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Multiobjective Programming Problem with Fuzzy Relational Equations

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Abstract —In this paper we propose a methodology for the solution of multiobjective programming problem with fuzzy relational equations (FRE's) as constraints. In the first part of the proposed methodology, we find the feasible solution set of FRE's and give an algorithm to compute minimal solutions and maximal solutions of objective functions which can also be computed by constructed computer program and then in the second part of the proposed methodology which works for the minimization of perpendicular distances between the parallel hyper planes at the optimal points of the objective functions. A compromise optimal solution is obtained as a result of minimization of supremum perpendicular distance. Suitable membership function has been defined with the help of supremum perpendicular distance. An example is given in last to support the model.

Keyword—Fuzzy relational equations, Multiobjective programming, Disjunction and conjunction operators, Fuzzy programming, Minimal solution.

1. INTRODUCTION

The use of fuzzy logic in optimization models has been attempted by Fang and Puthenpura (1993), Klir and Yuan (1995), in which problems are described with fuzzy numbers as constraints and fuzzy arithmetic operators are used for solution sets. The convexity of the feasible sets remain same as in classical problems, therefore the classical methods can be used to solve it. Another development in fuzzy set theory is the idea of using fuzzy relational by Nola, *et. al* (1991), Sanchez (1978), Dubois and Prade (1980), Zimmermann (1991), Wagenknecht, and Hartmann (1990), Pedrycrz (1983), Stamou and Tzafestas (2001), Pedrycz (1983), Pedrycz (1991), Higashi and Klir (1984), Yager (1979) and implementation in optimization models via FRE's and characterization of feasible domain by Prevot (1985), Sanchez (1978), Fang and Li (1999), Tamura *et.al* (1979), Wu (1986), Chung and Lee (1997), Li.(1994), Wang (1988) and recently by Pandey and Gaur (2004). This encourages researchers to use it in multi criteria decision making (MCDM) problems and many researchers cited as Wallenius (1975), Hanan (1979), Feng (1983), Chanas (1989) and Rommelfanger (1989) used/ or modified the concept of decision making in fuzzy environment. They have discussed different approaches to tackle the multiobjective programming problems. Gupta and Chakraborty (1997) suggested fuzzy approach for multi objective programming problems. Gauss and Roy (2003) discussed the compromise hyper sphere for multi objective linear programming problem. Then again Gupta and Chakraborty (2005) modified the concept of fuzzy approach and gave fuzzy mathematical programming for multi objective linear fractional programming problem. Jain and Lachhwani (2009) proposed a solution methodology for multiobjective linear fractional programming problem i.e. $\frac{f(X)}{g(X)}$ form in MOLFP. Jain and Lachhwani (2009) discussed solution of multi objective linear plus fractional program in which the objective function is the sum of the linear function and quotient function $f(X)+\frac{g(X)}{h(X)}$. Balbas and Galperin et al. (2005) gave a sensitivity analysis in multi objective optimization. Yan and Wei et al. (2005) constructed an efficient solution structure of multi objective linear programming. Afterwards Jain and Lachhwani (2009) suggested a fuzzy programming approach for solution of multi objective quadratic program.

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Here we consider an optimization problem as

$$\text{Max } \{ Z_1(x), Z_2(x), ..., Z_k(x) \} \tag{1}$$

where $Z_q(x) = C_q^T x \quad q = 1, 2, \ldots, k$

subject to $A o x = b, \quad x \geq 0$ (2)

The membership matrices A, b, x are denoted by $A = (a_{ij}),\ b = (b_i),\ x = (x_j)$ where a_{ij}, b_i, x_j are real numbers in the unit interval [0, 1] for all $i \in I$ and $j \in J$ and $C^T = (c_1, c_2, ..., c_n), I = \{1, 2, ..., m\}$ and $J = \{1, 2, ..., n\}$ are the index sets. The operator o signifies max-min composition as

$$\underset{j \in J}{\text{Max}} \min(a_{ij}, x_j) = b_i \quad \forall\ i \in I \tag{3}$$

