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An Improved Recurrence Method for Simple Continuous Linear Programs

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Abstract—In this paper we discuss a special class of continuous linear programs called simple continuous linear programs (*SP*). In our recent paper [Wen et al. (2009)], we proposed a recurrence algorithm for solving (*SP*). The major computational works in the proposed algorithm are finding the global minimal values of the given continuous functions on 2^n intervals. However, when n becomes large, it could be rather time-consuming. An improved numerical method for finding approximate solutions for (*SP*) is proposed to overcome the computational bottleneck.

Keyword—Infinite-dimensional optimization problems; Continuous linear program;

1. INTRODUCTION

In this paper we discuss a class of infinite-dimensional optimization problems called simple continuous linear programs (*SP*) and described below. Let $T > 0$, and let $L_+^\infty[0,T]$ be the set of nonnegative real valued, Lebesgue measurable, essentially bounded functions on the closed interval [0, *T*]. The simple continuous linear programs are defined as follows:

$$(SP): \quad \text{maximize} \quad \int_0^T f(t)x(t)dt$$
$$\text{subject to} \quad \beta \cdot x(t) - \int_0^t \gamma \cdot x(s)ds \le g(t), \forall\, t \in [0,T]$$
$$x(t) \in L_+^\infty[0,T],$$

where $x(t)$ is the decision variable, $f(t)$ and $g(t)$ are given functions and β, γ are given constants. The dual problem (*DSP*) of (*SP*) is defined as follows:

$$(DSP): \quad \text{minimize} \quad \int_0^T g(t)w(t)dt$$
$$\text{subject to} \quad \beta \cdot w(t) - \int_t^T \gamma \cdot x(s)ds \ge f(t), \forall t \in [0,T]$$
$$w(t) \in L_+^\infty[0,T].$$

The simple continuous linear programs are special cases of the so called continuous time linear programs (*CLP*) which were first proposed by [Bellman (1957)]. In the literature, (*CLP*) has been studied by a number of authors whose work can be loosely divided into two areas, those concerned with establishing strong duality theorems and those concerned with computational methods. One can consult [Tyndall (1965), Levinson (1966), Grinold (1969), Lehman (1954), Drews (1974), Hartberger (1974), Segers (1974), Anstreicher (1983), Buie and Abrham (1973)]. The model of (*CLP*) has wide range applications, but is notoriously difficult to solve in general. Recently, [Wen et al. (2009)] proposed a practical and efficient method for finding approximate solutions of (*SP*) with $\beta = \gamma = 1$. Like the approach in [Buie and Abrham (1973)], it is a discrete approximation algorithm. The major computational works in each iteration of proposed algorithm are finding the global minimal values of the given functions *f* and *g* on each subinterval $[\frac{i-1}{2^n}T, \frac{i}{2^n}T]$ for $1 \le i \le 2^n$. However, when n becomes large, it could be rather time-consuming. To overcome the computational bottleneck, we will propose an improved numerical method for finding the approximate values of problems (*SP*). Besides, we will further show how to construct the feasible approximate solution of (*SP*).

The remainder of this paper is organized as follows. In section 2, we extend the method proposed in [Wen et al. (2009)] to the present problem (*SP*). In section 3, we propose an improved method for finding the approximate solutions of (*SP*).

For the reader's convenience, we adopt the following notations. Let *F*(*P*) and *V*(*P*) denote the feasible set and optimal

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value of a linear programming problem (*P*), respectively.; and the superscript "T" denotes the transpose operation.

2. PRELIMINARY RESULTS

For the remainder of this paper, we make the following assumptions for the given constant β, γ and the given functions *f*(*t*) and *g*(*t*) .

Assumption 1.

(1) $\beta > 0$ and $\gamma \geq 0$;

(2) *f*(*t*) and *g*(*t*) are given continuous functions and $g(t) > 0$ for all $t \in [0, T]$.

We note that the results developed in [Wen et al. (2009)] can be easily extended to the present problem (*SP*). As in [Wen et al. (2009)], (*SP*) and (*DSP*) have the weak duality property described as follows.

Lemma 1. (Weak duality)

(1) *F*(*SP*)≠∅ and *F*(*DSP*)≠∅.

(2) If *x*(*t*) and *w*(*t*) are feasible for (*SP*) and (*DSP*), respectively, then $\int_0^T f(t)x(t)dt \leq \int_0^T g(t)w(t)dt.$

To solve (*SP*), following [Wen et al. (2009)], for each $n \in \mathbb{N}$, let $P_{2^n} = \{0, \frac{1}{2^n}T, \cdots, \frac{2^n-1}{2^n}T, T\}$ be a partition on [0,T]. Let $b_i^{(n)} = \min\{g(x) : x \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T]\}$ and $c_i^{(n)} = \min\{f(x) : x \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T]\}$ for $1 \leq i \leq 2^n$. Consider the following finite linear program:

(*P_n*): maximize $\sum_{i=1}^{2^n} \frac{Tc_i^{(n)} x_i}{2^n}$

subject to
$$\begin{bmatrix} \beta & & 0 \\ & \ddots & \\ \frac{-\gamma}{2^n}T & & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{2^n} \end{bmatrix} \leq \begin{bmatrix} b_1^{(n)} \\ \vdots \\ b_{2^n}^{(n)} \end{bmatrix}$$
$$x_i \geq 0, \forall i = 1, \cdots, 2^n.$$

The dual problem (D_n) of (P_n) is defined as follows:

(D_n): minimize $\sum_{i=1}^{2^n} \frac{Tb_i^{(n)} y_i}{2^n}$

subject to
$$\begin{bmatrix} \beta & & \frac{-\gamma}{2^n}T \\ & \ddots & \\ 0 & & \beta \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_{2^n} \end{bmatrix} \geq \begin{bmatrix} c_1^{(n)} \\ \vdots \\ c_{2^n}^{(n)} \end{bmatrix}$$
$$y_i \geq 0, \forall i = 1, \cdots, 2^n.$$

By the same method as in [Wen et al. (2009)], we can derive a recurrence method for solving (D_n) described as follows.

