International Journal of
Operations Research

International Journal of Operations Research Vol. 7, No. 1, 1–10 (2010)

A Note on Bottleneck Product Rate Variation Problem with Square-Deviation Objective

Tanka Nath Dhamala [1,*] **Shree Ram Khadkab** [2] **and Moon Ho Lee** [3]

[1, 2] Central Department of Mathematics, Institute of Science and Technology, Tribhuvan University,

P.O.Box. 13143, Kathmandu, Nepal

[3] Division of Electronics and Information Engineering, Chonbuk National University,

Chonju-561 756, South Korea

Received August 2008; *Revised April 2009*; *Accepted June 2010*

Abstract — In this note, the bottleneck product rate variation problem with square-deviation objective is considered. Bounds for the feasible solution of this problem are established explicitly. Existence of a perfect matching on the corresponding bipartite graph again turns out to be sufficient for the 1-bounded feasible solution to this problem as in the case of the problem with absolute-deviation objective. Furthermore, a pseudo-polynomial optimization algorithm that improves existing solution approaches is proposed. Like in the problem with absolute-deviation objective or the product rate variation problem with sum-deviation objective, the cyclic sequences are optimal.

Keywords — Scheduling, mixed-model just-in-time, bottleneck objective, integer programming.

1. INTRODUCTION

Mixed-model just-in-time production system has been developed with a goal of reducing cost of diversified small-lot instead large-lot minimizing large inventories and large shortages. Competitive industrial challenges of providing a variety of products at a very low cost by smoothing products (e.g. Toyota production system) and increasing computer applications (e.g. hard real-time) motivate researchers on the concept of penalizing jobs both for being early and for being tardy. This requires designing and controlling the system in such a way that the only required parts are produced in the necessary quantity when needed.

The main aim of the system is obtaining a sequence of a number of products that minimizes deviation throughout the time, between the actual and the ideal (desired) production. This maintains the assembly line keeping rate of parts usage as constant as possible. The problem has been referred as levelling or balancing the schedule.

Miltenburg (1989), and Miltenburg and Sinnamon (1989) formulate non-linear integer programming of the problem. The problem is known as the Product Rate Variation Problem (PRVP), Kubiak (1993). We call bottleneck PRVP for the PRVP with bottleneck objective and total PRVP in the case of sum objective.

Kubiak and Sethi (1994) reduce the total PRVP to an assignment problem and solve determining ideal position of the product and penalizing equally for either early or tardy production, the cost for the corresponding assignment

[*] Corresponding author's email: dhamala@yahoo.com

problem in pseudo-polynomial time. The assignment problem is note-worthy since this approach can be applied to solve the bottleneck PRVP with appropriate norms, see Dhamala and Kubiak (2005).

Steiner and Yeomans (1993) consider the bottleneck PRVP with absolute-deviation objective with the argument that the problem smoothes the schedule in every time period. They reduce the problem into a single machine scheduling problem with release times and due dates, which can be solved by finding a perfect matching. They obtain a feasible sequence via perfect matching and apply binary search to obtain an optimal solution with absolute-deviation less than one in pseudo-polynomial time. Note however that Still (1979) suggests some apportionment procedures that would give one or more sequences with the deviation less than one. Brauner and Crama (2004) study structural properties of the problem and give a set of algebraic necessary and sufficient conditions for the existence of a solution for a given objective value. They show that the problem is in Co-NP and leave unresolved situation of its exact complexity. Further, they show that bottleneck PRVP with absolute-deviation and with value less than a half has optimal sequence if and only if the demands are successive powers of two. Kubiak (2003a) gives its geometric proof.

Kubiak (2003b) shows that the cyclic sequences of total PRVP are optimal. Investigating structural properties for the reduction of computational time, Steiner and Yeomans (1996) show that the cyclic sequences for the bottleneck PRVP are also optimal. Time complexity can be substantially reduced when cyclic sequences exist.

Steiner and Yeomans (1994) develop an algorithm that determines all Pareto optimal solutions of the PRVP with bi-criterion objective of bottleneck absolute-deviation and sum deviations.

Corominas and Moreno (2003) study relations between optimal sequences for different total PRVP. They prove that total PRVP with absolute-deviation and with square-deviation are equivalent on the set of solutions with the deviation less than one. The result cannot be generalized. However, solution with this property for absolute-deviation will be sufficient for both cases.