Using our proposed methodology in section 3 the vector maximum problem (1) can be reduced to

$$Max\ \lambda$$

$$\text{subject to} \quad -Z_q(x) + p.\{\sum c_{qj}^2\}^{1/2} \lambda \leq p.\{\sum_{j=1}^{n} c_{qj}^2\}^{1/2} - \overline{Z_q} \qquad \forall\ q = 1, 2, ..., k$$

$$A o\, x = b$$

$$\text{and} \qquad x \geq 0 \tag{4}$$

The present paper is organized as follows: In section 2, we discuss solution sets for the maximal and minimal solutions of FRE's with their existence in the feasible sets. In section 3, we propose a solution methodology into two parts. In the first part, we characterize the objective functions of the problem and in the second part, we obtain a compromise optimal solution of problem (1) by minimizing the perpendicular distances between two hyper planes $Z_q(x) = \overline{Z_q}$ and $Z_q(x) = \underline{Z_q}$ where $\overline{Z_q}$ and $\underline{Z_q}$ are maximum and minimum value of the function $Z_q(x)$. Defining a suitable membership functions and then minimization of perpendicular distances affect the fuzzy parameters used. As a result a compromise optimum solution is obtained. In section 4, we prove that the solution obtained by proposed methodology is also a pareto optimal solution of the considered problem. In section 5, stepwise description of algorithm with corresponding computer program for obtaining maximal solutions are given. An example and conclusion are given in the last section.

2. THE FEASIBLE SET OF FUZZY RELATIONAL EQUATIONS

Definition: A vector x is a solution of matrix equation in Eq.(1) if and only if all its equations of the form Eq.(2) are satisfied. Let X (A,b) be the solution set, then

$$X(A,b) = \{x = (x_1, x_2, ..., x_n) \mid A o x = b,\ x_j \in [0,1], \forall j \in J\} \tag{5}$$

This will have a unique maximum solution $\hat{x}$ and a finite set of minimal solutions $\breve{x}$ provided $X(A,b) \neq \phi$ as by Adamopoulos and Pappis (1993).

Lemma 1. For any $i \in I$, $\exists$ some $j \in J$, such that $a_{ij} \geq b_i$, then $X(A,b) \neq \phi$.

Proof. Let $X(A,b) = \phi$, then it will have no solution if $\max a_{ij} < b_i$ for some $i \in I$ as by Klir and Yuan (1995), contrary to this $X(A,b) \neq \phi$, if for any $i \in I$, $\exists$ some $j \in J$ such that $a_{ij} \geq b_i$.

Lemma 2. If $x \in X(A,b)$ then $\min(a_{ij}, x_j) = b_i$ for some $j \in J$ and $\min(a_{ij}, x_j) \leq b$ for other $j \in J$.

Proof. By Eq.(3), max-min $(a_{ij}, x_j) = b_i$. This implies $\min(a_{ij}, x_j) \leq b_i$. Since $x \in X(A,b)$, therefore there exists at least one $j \in J$ such that $\min(a_{ij}, x_j) = b_i$.

Now $\breve{x} \in X(A,b)$ is called minimal solution if for all $x \in X(A,b)$, $x \geq \breve{x}$, implies that $x = \breve{x}$. Similarly $\hat{x} \in X(A,b)$ is a maximum solution if for any $x \in X(A,b)$, $x \leq \hat{x}$. Thus the solution set is the union of all lattices between each minimal and the unique maximum solution as

$$\text{Hence} \quad X(A,b) = \bigcup_{\breve{x} \in X(A,b)} \{x \in X, | \breve{x} \leq x \leq \hat{x}\} \tag{6}$$

I. The maximum solution $\hat{x} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ can be obtained as given by Chanas (1989):

$$\hat{x}_j = \bigwedge_{i=1}^{m} (a_{ij} @ b_i), \qquad \forall j \in J \tag{7}$$

$$\text{where} \quad a_{ij} @ b_i = \begin{cases} 1 & \text{if } a_{ij} \leq b_i \\ b_i & \text{if } a_{ij} > b_i \end{cases} \tag{8}$$

and $\hat{x}$ thus obtained is feasible if $A o \hat{x} = b$ and $J_i = \{j,\ a_{ij} o \hat{x}_j = b_i\}, \forall\ i \in I$. Where J_i is adjoin partition of I.