Lemma 2.

Suppose that Assumption 1 holds. Then

(i). $F(P_n)$≠∅, $F(D_n)$≠∅ and $V(P_n) = V(D_n)$.

(ii). Let the vector $\overline{\mathbf{w}}^{(n)} = (\overline{w}_1^{(n)}, \overline{w}_2^{(n)}, \cdots, \overline{w}_{2^n}^{(n)})^{\mathrm{T}}$ be defined by

$$\begin{cases} \overline{w}_{2^n}^{(n)} = \max\left\{\dfrac{c_{2^n}^{(n)}}{\beta}, 0\right\} \\ \overline{w}_i^{(n)} = \max\left\{\dfrac{c_i^{(n)}}{\beta} + \dfrac{\gamma T}{2^n \beta}\displaystyle\sum_{l=i+1}^{2^n} \overline{w}_l^{(n)}, 0\right\}, i = 2^n - 1, 2^n - 2, \cdots, 2, 1. \end{cases} \tag{1}$$

Then $\overline{\mathbf{w}}^{(n)}$ is an optimal solution of (D_n), and for $1 \leq i \leq 2^n$

$$0 \leq \overline{w}_i^{(n)} \leq \frac{L}{\beta}(1 + \frac{\gamma T}{2^n \beta})^{2^n - i} \leq \frac{L}{\beta}(1 + \frac{\gamma T}{2^n \beta})^{2^n} \leq \frac{L}{\beta} \exp\left(\frac{\gamma T}{\beta}\right), \tag{2}$$

where $L = \max_{0 \leq t \leq T}\{f(t), 0\}$

Besides, we also have the following result.

Lemma 3. We have $\lim_{n\to\infty} V(P_n) = V(SP)$, $\lim_{n\to\infty} V(D_n) = V(DSP)$ and *V*(*SP*) = *V*(*DSP*).

Moreover, we let

$$\delta_{2^n} = \max\{\frac{T}{2^n}\overline{w}_i^{(n)} : i = 1, \cdots, 2^n\}, \tag{3}$$

$$\rho = \max\left\{\frac{\gamma}{\beta}, \frac{1}{\beta}\right\}, \tag{4}$$

$$\sup_{t\in[0,T]} \{ f(t) - f_n(t)\} = \varepsilon_n , \tag{5}$$

and

$$\sup_{t\in[0,T]} \{g(t) - g_n(t)\} = \bar{\varepsilon}_n , \tag{6}$$

where $f_n(t)$ and $g_n(t)$ are step functions defined as follows:
$f_n(T) = b_{2^n}^{(n)}$, $f_n(t) = b_i^{(n)}$, if $t \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T)$ for $1 \le i \le 2^n$, and $g_n(T) = b_{2^n}^{(n)}, g_n(t) = b_i^{(n)}$, if $t \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T)$ for $1 \le i \le 2^n$. Note that, by (2) and (3), we have

$$0 \le \delta_{2^n} \le \frac{LT}{2^n\beta}(1+\frac{\gamma T}{2^n\beta})^{2^n} \le \frac{LT}{2^n\beta}\exp\left(\frac{\gamma T}{\beta}\right). \tag{7}$$

Then, by the same arguments as in [Wen et al. (2009)], it is not difficult to obtain the following results.
Lemma 4. Under Assumption 1, we have $0 \le V(SP) - V(P_n) = V(DSP) - V(D_n) \le \xi_n$, where

$$\xi_n = \bar{\varepsilon}_n\delta_{2^n}\left(2^n + \exp(\rho T) - 1\right) + (\varepsilon_n + \delta_{2^n})\int_0^T \rho\exp(\rho(T-t))g(t)dt . \tag{8}$$

Using the constructive method proposed in [Wen et al. (2009)], we can find the approximate solutions of (*SP*) by virtue of the optimal solution of (P_n). Besides, the error bounds between the optimal value of (*SP*) and the objective value of the approximate solutions can be estimated.
Lemma 5. Let $\bar{\mathbf{x}}^{(n)} = (\bar{x}_1^{(n)}, \bar{x}_2^{(n)}, \cdots, \bar{x}_{2^n}^{(n)})^{\mathrm{T}}$ be an optimal solution of (P_n). Define $\bar{x}^{(n)}(t)$ by

$$\bar{x}^{(n)}(t) = \begin{cases} \bar{x}_i^{(n)}, \text{ if } t \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T) \text{ for } 1 \le i \le 2^n \\ \bar{x}_{2^n}^{(n)}, \text{ if } t = T. \end{cases} \tag{9}$$

Then $\bar{x}^{(n)}(t)$ is a feasible solution of (*SP*) and $0 \le V(SP) - \int_0^T f(t)\bar{x}^{(n)}(t)dt \le \xi_n$.

To summarize the above results we have the following algorithm:
Algorithm 1:
Let μ be a prescribed small positive number and an initial number $n_0 \in \mathbb{N}$ be given.
Step 1. Set $n \leftarrow n_0$.
Step 2. Calculate the vector $\bar{w}^{(n)}$ defined as in Eq.(1) and evaluate the error bound ξ_n defined as in Eq.(8).
Step 3. If $\xi_n \le \mu$ then stop and evaluate the value $\sum_{i=1}^{2^n}\frac{T}{2^n}b_i^{(n)}\bar{w}_i^{(n)}$ as an approximate value with error bound ξ_n. And the function $\bar{x}^{(n)}(t)$ defined as in Eq.(9) is an approximate solution of (*SP*) with error bound ξ_n. Otherwise, update *n*← *n*+1 and go to Step 2.

