^* \in I$ and $z^* \in R_+^m$ such that $\left(x^*, y^*, z^*\right)$ is an optimal solution of (WD) and the optimal values of (P) and (WD) are equal.

Proof. If x^* is an optimal solution of (P), then by necessary conditions there exist $y^* \in I$ and $z^* \in R_+^m$ which satisfy the constraints of (WD), so that $\left(x^*, y^*, z^*\right) \in K_{WD}$. Furthermore

$$\frac{\Phi\left(x^*, y^*\right) + \Omega\left(x^*, z^*\right)}{\Psi\left(x^*, y^*\right)} = \frac{\Phi\left(x^*, y^*\right)}{\Psi\left(x^*, y^*\right)} = \varphi\left(x^*\right),$$

since $\Omega\left(x^*, z^*\right) = 0$. Hence $\left(x^*, y^*, z^*\right)$ is an optimal solution of (WD). Consequently, from Eq.(23), we see that (P) and (WD) have the same optimal values.□

5. MOND-WEIR TYPE DUAL

We introduce the Mond-Weir dual problem as follows:

(MD) Maximise $\dfrac{\Phi(u, y)}{\Psi(u, y)}$

subject to $(u, y) \in PS\left(T, R^n\right) \times I$, $\quad u(a) = \alpha$, $\quad u(b) = \beta$,

$$\Psi(u, y)\Phi_I'(u, y) - \Phi(u, y)\Psi_I'(u, y) + \Psi(u, y)\Omega_I'(u, y) = 0,$$

$$\Omega(u, z) \ge 0, \ z \in R_+^m.$$

Denote by K_{MD} the set of all feasible solutions of problem (MD). We assume that $\Phi(u, y) \ge 0$ and $\Psi(u, y) > 0$ for all $(u, y, z) \in K_{MD}$.

Theorem 8. (Weak duality). Let $x \in \Im_P$, $(u,y,z) \in K_{MD}$. If for each $y \in I$, $z \in R_+^m$, the problem (P) is type I with respect to η. Then

$$\varphi(x) \ge \frac{\Phi(u, y)}{\Psi(u, y)}. \tag{29}$$

Proof. If Eq.(29) does not hold, then

$$\phi(x) < \frac{\Phi(u, y)}{\Psi(u, y)}. \tag{30}$$

It follows from Eq.(2) that for any $y \in I$,

$$\frac{\Phi(x, y)}{\Psi(x, y)} \le \max_{\beta \in I} \frac{\Phi(x, \beta)}{\Psi(x, \beta)} = \varphi(x) < \frac{\Phi(u, y)}{\Psi(u, y)},$$

or

$$\Phi(x, y)\Psi(u, y) - \Phi(u, y)\Psi(x, y) < 0. \tag{31}$$

Since the problem (P) is type I with respect to η, for each $y \in I,\ z \in R_+^m$, we get

$$\Phi(x,y) - \Phi(u,y) \ge \Phi_I'(u,y)\eta(x,u), \tag{32}$$

$$-\left(\Psi(x,y) - \Psi(u,y)\right) \ge -\Psi_I'(u,y)\eta(x,u), \tag{33}$$

$$-\Omega(u,z) \ge \Omega_I'(u,z)\eta(x,u). \tag{34}$$

Multiplying Eq.(32) by $\Psi(u, y) > 0$, Eq.(33) by $\Phi(u, y) \ge 0$ and Eq.(34) by $\Psi(u, y) > 0$, adding up the resulting inequalities using Eq.(31) and the duality constraint $\Omega(u, z) \ge 0,\ z \in R_+^m$, we get

$$\left\{\Psi(u, y)\Phi_I'(u, y) - \Phi(u, y)\Psi_I'(u, y) + \Psi(u, y)\Omega_I'(u, z)\right\}\eta(x,u) < 0,$$

which contradicts the duality constraint

$$\Psi(u, y)\Phi_I'(u, y) - \Phi(u, y)\Psi_I'(u, y) + \Psi(u, y)\Omega_I'(u, y) = 0,$$