II. The set $\breve{x} \in X(A,b)$ of minimal solutions can be computed from algorithm given by Fang and Puthenpura (1993), and Pedrycz (1991) as:

Step 1. Compute Ω using

$$\Omega = \prod_{i \in I,\, b_i \neq 0} \sum_{j \in J} \frac{b_i}{j} \tag{9}$$

Step 2. Now convert Ω into conjunctive normal form.

Step 3. For simplification of Ω

Using conjunctive operator $$\frac{c}{l} . \frac{d}{k} = \begin{cases} \max\ {}^{(c,d)}\!/_{l} & \text{if } l = k \\ \text{unchanged} & \text{if } l \neq k \end{cases} \tag{10}$$

and using disjunctive operator

$$\frac{c_1}{l_1} . \frac{c_2}{l_2} .... \frac{c_m}{l_m} + \frac{d_1}{l_1} . \frac{d_2}{l_2} ..... \frac{d_m}{l_m} = \begin{cases} \frac{c_1}{l_1} . \frac{c_2}{l_2} .... \frac{c_m}{l_m} \text{ if } c_i \le d_i, & \forall\, i \in I \\ \text{unchanged, otherwise} \end{cases} \tag{11}$$

Step 4. Suppose Ω has s step after the step 3, then $\breve{X}(A,b)$ has s elements which can be computed as:

$$\breve{x}^{(p)} = (x_1^{(p)}, x_2^{(p)}, ..., x_n^{(p)}) \tag{12}$$

where $x_j^{(p)} = c_j^{(p)}$, $j \in J$ and $p = 1,2, ...,s$

3. THE PROPOSED METHODOLOGY

The proposed methodology can be divided into two parts as:

Part I : In this part, we characterize the objective function

$$Z = \{z = c^T x, \mid x \in X(A,b)\} \tag{13}$$

If $c^T \ge 0$, then the linear function $z = c^T\ x$ is monotonically increasing and positive say $\dot{z}$ and if $c^T \le 0$ is a monotonically decreasing and negative over $X(A,b)$ say $\ddot{z}$, then $z = \dot{z} + \ddot{z}$. For any given $c^T = (c_1, c_2, ..., c_n) \in R^n$, we define $\dot{c}^T = (\dot{c}_1, \dot{c}_2, ..., \dot{c}_n)$ and $\ddot{c}^T = (\ddot{c}_1, \ddot{c}_2, ..., \ddot{c}_n)$ such that

$$\dot{c}_j = \begin{cases} c_j & \text{if} \quad c_j \ge 0 \\ 0 & \text{if} \quad c_j < 0 \end{cases}$$

$$\ddot{c}_j = \begin{cases} 0 & \text{if} \quad c_j \ge 0 \\ c_j & \text{if} \quad c_j < 0 \end{cases} \tag{14}$$

obviously $c^T = \dot{c}^T + \ddot{c}^T$, thus for any $x \in X(A,b)$, $z = \dot{z} + \ddot{z}$ and hence

$$\max z = \max \dot{z} + \max \ddot{z} \tag{15}$$

Lemma 3. If $c_j \le 0, \forall j \in J$, then $\max \ddot{z} = \ddot{c}^T \breve{x}^*$.

Proof. Since $\breve{x}_0 \le x \le \hat{x}$ and $c^T \le 0$, therefore $c^T \hat{x} \le c^T x \le c^T \breve{x}_0$ and $\breve{x}^*$ will be such that $c^T \breve{x}^* = \max\{c^T \breve{x} , \mid \breve{x} \in X(A,b)\}$, so $\max \ddot{z} = \ddot{c}^T \breve{x}^*$.

Lemma 4. If $c_j \ge 0, \forall j \in J$, then $\max \dot{z} = \dot{c}^T \hat{x}$.