3. AN IMPROVED APPROXIMATION METHOD FOR SOLVING (SP)

In the above approach, the major computational works in each iteration are finding the global minimal values of *f* and *g* on each subinterval $[\frac{i-1}{2^n}T, \frac{i}{2^n}T]$ for $1 \le i \le 2^n$. However, when *n* becomes large, it could be rather time-consuming. To overcome this computational bottleneck, instead of taking the minimal values of *f* and *g* on every $[\frac{i-1}{2^n}T, \frac{i}{2^n}T]$, we take the values at middle points of $[\frac{i-1}{2^n}T, \frac{i}{2^n}T]$. That is, we let, for $1 \le i \le 2^n$,

$$\tilde{b}_i^{(n)} = g(\frac{2i-1}{2^{n+1}}T) \text{ and } \tilde{c}_i^{(n)} = f(\frac{2i-1}{2^{n+1}}T). \tag{10}$$

Consider the corresponding linear programming problem defined as follows:

$$(\tilde{P}_n) : \text{maximize} \quad \sum_{i=1}^{2^n}\frac{T\,\tilde{c}_i^{(n)}\,x_i}{2^n}$$

subject to $$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ \frac{-\gamma}{2^n}T & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{2^n} \end{bmatrix} \le \begin{bmatrix} \tilde{b}_1^{(n)} \\ \vdots \\ \tilde{b}_{2^n}^{(n)} \end{bmatrix}$$

$$x_i \ge 0, \forall i = 1, \cdots, 2^n.$$

The dual problem $(\tilde{D}_n)$ of $(\tilde{P}_n)$ is defined as follows:

$(\tilde{D}_n)$: minimize $$\sum_{i=1}^{2^n} \frac{T\,\tilde{b}_i^{(n)}\, y_i}{2^n}$$

subject to $$\begin{bmatrix} 1 & & \frac{-\gamma}{2^n}T \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_{2^n} \end{bmatrix} \ge \begin{bmatrix} \tilde{c}_1^{(n)} \\ \vdots \\ \tilde{c}_{2^n}^{(n)} \end{bmatrix}$$

$$y_i \ge 0, \forall i = 1, \cdots, 2^n.$$

There is a recurrence method for solving $(\tilde{D}_n)$ which is similar with lemma 2.

Theorem 1. Suppose that Assumption 1 holds. Then

(i). $F(\tilde{P}_n) \ne \varnothing$, $F(\tilde{D}_n) \ne \varnothing$ and $V(\tilde{P}_n) = V(\tilde{D}_n)$.

(ii). The vector $\tilde{\mathbf{w}}^{(n)} = (\tilde{w}_1^{(n)}, \tilde{w}_2^{(n)}, \cdots, \tilde{w}_{2^n}^{(n)})$ defined by

$$\begin{cases} \tilde{w}_{2^n}^{(n)} = \max\left\{ \dfrac{\tilde{c}_{2^n}^{(n)}}{\beta}, 0 \right\} \\ \tilde{w}_i^{(n)} = \max\left\{ \dfrac{\tilde{c}_i^{(n)}}{\beta} + \dfrac{\gamma T}{2^n \beta} \displaystyle\sum_{l=i+1}^{2^n} \tilde{w}_l^{(n)}, 0 \right\}, i = 2^n - 1, 2^n - 2, \cdots, 2, 1, \end{cases} \tag{11}$$

is an optimal solution of $(\tilde{D}_n)$, and for $1 \le i \le 2^n$

$$0 \le \tilde{w}_i^{(n)} \le \frac{L}{\beta}(1 + \frac{\gamma T}{2^n \beta})^{2^n - i} \le \frac{L}{\beta}(1 + \frac{\gamma T}{2^n \beta})^{2^n} \le \frac{L}{\beta} \exp\left(\frac{\gamma T}{\beta}\right), \tag{12}$$

where L is defined as in Eq.(2).

We now discuss the relation between $\tilde{\mathbf{w}}^{(n)}$ and $\overline{\mathbf{w}}^{(n)}$. It is obvious that $\tilde{w}_i^{(n)} \ge \overline{w}_i^{(n)}$ for all *i*, since $\tilde{c}_i^{(n)} \ge c_i^{(n)}$. Owing to $\tilde{b}_i^{(n)} \ge b_i^{(n)} > 0$ for all *i*, we have $V(\tilde{D}_n) \ge V(D_n)$. Define

$$d_c^{(n)} = \max_{1 \le i \le 2^n} \Delta c_i^{(n)}, . \tag{13}$$

where, $\Delta c_i^{(n)} = \tilde{c}_i^{(n)} - c_i^{(n)} \ge 0$. Then

$$\tilde{w}_{2^n}^{(n)} = \max\left\{ \frac{c_{2^n}^{(n)} + \Delta c_{2^n}^{(n)}}{\beta}, 0 \right\} \le \max\left\{ \frac{c_{2^n}^{(n)}}{\beta}, 0 \right\} + \frac{\Delta c_{2^n}^{(n)}}{\beta} \le \overline{w}_{2^n}^{(n)} + \frac{d_c^{(n)}}{\beta}$$