Lebacque et al. (2007) compare PRVP with different objective functions and give structures in which some sequences optimize several objective functions simultaneously. The result is valid particularly for total PRVP with absolute-deviation and square-deviation. They also show that it is not possible for bottleneck PRVP in general. Further, they notice that minimizing the maximum absolute-deviation is equivalent to minimize the maximum deviation of more general functions, in particular, the square-deviation. This approach yields optimal solution for the bottleneck PRVP with square-deviation if the solution for the bottleneck PRVP with absolute-deviation is optimal. This note investigates optimal solution of the bottleneck PRVP with square-deviation that is independent of other objectives. Since there is no single objective that is better than others, like the total PRVP with square-deviation, the bottleneck PRVP with square-deviation would be interesting because of its strong theoretical feature and real world applications like the case with other objectives.

The plan of the paper is as follows. Section 2 reviews the mathematical models. In Section 3, a new solution method for bottleneck PRVP with square-deviation is presented. The final section concludes the paper.

2. MATHEMATICAL MODEL

2.1 Constraints

Let d_i be the demand for product i , $i=1,\ldots,n$, where n denotes the number of different products. Let $D=\sum_{i=1}^{n} d_i$ be the total demands with demand rate $r_i=\frac{d_i}{D}$ and then $\sum_{i=1}^{n} r_i=1$. The time horizon is partitioned into D equal units.

Let x_{ik} be the cumulative production of product i produced during the time units 1 through k . For $i=1,\ldots,n$,

$$\sum_{i=1}^{n} x_{ik}=k\,,\qquad k=1,\ldots,D \tag{1}$$

$$x_{i(k-1)}\le x_{ik}\,,\qquad k=2,\ldots,D \tag{2}$$

$$x_{iD} = d_i \,;\; x_{i0} = 0 \tag{3}$$

$$x_{ik} \geq 0 \text{, integer,} \qquad k = 1,\ldots,D \tag{4}$$

Constraint (1) ensures that exactly k units of products are produced during the periods 1 through k. Constraint (2) states that the total production is a non-decreasing function of k. (3) guarantees the demands are met exactly. (1), (2) and (4) ensure that exactly one unit of a product is sequenced during a time unit.

2.2 Objective functions

Let w_i be a weighting factor, which reflects the relative importance of balancing the schedule for product i, $i = 1,\ldots,n$. Assume that f_i, $i = 1,\ldots,n$, be unimodal, convex, symmetric, and non-negative function of deviation between the actual and the ideal production having zero at zero deviation. The problem is to extract the sequence $s = s_1 s_2 \ldots s_D$, satisfying the constraints (1)- (4) that minimizes the objective function

$$\max_{i,k} \; f_i(x_{ik} - kr_i) \tag{5}$$

Usually considered objective functions are absolute-deviation i.e.

$$\max_{i,k} \; w_i \left| x_{ik} - kr_i \right| \tag{6}$$

and square-deviation i.e.

$$\max_{i,k} \; w_i (x_{ik} - kr_i)^2 \tag{7}$$

We assume $w_i = 1$, $i = 1,\ldots,n$, for unweighted case of the problem.

Let the bottleneck PRVP with absolute-deviation objective and the bottleneck PRVP with square-deviation objective subject to the constraints (1)-(4) be denoted by the Problem A and the Problem S, respectively.

A solution X is called B-feasible if $\max_{i,k} f_i(x_{ik} - kr_i) \leq B$ for given B, and satisfies the constraints (1)-(4). The decision version of the problem is whether there exists a B-feasible solution. The terms “solution”, “sequence” and “schedule” are used for the same meaning.

Further, the ceiling function, the floor function and the fractional part of a real number y are denoted by $\lceil y \rceil$, $\lfloor y \rfloor$ and (y), respectively.

3. BOTTLENECK PRVP WITH SQUARE-DEVIATION

The solution method for the bottleneck PRVP with absolute-deviation, appeared in the literature, can also be applied for the bottleneck PRVP with square-deviation applying necessary modifications.

Consider the V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$ with

$V_1 = \{1,\ldots,D\}$.

$V_2 = \{(i,j) \mid i = 1,\ldots,n;\; j = 1,\ldots,d_i\}$.

$E = \{\{k,(i,j)\} \mid k \in [E(i,j), L(i,j)]\}$.

where $E(i,j)$ and $L(i,j)$, respectively, denote the earliest and the latest starting time for (i,j), the j^{th} copy of product i. A perfect matching constructed in the V_1-convex bipartite graph G gives rise a feasible solution. Optimal solution can be obtained using the bisection search algorithm.