Proof. $\forall\ x \in X(A,b)$, $0 \le x \le \hat{x}$ and $c^T \hat{x} \ge c^T x$, therefore $\max \dot{z} = \dot{c}^T \hat{x}$.

Hence $$\max z = \max \dot{z} + \max \ddot{z} = \dot{c}^T \hat{x} + \ddot{c}^T \breve{x}^* \tag{16}$$

Where optimal solution $x^* = \{x_j \mid j \in J\}$ is the combination of $\breve{x}^*$ and $\hat{x}$ as:-

$$x_j^* = \begin{cases} \hat{x}_j & \text{if} \quad c_j < 0 \\ \breve{x}^* & \text{if} \quad c_j \ge 0 \end{cases} \qquad \forall\, j \in J \tag{17}$$

Lemma 5. If $X(A,b) \neq \varphi$ and x^* is computed according to Eq.(16) then x^* is an optimal solution of Eq.(1) with an optimal value

$$z^* = c^T x = \sum_{j=1}^{n} (\dot{c}^T \hat{x}_j + \ddot{c}^T \breve{x}_j^*) \tag{18}$$

Proof. Since $\breve{x} \le x \le \hat{x}$, then $z = \dot{z} + \ddot{z} = \dot{c}^T x + \ddot{c}^T x \le \dot{c}^T \hat{x} + \ddot{c}^T \breve{x} \le \dot{c}^T \hat{x} + \ddot{c}^T \breve{x}^*$, therefore x^* is an optimal solution of Eq.(1) with value $z^* = c^T x^*$.

Part II: In this part, we use fuzzy programming approach for the solution of multiobjective programming problem Eq.(1) with FRE's as constraints. For problem Eq.(1), we define the distance function d as

$$d_q(x) = \left|\frac{\overline{Z}_q - Z_q(x)}{\{\sum c_{qj}^2\}^{1/2}}\right| = \left|g_q^* - g_q(x)\right| \qquad (19)$$

where $g_q^* = \dfrac{\overline{Z}_q}{\{\sum c_{qj}^2\}^{1/2}}$, $\quad g_q(x) = \dfrac{Z_q(x)}{\{\sum c_{qj}^2\}^{1/2}}$

$\overline{Z}_q$ is the maximum value of $Z_q(x)$. At $x = \overline{x}$ (ideal point x – space) as taken by Gupta and Chakraborty (1997), $d_q = 0$ and $x = \underline{x}$ (nadir point in x – space), we get the maximum value of d_q as:-

$$\overline{d}_q = \left|\frac{\overline{Z}_q - \underline{Z}_q}{\{\sum c_{qj}^2\}^{1/2}}\right| \qquad q = 1,2,\ldots,k$$

Thus vector maximum problem Eq.(1) can be modeled as

Find an action $x \in S$

which minimize $\quad \text{Max}\ \{\left|g_q^* - g_q(x)\right|, \quad q = 1, 2, \ \ldots\ldots k\}$

where $\quad S = \{\ x : A o x = b,\ x \geq 0\ \}$ (20)

To reduce further, we define the membership function $\mu_q(d_q(x))$ as

$$\mu_q(d_q(x)) = \begin{cases} 0 & \text{if } d_q(x) \geq p \\ \dfrac{(p - d_q(x)}{p} & \text{if } 0 \leq d_q(x) < p \\ 1 & \text{if } d_q(x) \leq 0 \end{cases}$$

where $\quad p = \text{Sup}\ \{\overline{d}_q\ \} \quad \forall\ q = 1,2,\ldots,k$

If λ be the maximum value of all $\mu_q(d_q(x))$, then

$$d_q(x) \leq -p.\lambda + p$$

i.e. $$\frac{\overline{Z}_q - Z_q(x)}{\{\sum c_{ij}^2\}^{1/2}} \leq -p.\lambda + p$$

$$-Z_q(x) + p.\{\sum c_{ij}^2\}^{1/2}.\lambda \leq p.\{\sum c_{ij}^2\}^{1/2} - \overline{Z}_q \qquad \forall\ q = 1,2,\ldots,k$$