and

$$\tilde{w}_{2^n-1}^{(n)} = \max\left\{ \frac{c_{2^n-1}^{(n)} + \Delta c_{2^n-1}^{(n)}}{\beta} + \frac{\gamma T}{2^n \beta} \tilde{w}_{2^n}^{(n)}, 0 \right\}$$

$$\le \max\left\{ \frac{c_{2^n-1}^{(n)} + d_c^{(n)}}{\beta} + \frac{\gamma T}{2^n \beta} (\overline{w}_{2^n}^{(n)} + \frac{d_c^{(n)}}{\beta}), 0 \right\}$$

$$\le \max\left\{ \frac{c_{2^n-1}^{(n)}}{\beta} + \frac{\gamma T}{2^n \beta} \overline{w}_{2^n}^{(n)}, 0 \right\} + \frac{1}{\beta}(1 + \frac{\gamma T}{2^n \beta}) d_c^{(n)}$$

$$= \overline{w}_{2^n-1}^{(n)} + \frac{1}{\beta}(1 + \frac{\gamma T}{2^n \beta}) d_c^{(n)}.$$

Continuous this process, we see that $\overline{w}_{2^n-k}^{(n)} \le \tilde{w}_{2^n-k}^{(n)} \le \overline{w}_{2^n-k}^{(n)} + \frac{1}{\beta}\left(1 + \frac{\gamma T}{2^n \beta}\right)^k d_c^{(n)}, \forall\, k = 0, 1, \cdots, 2^n - 1,$ or equivalently,

$$\overline{w}_i^{(n)} \le \tilde{w}_i^{(n)} \le \overline{w}_i^{(n)} + \frac{1}{\beta}\left(1 + \frac{\gamma T}{2^n \beta}\right)^{2^n - i} d_c^{(n)}, \forall\, i = 1, 2, \cdots, 2^n. \tag{14}$$

Define

$$d_b^{(n)} = \max_{1\le i\le 2^n} \Delta b_i^{(n)}, \tag{15}$$

where, $\Delta b_i^{(n)} = \tilde{b}_i^{(n)} - b_i^{(n)} \ge 0$,

then, by Eq.(14), we have

$$\begin{aligned} V(\tilde{D}_n) &= \frac{T}{2^n}\sum_{i=1}^{2^n} \tilde{b}_i^{(n)}\tilde{w}_i^{(n)} = \frac{T}{2^n}\sum_{i=1}^{2^n}(b_i^{(n)}+\Delta b_i^{(n)})\tilde{w}_i^{(n)} \\ &= \frac{T}{2^n}\sum_{i=1}^{2^n} b_i^{(n)}\tilde{w}_i^{(n)} + \frac{T}{2^n}\sum_{i=1}^{2^n}\Delta b_i^{(n)}\tilde{w}_i^{(n)} \\ &\le \frac{T}{2^n}\sum_{i=1}^{2^n} b_i^{(n)}\overline{w}_i^{(n)} + \frac{T}{2^n}\sum_{i=1}^{2^n} b_i^{(n)}\frac{1}{\beta}(1+\frac{\gamma T}{2^n\beta})^{2^n-i} d_c^{(n)} + \frac{T}{2^n}\sum_{i=1}^{2^n}\Delta b_i^{(n)}\tilde{w}_i^{(n)} \\ &= V(D_n) + \frac{T}{2^n}\sum_{i=1}^{2^n} b_i^{(n)}\frac{1}{\beta}(1+\frac{\gamma T}{2^n\beta})^{2^n-i} d_c^{(n)} + \frac{T}{2^n}\sum_{i=1}^{2^n}\Delta b_i^{(n)}\tilde{w}_i^{(n)}, \end{aligned}$$

which implies that

$$\begin{aligned} & 0 \le V(\tilde{D}_n) - V(D_n) \\ & \le \frac{T}{2^n}\sum_{i=1}^{2^n} b_i^{(n)}\frac{1}{\beta}(1+\frac{\gamma T}{2^n\beta})^{2^n-i} d_c^{(n)} + \frac{T}{2^n}\sum_{i=1}^{2^n}\Delta b_i^{(n)}\tilde{w}_i^{(n)} \\ & \le \frac{T}{2^n}\sum_{i=1}^{2^n} b_i^{(n)}\frac{1}{\beta}(1+\frac{\gamma T}{2^n\beta})^{2^n-i} d_c^{(n)} + \frac{T}{2^n}\sum_{i=1}^{2^n}\Delta b_i^{(n)}\frac{L}{\beta}(1+\frac{\gamma T}{2^n\beta})^{2^n-i} \quad \text{(by (12))} \\ & \le \max_{t\in[0,T]} g(t)\cdot d_c^{(n)}\frac{1}{\beta}\left[(1+\frac{\gamma T}{2^n\beta})^{2^n}-1\right] + d_b^{(n)}\frac{TL}{2^n\beta}\sum_{i=1}^{2^n}(1+\frac{\gamma T}{2^n\beta})^{2^n-i} \\ & \le \max_{t\in[0,T]} g(t)\cdot d_c^{(n)}\frac{1}{\beta}\left[(1+\frac{\gamma T}{2^n\beta})^{2^n}-1\right] + d_b^{(n)}\frac{L}{\gamma}\left[(1+\frac{\gamma T}{2^n\beta})^{2^n}-1\right] \\ & = \left(\frac{1}{\beta}\max_{t\in[0,T]} g(t)\cdot d_c^{(n)} + \frac{L}{\gamma}\cdot d_b^{(n)}\right)\left[(1+\frac{\gamma T}{2^n\beta})^{2^n}-1\right] \end{aligned} \tag{16}$$

Under Assumption 1, $f(t)$ and $g(t)$ are uniformly continuous on [0, T], hence, $d_c^{(n)} \to 0$ and $d_b^{(n)} \to 0$, as $n\to\infty$. Thus $\lim_{n\to\infty} V(\tilde{D}_n) = \lim_{n\to\infty} V(D_n)$. Therefore, by Lemma 3, we have the following result.