3.1 Reduction to perfect matching

Let B be the target value (bound) for the problem. The starting times $E(i,j)$ and $L(i,j)$ for a given B can be determined by the integral adjustment of the points where the bound B and the curves $f_i(j-kr_i)=(j-kr_i)^2$, $i=1,\dots,n$; $j=0,\dots,d_i$ intersect. The points that the line B and the deviation curve $(j-kr_i)^2$ intersect correspond to some real values on the time horizon. The points are adjusted to the nearest integer in such a way that $(j-kr_i)^2$ does not exceed B. The main idea is to look for the smallest B with this property.

The starting times $E(i,j)$ and $L(i,j)$ are the following unique positive integers.

Lemma 1.
Let B be the given target value. Then $E(i,j)=\left\lceil \frac{j-\sqrt{B}}{r_i} \right\rceil$ and $L(i,j)=\left\lfloor \frac{j-1+\sqrt{B}}{r_i}+1 \right\rfloor$, for $i=1,\dots,n$; $j=1,\dots,d_i$ the unique integers.

Proof:
If the j^{th} copy (i,j) of product i is produced in the time unit k, then the penalty due to this unit product is calculated as $(x_{ik}-kr_i)^2=(j-kr_i)^2$, $i=1,\dots,n$; $j=0,\dots,d_i$; $k=1,\dots,D$. Let $j=0$ for $x_{ik}=0$.

On one hand, the earliest starting time $E(i,j)$ for (i,j) must satisfy the two inequalities $[j-(E(i,j)-1)r_i]^2>B$ and $[j-E(i,j)r_i]^2\le B$. This implies $\frac{j-\sqrt{B}}{r_i}\le E(i,j)\le \frac{j-\sqrt{B}}{r_i}+1$.

Therefore, $E(i,j)=\left\lceil \frac{j-\sqrt{B}}{r_i} \right\rceil$ holds.
On the other hand, the latest starting time $L(i,j)$ must satisfy the two inequalities $[(L(i,j)-1)r_i-(j-1)]^2\le B$ and $[L(i,j)r_i-(j-1)]^2>B$. This implies $\frac{j-1+\sqrt{B}}{r_i}<L(i,j)\le \frac{j-1+\sqrt{B}}{r_i}+1$.

Therefore, $L(i,j)=\left\lfloor \frac{j-1+\sqrt{B}}{r_i}+1 \right\rfloor$ holds.
Also, (i,j) may start at $k=1$ i.e. $E(i,j)=1$ if $(j-r_i)^2\le B$ and at $k=D$ i.e. $L(i,j)=D$ if $(d_i-r_i-j+1)^2\le B$.

In the weighted case, $E(i,j)=\left\lceil \frac{j-\sqrt{\frac{B}{w_i}}}{r_i} \right\rceil$ since $[j\sqrt{w_i}-(E(i,j)-1)r_i\sqrt{w_i}]^2>B$ and $[j\sqrt{w_i}-E(i,j)r_i\sqrt{w_i}]^2\le B$. Moreover, $L(i,j)=\left\lfloor \frac{j-1+\sqrt{\frac{B}{w_i}}}{r_i}+1 \right\rfloor$ since $E(i,j)$ and $[L(i,j)r_i\sqrt{w_i}-(j-1)\sqrt{w_i}]^2>B$.

Note that like in the Problem A, $E(i,j)$ and $L(i,j)$ can be calculated in $O(D)$ time.

V_1-convex bipartite graph G relies on $E(i,j)$ and $L(i,j)$ and has a perfect matching if and only if $|N(K)|\ge|K|$ for all K, where K is either an interval in V_1 or the neighborhood of an interval in V_1 and $N(K)=\{(i,j):(i,j)\in V_2, \exists k\in K s.t.(k,(i,j))\in E\}$, Brauner and Crama (2004). The necessary and sufficient condition for the existence of a perfect matching is the following.

Theorem 1.

The graph $G=(V_1 \cup V_2, E)$ formed by the Problem S has a perfect matching if and only if, for all $k_1, k_2 \in V_1$, $k_1 \le k_2$ and $[E(i,j), L(i,j)] \cap [k_1,k_2] \ne j$, the following inequalities hold

$$\sum_{i=1}^{n}(\lfloor k_2 r_i + \sqrt{B} \rfloor - \lceil (k_1-1)r_i - \sqrt{B} \rceil) \ge k_2 - k_1 + 1 \text{ and } \sum_{i=1}^{n}(\lceil k_2 r_i - \sqrt{B} \rceil - \lfloor (k_1-1)r_i + \sqrt{B} \rfloor) \le k_2 - k_1 + 1$$

Proof:

Let $K=[k_1,k_2] \subseteq V_1$. Then $(i,j) \in N(K)$

$\Leftrightarrow$ $E(i,j) \le k_2$ and $L(i,j) \ge k_1$

$\Leftrightarrow$ $\frac{j-\sqrt{B}}{r_i} \le k_2$ and $\frac{j-1+\sqrt{B}}{r_i} + 1 \ge k_1$

$\Leftrightarrow$ $\lceil (k_1-1)r_i + 1 - \sqrt{B} \rceil \le j \le \lfloor k_2 r_i + \sqrt{B} \rfloor$.