Hence the problem Eq.(1) reduces to

Max λ

subject to $-Z_q(x) + p.\{\sum c_{ij}^2\}^{1/2}.\lambda \leq p.\{\sum c_{ij}^2\}^{1/2} - \overline{Z}_q, \qquad \forall\ q = 1,2,\ldots,k$

$A\ o\ x = b$

and $\quad \lambda, x \geq 0$ (21)

4. THE PROPOSED METHODOLOGY AND PARETO OPTIMAL SOLUTION

Within the scope of multiobjective decision making theory, the pareto optimality is a necessary condition in order to guarantee the rationality of a decisions. Therefore a "reasonable' solution to multiobjective programming problem should be pareto optimal.

Here the problem of solving multiobjective programming problem with fuzzy relational equations is addressed. Several methods have been proposed in literature for obtaining efficient solution (pareto solution) of this problem. We propose here a general procedure to obtain compromise optimal solution (Compromise efficient solution) which is also a pareto optimal solution of the considered problem.

In order to prove this fact, let us consider some related equivalent definitions as:

Definition 1. Pareto Optimal Solution (Efficient Solution)

$x^0 \in X$ is an efficient solution to multi objective programming problem Eq.(1) if and only if there exists no other $x \in X$ such that $Z_q \geq Z_q^0$ for all *q=1,2,…,k* and $Z_q > Z_q^0$ for at least one *q*.

Definition 2. Compromise Efficient Solution

For problem Eq.(1), a compromise efficient solution is an efficient solution selected by the decision maker (DM) as being the best solution where the selection is based on the DM's explicit or implicit criteria.

Zeleny (1982) as well as most authors describes the act of finding a compromise solution to problem Eq.(2) as "… an effort or emulate the ideal solution as closely as possible".

In present problem it is well established that the solution set X contains a unique maximum solution $\hat{x}$ and it may contain several minimal solutions $\check{x} \in X(A,b)$ corresponding to each objective function. Thus the solution set X is fully characterized by the set of all maximum solutions $\hat{x}_j$ $\forall j \in J$ and all minimal solutions $x_j^{(p)}$ for all objective functions.

Since in the proposed methodology which works for the minimization of the perpendicular distances between two hyper planes where $\overline{Z_q}$ and $\underline{Z_q}$ are the maximum and minimum value of the function $Z_q(x)$ and can be calculated using maximum and minimal solutions respectively. Similarly distance function $d_q(x)$ can be calculated and a suitable membership function has been defined. Then minimization of perpendicular distances affect the fuzzy parameters used. As a result the compromise optimal solution is obtained.

Compromise solution depends on the choice of lowest point (lowest justifiable value) of the objective functions. When the justifiable value changes, the compromise solution also changes. In the figure 1, if Z^0 be the ideal point and N and N' be the two different minimum aspiration levels and their compromise solution are Z_1 and Z_2 respectively, because NZ^0 and $N'Z^0$ are the direction in which the decision parameter λ maximizes. In our methodology to find minimum aspiration level we have used minimum value of each objective function. This point is the ideal point of the vector minimization problem of the same objective functions with same constraints which generally lies outside the feasible region. Knowing the lowest point and zenith point (ideal point), we can find the direction of the decision parameter λ in which λ maximizes so that each objective function gets equal importance in the optimization process.