Theorem 2. Suppose that Assumption 1 holds, then $\lim_{n\to\infty} V(\tilde{P}_n) = V(SP)$ and $\lim_{n\to\infty} V(\tilde{D}_n) = V(DSP)$.

Moreover, we can also provide an error bound for every approximate value $V(\tilde{D}_n)$ of $V(DSP)$. To see this we make the following assumption.

Assumption 2. f and g satisfy Lipschitz conditions, that is, there exist $\overline{L}$ and $\overline{M}$ such that $|f(t_1)-f(t_2)| \le \overline{L}|t_1-t_2|$ and $|g(t_1)-g(t_2)| \le \overline{M}|t_1-t_2|$ for every $t_1, t_2 \in [0,T]$.

Let L be defined as in Eq.(2) and $M = \max_{t\in[0,T]} g(t)$. Then we have the following result.

Theorem 3. Suppose that Assumption 1 and 2 hold. Then $0 \le |V(\tilde{D}_n) - V(DSP)| \le \max\{\alpha_n, \theta_n\}$, where

$$\alpha_n = \frac{T}{2^{n+1}}(\frac{1}{\beta}\overline{L}M + \frac{1}{\gamma}L\overline{M})\left[(1+\frac{\gamma T}{2^n\beta})^{2^n}-1\right], \tag{17}$$

and

$$\theta_n = \left(\frac{T}{2^n}\right)^2 L\overline{M}\left(1+\frac{\gamma T}{2^n\beta}\right)^{2^n}\left(2^n+\exp(\rho T)+1\right) + \frac{T}{2^n}\left[\overline{L} + \frac{L}{\beta}(1+\frac{\gamma T}{2^n\beta})^{2^n}\right]\int_0^T \rho\exp(\rho(T-t))g(t)dt. \tag{18}$$

Proof. It is easy to see that $0 \le |V(\tilde{D}_n) - V(DSP)| \le \max\{V(DSP)-V(D_n), V(\tilde{D}_n)-V(D_n)\}$, since $V(D_n) \le V(\tilde{D}_n)$ and $V(D_n) \le V(DSP)$. We will show that $V(\tilde{D}_n) - V(D_n) \le \alpha_n$ and $V(DSP) - V(D_n) \le \theta_n$. It is obvious that, under Assumption 2, $d_c^{(n)} \le \frac{T}{2^{n+1}}\overline{L}$ and $d_b^{(n)} \le \frac{T}{2^{n+1}}\overline{M}$. Hence, by Eq.(16), we have $V(\tilde{D}_n) - V(D_n) \le \alpha_n$. On the other hand, recall the definitions of $f^{(n)}(t)$, $g^{(n)}(t)$, ε_n and $\overline{\varepsilon}_n$, we see, under Assumption 2, that

$$0 \le f(t) - f_n(t) \le \frac{T}{2^n}\overline{L} \quad \text{and} \quad 0 \le g(t) - g_n(t) \le \frac{T}{2^n}\overline{M} \quad \text{for every } t\in[0,T],$$

which implies that $\varepsilon_n (= \max_{t\in[0,T]}\{f(t)-f_n(t)\}) \le \frac{T}{2^n}\overline{L}$, $\overline{\varepsilon}_n (= \max_{t\in[0,T]}\{g(t)-g_n(t)\}) \le \frac{T}{2^n}\overline{M}$. Hence, by Lemma 4 and the inequality Eq.(7), we have

$$
\begin{aligned}
&0 \le V(DSP) - V(D_n) \\
&\le \bar{\varepsilon}_n \delta_{2^n} \left(2^n + \exp(\rho T) - 1\right) + (\varepsilon_n + \delta_{2^n}) \int_0^T \rho \exp(\rho(T-t)) g(t) dt \\
&\le \left(\frac{T}{2^n}\right)^2 L\bar{M} \left(1 + \frac{\gamma T}{2^n \beta}\right)^{2^n} \left(2^n + \exp(\rho T) + 1\right) + \frac{T}{2^n}\left[\bar{L} + \frac{L}{\beta}(1 + \frac{\gamma T}{2^n \beta})^{2^n}\right] \int_0^T \rho \exp(\rho(T-t)) g(t) dt \\
&= \theta_n .
\end{aligned}
$$

We complete this proof.

It is remarkable that we can find an optimal solution of $(\tilde{P}_n)$ by virtue of the dual optimal solution $\bar{\mathbf{w}}^{(n)}$ derived by the recursion Eq.(1). By the complementary slackness theorem, if $\mathbf{x}^{(n)} \in F(\tilde{P}_n)$ and $\mathbf{w}^{(n)} \in F(\tilde{D}_n)$, then $\mathbf{x}^{(n)}$ and $\mathbf{w}^{(n)}$ become an optimal solution pair if and only if they satisfy the following equations:

$$
\begin{aligned}
&\left(b_i^{(n)} - \beta x_i^{(n)} + \frac{\gamma T}{2^n} \sum_{l=1}^{i-1} x_l^{(n)} \right) w_i^{(n)} = 0 \text{ and} \\
&\left(\beta w_i^{(n)} - \frac{\gamma T}{2^n} \sum_{l=i+1}^{2^n} w_l^{(n)} - c_i^{(n)} \right) x_i^{(n)} = 0, \qquad (19)\\
&\forall\, i = 1, 2, \cdots, 2^n .
\end{aligned}
$$