Hence Eq.(30) does not hold, and Eq.(29) does hold.□

Theorem 9. (Weak duality). Let $x \in \Im_P$, $(u,y,z) \in K_{MD}$. If for each $y \in I,\ z \in R_+^m$, the problem (P) is pseudo-quasi-type I with respect to η. Then

$$\varphi(x) \ge \frac{\Phi(u, y)}{\Psi(u, y)}. \tag{35}$$

Proof. If Eq.(29) does not hold, then

$$\varphi(x) < \frac{\Phi(u, y)}{\Psi(u, y)}. \tag{36}$$

It follows from Eq.(2) that for any $y \in I$,

$$\frac{\Phi(x, y)}{\Psi(x, y)} \le \max_{\beta \in I} \frac{\Phi(x, \beta)}{\Psi(x, \beta)} = \varphi(x) < \frac{\Phi(u, y)}{\Psi(u, y)},$$

or

$$\Phi(x, y)\Psi(u, y) - \Phi(u, y)\Psi(x, y) < 0. \tag{37}$$

Note that

$$\Phi(u, y)\Psi(u, y) - \Phi(u, y)\Psi(u, y) = 0. \tag{38}$$

Since the problem (P) is pseudo-quasi-type I with respect to η, for each $y \in I,\ z \in R_+^m$, from duality constraint $\Omega(u, z) \ge 0,\ z \in R_+^m$, inequalities Eq.(37) and Eq.(38), we get

$$\left\{\Psi(u,y)\Phi_I'(u,y) - \Phi(u,y)\Psi_I'(u,y)\right\}\eta(x, u) < 0, \tag{39}$$

$$\Omega_I'(u,z)\eta(x, u) \le 0. \tag{40}$$

Since $\Psi(u, y) > 0$, multiplying Eq.(40) by $\Psi(u, y) > 0$ and adding up the resulting inequality with Eq.(39), we get

$$\left\{\Psi(u, y)\Phi_I'(u, y) - \Phi(u, y)\Psi_I'(u, y) + \Psi(u, y)\Omega_I'(u, z)\right\}\eta(x,u) < 0,$$

which contradicts the duality constraint

$$\Psi(u, y)\Phi_I'(u, y) - \Phi(u, y)\Psi_I'(u, y) + \Psi(u, y)\Omega_I'(u, y) = 0,$$

Hence Eq.(36) does not hold, and Eq.(35) does hold.□

Theorem 10. (Weak duality). Let $x \in \Im_P$, $(u, y, z) \in K_{MD}$. If for each $y \in I,\ z \in R_+^m$, the problem (P) is quasi-strictly-pseudo-type I with respect to η. Then

$$\varphi(x) \geq \frac{\Phi(u, y)}{\Psi(u, y)}.$$

Proof. The proof can be given as the proof of the Theorem 9.□

Theorem 11. (Strong duality). If x^* is an optimal solution for (P) satisfying the conditions of any of the Theorems 8, 9 or 10 then there exist $y^* \in I$ and $z^* \in R_+^m$ such that (x^*, y^*, z^*) is an optimal solution of (MD) and the optimal values of (P) and (MD) are equal.

Proof. If x^* is an optimal solution of (P), then by necessary conditions there exist $y^* \in I$ and $z^* \in R_+^m$ which satisfy the constraints of (MD), so that $(x^*, y^*, z^*) \in K_{WD}$. Furthermore

$$\varphi(x^*) = \frac{\Phi(x^*, y^*)}{\Psi(x^*, y^*)} = \max_{y \in I} \frac{\Phi(x^*, y)}{\Psi(x^*, y)}.$$

It follows from any one of the weak duality Theorems 8, 9 or 10 that (x^*, y^*, z^*) is an optimal solution of (MD) and $\min(\mathrm{P}) = \max(\mathrm{MD})$. □

Theorem 12. (Strict converse duality). Let x_1 and (x^*, y_0, z_0) be optimal solutions of (P) and (MD), respectively. Assume that the conditions of Theorem 11 hold and the problem (P) is strictly-pseudo-quasi-type I with respect to η. Then $x_1 = x^*$ is an optimal solution for (P), and (P) and (MD) have the same optimal values $\varphi(x_1) = \dfrac{\Phi(x^*, y_0)}{\Psi(x^*, y_0)}$.

Proof. The proof can be given along the lines of the proof of Theorem 5.3 due to Lai and Liu (2003) in the light of the discussions above in this paper.□

6. CONCLUSION

In this paper, the convexity assumptions for a minimax fractional programming problem of variational type are relaxed to those type I and related functions. Sufficient optimality conditions and duality results are established under type I and related functions. Several works in the literature are a particular case of the results obtained in this paper.

Further it will be interesting to investigate similar results for nondifferentiable minimax fractional programming problems of variational type under the aforesaid assumptions.

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