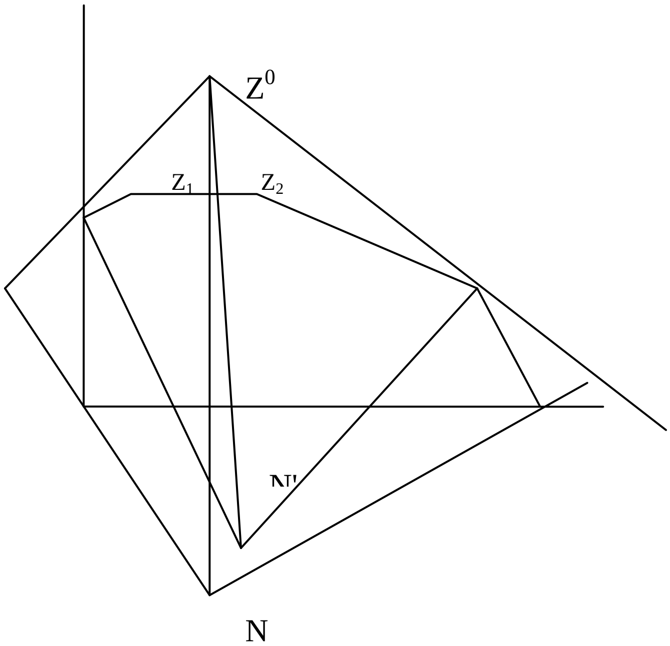

Figure 1. Selection of ideal point

It is obvious from figure that there is no feasible solution with $Z_q \geq Z_q^0$ for all *q=1,2,…,k* and $Z_q > Z_q^0$ for at least one *q*. This proves that the proposed methodology gives pareto optimal solution for multiobjective programming problem with FRE's.

5. ALGORITHM

Step 1. Compute $\hat{x}_j = \bigwedge_{i=1}^{m} (a_{ij} @ b_i), \quad \forall j \in J$ according to Eq.(7) and construct maximum solution $\hat{x}$.

This step is also constructed by the following computer program in C programming:

```
#include<stdio.h>
#include<conio.h>
void main()
{
float a[10][10],b[10],c[10],x[10];
int m,n,i,j,k;
clrscr();
```

```
printf("Enter an integer value for rows in matrix : ");
scanf("%d",&m);
printf("Enter an integer value for cols in matrix : ");
scanf("%d",&n);
printf("Enter %d values for first matrix :\n",m*n);
for(i=0;i<m;i++)
{
for(j=0;j<n;j++)
{
scanf("%f",&a[i][j]);
}
}
printf("Enter %d values for second matrix :\n",m);
for(i=0;i<m;i++)
scanf("%f",&b[i]);
for(j=0;j<n;j++)
{
for(i=0;i<m;i++)
{
if(a[i][j]<=b[i])
{
x[i]=1;
}
else
x[i]=b[i];
}
c[j]=x[0];
for(i=1;i<m;i++)
{
if(c[j]>x[i])
c[j]=x[i];
}
}
printf("The Answer Matrix is following ..........\n");
for(j=0;j<n;j++)
printf("%f\t",c[j]);
getch();
}
```

Step 2. Check the feasibility $A o \hat{x} = b$, if yes, go to next step, otherwise stop and the problem has no solution.

Step 3. Find the index set $J_i = \{ j \in J \mid a_{ij} o x_j = b_i \}, \forall i \in I.$

Step 4. Find the minimal solution set $\breve{X}(A,b)$ as given in section 3.

Step 5. Define average cost vector $\dot{c}^T$ and $\ddot{c}^T$ according to (14).

Step 6. Compute $\dot{z}_q = \dot{c}_q^T \hat{x}, \quad \forall q = 1, 2, ...k$ and $\ddot{z}_q = \ddot{c}_q^T \breve{x}$ for all minimal solutions.

Step 7. Calculate for all minimal solutions $Z_q = \dot{z}_q + \ddot{z}_q$.

Step 8. Compute $\overline{Z}_q = \dot{z}_q + \ddot{z}_q$ for maximum $\ddot{z}_q$ and $\underline{Z_q} = \dot{z}_q + \ddot{z}_q$ for minimum $\ddot{z}_q$.

Step 9. Calculate $\dfrac{\overline{Z}_q - Z_q(x)}{\{\sum c_{ij}^2\}^{1/2}} \qquad \forall q = 1, 2, ..., k$

Step 10. Calculate $p = \sup \{ \overline{d}_q \}$

Step 11. Solve the reduced problem

$$\begin{aligned}
&\text{Max } \lambda \\
&\quad \text{subject to } -Z_q(x) + p.\{\textstyle\sum c_{ij}^2\}^{1/2}.\lambda \le p.\{\textstyle\sum c_{ij}^2\}^{1/2} - \overline{Z}_q, \qquad \forall q = 1,2,...,k \\
&\qquad\qquad A \; o \; x = b \\
&\quad \text{and} \qquad \lambda, x \ge 0
\end{aligned}$$

and required compromise optimal solution of MOLP obtained.