Hence, if $\tilde{\mathbf{w}}^{(n)}$ is an optimal solution of $(\tilde{D}_n)$, and the vector $\tilde{\mathbf{x}}^{(n)} = \left(\tilde{x}_1^{(n)}, \tilde{x}_2^{(n)}, \cdots, \tilde{x}_{2^n}^{(n)}\right)^{\mathrm{T}}$ is constructed by the following recursion:

$$
\tilde{x}_i^{(n)} = \begin{cases} 0\ ,\ \text{if } i \notin \Lambda \\ \dfrac{b_i^{(n)}}{\beta} + \dfrac{\gamma T}{2^n \beta} \displaystyle\sum_{l=1}^{i-1} x_l^{(n)},\ \text{if } i \in \Lambda \end{cases} \quad \text{and} \quad \Lambda = \left\{ i : 1 \le i \le 2^n,\ \tilde{w}_i^{(n)} > 0 \right\}.
$$

Then it is not difficult to show that $\tilde{\mathbf{x}}^{(n)}$ is feasible for $(\tilde{P}_n)$ and $\tilde{\mathbf{x}}^{(n)}$ as well as $\tilde{\mathbf{w}}^{(n)}$ satisfy Eq.(19). Therefore, $\tilde{\mathbf{x}}^{(n)}$ is an optimal solution of $(\tilde{P}_n)$.

Moreover, by using $\tilde{\mathbf{x}}^{(n)}$, we can find a corresponding solution of (*SP*). To see this, for $n \in \mathbb{N}$, we define step functions $\tilde{g}_n$ and $\tilde{f}_n$ as follows:

$$
\tilde{g}_n(t) = \begin{cases} \tilde{b}_i^{(n)},\ \text{if } t \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T), 1 \le i \le 2^n \\ \tilde{b}_{2^n}^{(n)},\ \text{if } t = T, \end{cases} \qquad (20)
$$

and

$$
\tilde{f}_n(t) = \begin{cases} \tilde{c}_i^{(n)},\ \text{if } t \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T), 1 \le i \le 2^n \\ \tilde{c}_{2^n}^{(n)},\ \text{if } t = T. \end{cases} \qquad (21)
$$

Note that, under Assumption 2, $0 \le \tilde{f}_n(t) - f_n(t) \le \frac{T\bar{L}}{2^{n+1}}$ and $0 \le \tilde{g}_n(t) - g_n(t) \le \frac{T\bar{M}}{2^{n+1}}$. Define a function $\tilde{x}^{(n)}(t) : [0, T] \to \mathbb{R}$ by

$$
\tilde{x}^{(n)}(t) = \begin{cases} \tilde{x}_i^{(n)},\ \text{if } t \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T), 1 \le i \le 2^n \\ \tilde{x}_{2^n}^{(n)},\ \text{if } t = T. \end{cases}
$$

Then, by the following two cases, we have $\beta \tilde{x}^{(n)}(t) - \int_0^t \gamma \tilde{x}^{(n)}(s) ds \le \tilde{g}_n(t)$ for all $t \in [0, T]$.

Case 1. $t \in [\frac{i-1}{2^n}T, \frac{i}{2^n}T)$ for some $1 \le i \le 2^n$. Then

$$
\beta \tilde{x}^{(n)}(t) - \int_0^t \gamma \tilde{x}^{(n)}(s) ds = \beta \tilde{x}_i^{(n)} - \sum_{j=1}^{i} \frac{\gamma T}{2^n} \tilde{x}_j^{(n)} - \int_{\frac{i-1}{2^n}T}^{t} \gamma \tilde{x}^{(n)}(s) ds \le \beta \tilde{x}_i^{(n)} - \sum_{j=1}^{i} \frac{\gamma T}{2^n} \tilde{x}_j^{(n)} \le \tilde{b}_i^{(n)} = \tilde{g}_n(t).
$$

Case 2. $t = T$. Then

$$
\beta \tilde{x}^{(n)}(T) - \int_0^T \gamma \tilde{x}^{(n)}(s) ds = \beta \tilde{x}_{2^n}^{(n)} - \sum_{j=1}^{2^n} \frac{\gamma T}{2^n} \tilde{x}_j^{(n)} \le \beta \tilde{x}_{2^n}^{(n)} - \sum_{j=1}^{2^n - 1} \frac{\gamma T}{2^n} \tilde{x}_j^{(n)} \le \tilde{b}_{2^n}^{(n)} = \tilde{g}_n(T).
$$

For further work, we let $E=\left\{t\in[0,T]:0\le \tilde{x}^{(n)}(t)\le \frac{Te^T\bar{M}}{2^{n+1}}\right\}$ and construct a measurable function

$$\tilde{\tilde{x}}^{(n)}(t)=\begin{cases}\tilde{x}^{(n)}(t)+\frac{Te^T\bar{M}}{2^{n+1}}, & \text{if } t\in E\\ \tilde{x}^{(n)}(t), & \text{if } t\notin E,\end{cases} \tag{22}$$

Then $\tilde{\tilde{x}}^{(n)}(t)\ge \frac{Te^T\bar{M}}{2^{n+1}}$ for all *t*. Define

$$\hat{x}^{(n)}(t)=\tilde{\tilde{x}}^{(n)}(t)-\frac{T\bar{M}}{2^{n+1}}e^t,\quad \text{for all } t\in[0,T], \tag{23}$$

Then, under proper conditions, we can show that $\hat{x}^{(n)}(t)$ is a feasible solution of (*SP*).

Theorem 4. Suppose that Assumption 1 and 2 hold. If $n\in\mathbb{N}$ satisfying $2^n\ge\frac{\beta Te^T\bar{M}}{M}$, then the function $\hat{x}^{(n)}(t)$ defined in Eq.(23) is feasible for (*SP*). Moreover, we have

$$0\le V(SP)-\int_0^T f(t)\hat{x}^{(n)}(t)dt\le \max\{\alpha_n,\theta_n\}+\gamma_n,$$

where α_n and θ_n are defined as in Eq.(17) and Eq.(18), respectively, and

$$\gamma_n=\frac{\bar{L}T}{2^{n+1}}\int_0^T\tilde{x}^{(n)}(t)dt+\frac{T^2e^T\bar{M}}{2^{2n+1}}\sum_{i=1}^{2^n}\left|\tilde{c}_i^{(n)}\right|+\frac{T^2\bar{L}\bar{M}}{2^{2n+2}}(e^T-1)+\frac{T\bar{M}}{2^{n+1}}(e^{\frac{T}{2^n}}-1)\left|\sum_{i=1}^{2^n}\tilde{c}_i^{(n)}e^{\frac{i-1}{2^n}T}\right|.$$

Proof. Clearly $\hat{x}(t)\ge 0$ for all *t*. We are going to show that $\beta\tilde{\tilde{x}}^{(n)}(t)-\int_0^t\gamma\,\tilde{\tilde{x}}^{(n)}(s)ds\le\tilde{g}_n(t)$ for all $t\in[0,T]$ if *n* satisfies $2^n\ge\frac{\beta Te^T\bar{M}}{M}$. If $t\notin E$ then $\beta\tilde{\tilde{x}}^{(n)}(t)-\int_0^t\gamma\,\tilde{\tilde{x}}^{(n)}(s)ds\le\beta\tilde{x}^{(n)}(t)-\int_0^t\gamma\,\tilde{x}^{(n)}(s)ds\le\tilde{g}_n(t)$. On the other hand, if $t\in E$ then we have

$$\begin{aligned}&\beta\tilde{\tilde{x}}^{(n)}(t)-\int_0^t\gamma\,\tilde{\tilde{x}}^{(n)}(s)ds\le\beta\tilde{\tilde{x}}^{(n)}(t)\\&=\beta\tilde{x}^{(n)}(t)+\frac{\beta Te^T\bar{M}}{2^{n+1}}\\&\le\frac{\beta Te^T\bar{M}}{2^{n+1}}+\frac{\beta Te^T\bar{M}}{2^{n+1}}\\&=\frac{\beta Te^T\bar{M}}{2^n}\le\tilde{g}_n(t)\quad\left(\text{since }\frac{\beta Te^T\bar{M}}{2^n}\le M\le\tilde{g}_n(t).\right)\end{aligned}$$

Hence we obtain, under Assumption 2, that

$$\beta\hat{x}^{(n)}(t)-\int_0^t\gamma\hat{x}^{(n)}(s)ds=\beta\tilde{\tilde{x}}^{(n)}(t)-\int_0^t\gamma\tilde{\tilde{x}}^{(n)}(s)ds-\frac{T\bar{M}}{2^{n+1}}\le\tilde{g}_n(t)-(\tilde{g}_n(t)-g_n(t))=g_n(t)\le g(t).$$

Thus $\hat{x}^{(n)}(t)\in F(SP)$ for all *n* satisfying $2^n\ge\frac{\beta Te^T\bar{M}}{M}$. Note that $\int_0^T\tilde{f}_n(t)\tilde{x}^{(n)}(t)dt=\sum_{i=1}^{2^n}\frac{T\,\tilde{c}_i^{(n)}\,\tilde{x}_i^{(n)}}{2^n}=V(\tilde{P}_n)$ and observe that

$$\begin{aligned}&\left|\int_0^T f_n(t)\hat{x}^{(n)}(t)dt-V(\tilde{P}_n)\right|\\&=\left|\int_0^T f_n(t)\tilde{\tilde{x}}^{(n)}(t)dt-\frac{T\bar{M}}{2^{n+1}}\int_0^T f_n(t)e^tdt-V(\tilde{P}_n)\right|\\&=\left|\int_0^T\left[f_n(t)\tilde{\tilde{x}}^{(n)}(t)-\tilde{f}_n(t)\tilde{\tilde{x}}^{(n)}(t)+\tilde{f}_n(t)\tilde{\tilde{x}}^{(n)}(t)-\tilde{f}_n(t)\tilde{x}^{(n)}(t)\right]dt+\frac{T\bar{M}}{2^{n+1}}\left(-\int_0^T f_n(t)e^tdt+\int_0^T\tilde{f}_n(t)e^tdt-\int_0^T\tilde{f}_n(t)e^tdt\right)\right|\\&\le\left|\int_0^T\left[f_n(t)-\tilde{f}_n(t)\right]\tilde{\tilde{x}}^{(n)}(t)dt\right|+\left|\int_0^T\tilde{f}_n(t)\left[\tilde{\tilde{x}}^{(n)}(t)-\tilde{x}^{(n)}(t)\right]dt\right|+\frac{T\bar{M}}{2^{n+1}}\left|\int_0^T\left[\tilde{f}_n(t)-f_n(t)\right]e^tdt\right|+\frac{T\bar{M}}{2^{n+1}}\left|\int_0^T\tilde{f}_n(t)e^tdt\right|\\&\le\frac{MT}{2^{n+1}}\int_0^T\tilde{\tilde{x}}^{(n)}(t)dt+\frac{Te^T\bar{M}}{2^{n+1}}\int_0^T\left|\tilde{f}_n(t)\right|dt+\frac{T^2\bar{L}\bar{M}}{2^{2n+2}}\int_0^Te^tdt+\frac{T\bar{M}}{2^{n+1}}\left|\int_0^T\tilde{f}_n(t)e^tdt\right|\\&\le\frac{\bar{L}T}{2^{n+1}}\int_0^T\tilde{\tilde{x}}^{(n)}(t)dt+\frac{T^2e^T\bar{M}}{2^{2n+1}}\sum_{i=1}^{2^n}\left|\tilde{c}_i^{(n)}\right|+\frac{T^2\bar{L}\bar{M}}{2^{2n+2}}(e^T-1)+\frac{T\bar{M}}{2^{n+1}}(e^{\frac{T}{2^n}}-1)\left|\sum_{i=1}^{2^n}\tilde{c}_i^{(n)}e^{\frac{i-1}{2^n}T}\right|=\gamma_n.\end{aligned}$$