6. EXAMPLE

Consider the problem Eq.(1) with

$$A = \begin{bmatrix} 0.5 & 0.8 & 0.9 & 0.3 & 0.85 & 0.4 \\ 0.2 & 0.2 & 0.1 & 0.95 & 0.1 & 0.8 \\ 0.8 & 0.8 & 0.4 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$$

with $m = 4$ and $n = 6$ and $C_1^T = (3,4,1,1,-1,5)$, $C_2^T = (1,1,1,1,-1,1)$, $\mathrm{b} = (.85, 0.6, 0.5, 0.1)^T$

Step 1. For the above given A and b , using constructed program, we find $\hat{\mathrm{x}} = A@\mathrm{b} = (0.5, 0.5, 0.85, 0.6, 1.0, 0.6)$ which is supported by output as
Enter an integer value for rows in matrix: 4
Enter an integer value for cols in matrix: 6
Enter 24 values for first matrix:
.5 .8 .9 .3 .85 .4
.2 .2 .1 .95 .1 .8
.8 .8 .4 .1 .1 .1
.1 .1 .1 .1 .1 0
Enter 4 values for second matrix:
0.85 0.6 0.5 0.1
The Answer Matrix is following..........
0.500000 0.500000 0.850000 0.600000 1.000000 0.600000

Step 2. $\hat{x}$ is feasible, Since $A o \mathrm{x} = \mathrm{b}$.Thus $\mathrm{x}(A, b) \neq \phi$.

Step 3. For $I = \{1,2,3,4\}$, $J_1 = \{3,5\}$, $J_2 = \{4,6\}$, $J_3 = \{1,2\}$, $J_4 = \{1,2,3,4,5,6\}$

Step 4. $\mathrm{s} = 8$ and the minimal solutions are

$$\breve{\mathrm{x}}^{(1)} = (0.0, 0.5, 0.0, 0.0, 0.85, 0.6)$$
$$\breve{\mathrm{x}}^{(2)} = (0.0, 0.5, 0.0, 0.6, 0.85, 0.0)$$
$$\breve{\mathrm{x}}^{(3)} = (0.0, 0.5, 0.85, 0.0, 0.0, 0.6)$$
$$\breve{\mathrm{x}}^{(4)} = (0.0, 0.5, 0.85, 0.6, 0.0, 0.0)$$
$$\breve{\mathrm{x}}^{(5)} = (0.5, 0.0, 0.0, 0.0, 0.85, 0.6)$$
$$\breve{\mathrm{x}}^{(6)} = (0.5, 0.0, 0.0, 0.6, 0.85, 0.0)$$
$$\breve{\mathrm{x}}^{(7)} = (0.5, 0.0, 0.85, 0.0, 0.0, 0.6)$$
$$\breve{\mathrm{x}}^{(8)} = (0.5, 0.0, 0.85, 0.6, 0.0, 0.0)$$

Step 5. $\dot{\mathrm{c}}_1^T = (3,4,1,1,0,5)$ and $\ddot{\mathrm{c}}_1^T = (0,0,0,0,-1,0)$
$\dot{\mathrm{c}}_2^T = (1,1,1,1,0,1)$ and $\ddot{\mathrm{c}}_2^T = (0,0,0,0,-1,0)$

Step 6. $\dot{\mathrm{z}}_1 = \dot{c}_1^T \hat{\mathrm{x}} = (3,4,1,1,0,5)\,(0.5, 0.5, 0.85, 0.6, 1.0, 0.6) = 7.95$
$\dot{\mathrm{z}}_2 = \dot{c}_2^T \hat{\mathrm{x}} = (1,1,1,1,0,1)\,(0.5, 0.5, 0.85, 0.6, 1.0, 0.6) = 3.05$
For all minimal, solutions (computed in step 4)
the values of $\ddot{\mathrm{z}}_1 = \ddot{c}_1^T \breve{\mathrm{x}}$ are -0.85, -0.85, 0.0, 0.0, -0.85, -0.85, 0.0, 0.0. and the values of $\ddot{\mathrm{z}}_2 = \ddot{c}_2^T \breve{\mathrm{x}}$ are -0.85, -0.85, 0.0, 0.0, -0.85, -0.85, 0.0, 0.0 respectively.