Hence, by Theorem 3, we have $0\le V(SP)-\int_0^T f(t)\hat{x}^{(n)}(t)dt\le V(SP)-\int_0^T f_n(t)\hat{x}^{(n)}(t)dt\le\left|V(DSP)-V(\tilde{D}_n)\right|+$

$\left|V(\tilde{P}_n)-\int_0^T f_n(t)\hat{x}^{(n)}(t)dt\right|\le \max\{\alpha_n+\theta_n\}+\gamma_n$, and we complete this proof.

Remarks:

(1). Note that $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\theta_n=0$, since $\lim_{n\to\infty}(1+\frac{\gamma T}{2^n\beta})^{2^n}=\exp\left(\frac{\gamma T}{\beta}\right)$.

(2). Moreover, since

$$\lim_{n\to\infty}\frac{T^2e^T\bar{M}}{2^{2n+1}}\sum_{i=1}^{2^n}\left|\tilde{c}_i^{(n)}\right|=\lim_{n\to\infty}\frac{Te^T\bar{M}}{2^{n+1}}\int_0^T|f(t)|dt=0$$

and

$$\lim_{n\to\infty}\frac{T\bar{M}}{2^{n+1}}(e^{\frac{T}{2^n}}-1)\left|\sum_{i=1}^{2^n}\tilde{c}_i^{(n)}e^{\frac{i-1}{2^n}T}\right|\le\lim_{n\to\infty}\frac{T\bar{M}}{2^{n+1}}(e^{\frac{T}{2^n}}-1)\sum_{i=1}^{2^n}\left|\tilde{c}_i^{(n)}e^T\right|=\lim_{n\to\infty}\frac{\bar{M}}{2}e^T(e^{\frac{T}{2^n}}-1)\int_0^T|f(t)|dt=0,$$

we have $\lim_{n\to\infty}\gamma_n=0$.

Numerical Examples:

For illustration purpose, we use two examples to implement the improved method and to show the quality of the purposed error bound.

Example 1:

$$\begin{aligned}(SP1):\quad &\text{maximize}\quad \int_0^{3/2}t^2\sin(7t)x(t)dt\\ &\text{subject to}\quad 4\cdot x(t)-\int_0^t 3\cdot x(s)ds\le 2+\cos(5t),\forall\, t\in[0,3/2]\\ &\qquad\qquad x(t)\in L_+^\infty[0,3/2].\end{aligned}$$

Example 2: Let $g(t)=\begin{cases}t^2\sin\left(\frac{1}{t}\right)+1, & if\ t\in(0,1]\\ 1, & if\ t=0.\end{cases}$. We consider the following problem (*SP*2).

$$\begin{aligned}(SP2):\quad &\text{maximize}\quad \int_0^1 t^3x(t)dt\\ &\text{subject to}\quad 3\cdot x(t)-\int_0^t 6\cdot x(s)ds\le g(t),\forall\, t\in[0,1]\\ &\qquad\qquad x(t)\in L_+^\infty[0,1].\end{aligned}$$

To illustrate the convergence, we select the partition number *n* from 15 to 24. Using MATLAB Version 7.0.1 on a PC for the experiment, the result obtained by running the program which implement the proposed method are presented in the following tables, where *APV*(*n*) is the objective value of the approximate solution $\hat{x}^{(n)}(t)$, that is, $APV(n)=\int_0^T f(t)\hat{x}^{(n)}(t)dt$; and *EB*(*n*)= $\max\{\alpha_n+\theta_n\}+\gamma_n$ is the error bound between the objective value of the approximate solution $\hat{x}^{(n)}(t)$ and the optimal value of (*SP*). In Example 2, we note that the function *g*(*t*) oscillates as *t* tends to 0, hence it could be rather time-consuming to find the global minimum of *g*(*t*) on each subinterval $\left[\frac{i-1}{2^n}T,\frac{i}{2^n}T\right]$ $(1\le i\le 2^n)$ when *n* becomes large.

Table 1. Numerical results for example 1.

n	$APV(n)$	*EB*(*n*)
15	1.0409268	0.0045677
16	1.0410199	0.0022838
17	1.0410664	0.0011419
18	1.0410897	0.0005710
19	1.0411013	0.0002855
20	1.0411071	0.0001427
21	1.0411100	0.0000714
22	1.0411115	0.0000357
23	1.0411122	0.0000178
24	1.0411126	0.0000089

Table 2. Numerical results for example 2.

n	$APV(n)$	$EB(n)$
15	0.5216110	0.0014449
16	0.5216367	0.0007224
17	0.5216495	0.0003612
18	0.5216560	0.0001806
19	0.5216592	0.0000903
20	0.5216608	0.0000452
21	0.5216616	0.0000226
22	0.5216620	0.0000113
23	0.5216622	0.0000056
24	0.5216623	0.0000028

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