Step 7. The values of objective functions $\mathrm{z}_1 = \dot{\mathrm{z}}_1 + \ddot{\mathrm{z}}_1$ and $\mathrm{z}_2 = \dot{\mathrm{z}}_2 + \ddot{\mathrm{z}}_2$ with corresponding solutions respectively are

$x^{*(1)} = (0.5, 0.5, 0.85, 0.6, 0.85, 0.6), \quad z_1 = 7.10, \quad z_2 = 2.20$

$x^{*(2)} = (0.5, 0.5, 0.85, 0.6, 0.85, 0.6), \quad z_1 = 7.10, \quad z_2 = 2.20$

$x^{*(3)} = (0.5, 0.5, 0.85, 0.6, 0.0, 0.6), \quad z_1 = 7.95, \quad z_2 = 3.05$

$x^{*(4)} = (0.5, 0.5, 0.85, 0.6, 0.0, 0.6), \quad z_1 = 7.95, \quad z_2 = 3.05$

$x^{*(5)} = (0.5, 0.5, 0.85, 0.6, 0.85, 0.6), \quad z_1 = 7.10, \quad z_2 = 2.20$

$x^{*(6)} = (0.5, 0.5, 0.85, 0.6, 0.85, 0.6), \quad z_1 = 7.10, \quad z_2 = 2.20$

$x^{*(7)} = (0.5, 0.5, 0.85, 0.6, 0.0, 0.6), \quad z_1 = 7.95, \quad z_2 = 3.05$

$x^{*(8)} = (0.5, 0.5, 0.85, 0.6, 0.0, 0.6), \quad z_1 = 7.95, \quad z_2 = 3.05$

Step 8. Max.(z_1)= $\overline{z_1} = 7.95$; Max.(z_2)= $\overline{z_2} = 3.05$

Min.(z_2)= $\underline{z_1} = 7.10$; Min.(z_2)= $\underline{z_2} = 2.20$

Step 9. $\overline{d}_1 = \left| \frac{\overline{Z}_1 - \underline{Z}_1}{\{\sum c_{1j}^2\}^{1/2}} \right| = \left| \frac{7.95 - 7.10}{\sqrt{53}} \right| = 0.1167$

$\overline{d}_2 = \left| \frac{\overline{Z}_2 - \underline{Z}_2}{\{\sum c_{2j}^2\}^{1/2}} \right| = \left| \frac{305 - 2.20}{\sqrt{6}} \right| = 0.3470$

Step 10. $p = \sup \{ \overline{d}_q \} = 0.1167$

Step 11. According to Eq(21), the problem reduces to

Max. λ

Subject to $-3x_1 - 4x_2 - x_3 - x_4 + x_5 - x_6 + 0.85\lambda \le -7.10$

$-x_1 - x_2 - x_3 - x_4 + x_5 - x_6 + 0.286\lambda \le -2.764$

$A o\, x = b$

and $x \ge 0$

Solving this L.P.P using the solutions obtained in step 7, the compromise solution of MOLPP is $x^* = x^{*(3)}, x^{*(4)}, x^{*(7)}, x^{*(8)} = (0.5, 0.5, 0.85, 0.6, 0.0, 0.6)$, with values $z_1 = 7.95$, $z_2 = 3.05$, and λ=2.8235.

7. CONCLUSION

An effort has been made to solve a multiobjective programming problem with FRE's as constraints and the compromise optimal solution can be obtained by proposed methodology which depends upon the value of λ as well as the complexity of FRE's .The main difficulty with the proposed methodology is to obtain feasible solution set of FRE's. However, this computational work can be reduced using computer program. Also the complexity in equations can be reduced using stepwise procedure for finding out all minimal solutions for the given system of FRE's.

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