Therefore, for $K \subseteq V_1$, $|N(K)| \ge |K|$ if and only if $\sum_{i=1}^{n}(\lfloor k_2 r_i + \sqrt{B} \rfloor - \lceil (k_1-1)r_i - \sqrt{B} \rceil) \ge k_2 - k_1 + 1$

Let K be the neighborhood of an interval $[k_1,k_2]$ in V_1, i.e., let $N(K)=[k_1,k_2] \subseteq V_1$.
Then, $(i,j) \in K \subseteq V_2$

$\Leftrightarrow$ $k_1 \le E(i,j)$ and $L(i,j) \le k_2$

$\Leftrightarrow$ $k_1 \le \frac{j-\sqrt{B}}{r_i}$ and $\frac{j-1+\sqrt{B}}{r_i} + 1 \le k_2$

$\Leftrightarrow$ $\lfloor (k_1-1)r_i + 1 + \sqrt{B} \rfloor \le j \le \lceil k_2 r_i - \sqrt{B} \rceil$.

Thus, for K with $N(K)=[k_1,k_2] \subseteq V_1$, $|N(K)| \ge |K|$ if and only if $\sum_{i=1}^{n}(\lceil k_2 r_i - \sqrt{B} \rceil - \lfloor (k_1-1)r_i + \sqrt{B} \rfloor) \le k_2 - k_1 + 1$.

The starting times $E(i,j)$ and $L(i,j)$ are strictly monotonic, as we have $E(i,j) = \left\lceil \frac{j-\sqrt{B}}{r_i} \right\rceil < \left\lceil \frac{j+1-\sqrt{B}}{r_i} \right\rceil = E(i,j+1)$ and $L(i,j) = \left\lfloor \frac{j-1+\sqrt{B}}{r_i} + 1 \right\rfloor < \left\lfloor \frac{j+\sqrt{B}}{r_i} + 1 \right\rfloor = L(i,j+1)$ for $0 < r_i < 1$, $i = 1,\dots,n$, $n \ge 2$. Glover's (1967) EDD algorithm with complexity $O(|E|)$ assigns the lower numbered copies to earlier time units than the higher numbered copies. Thus, a perfect matching is order-preserving. Therefore,

Corollary 1.

A perfect matching in $G=(V_1 \cup V_2, E)$ is order-preserving.

Theorem 1 shows that existence of a perfect matching in G depends on the bound B. It is noteworthy to establish the lower and the upper bounds. The lower and the upper bounds for the Problem S can be established as

Theorem 2.

The lower and the upper bounds for the Problem S are $(1-r_{\max})^2$ and $(1-\frac{1}{D})^2$, respectively.

Proof:

As a copy of some product i must be assigned to the time unit $k=1$, it holds $\min(1-r_i)^2 \le B$, that is, $(1-r_{\max})^2 \le B$, for a B-feasible sequence.

For $B=(1-\frac{1}{D})^2$, we have $\lfloor k_2 r_i + \sqrt{B} \rfloor = \lfloor k_2 r_i + 1 - \frac{1}{D} \rfloor$.

$\lfloor k_2 r_i + 1 - \frac{1}{D} \rfloor = k_2 r_i$ if $k_2 r_i$ is an integer.
$k_2 r_i = \lfloor k_2 r_i \rfloor + (k_2 r_i)$ if $k_2 r_i$ is not an integer.
Since $(k_2 r_i) \geq \frac{1}{D}$, we have
$\lfloor k_2 r_i + 1 - \frac{1}{D} \rfloor > k_2 r_i$.

Therefore, $\lfloor k_2 r_i + 1 - \frac{1}{D} \rfloor \geq k_2 r_i$. Thus, $\sum_{i=1}^{n} (\lfloor k_2 r_i + \sqrt{B} \rfloor - \lceil (k_1 - 1) r_i - \sqrt{B} \rceil) \geq k_2 - k_1 + 1$.
Again, $\lceil k_2 r_i - \sqrt{B} \rceil = \lceil k_2 r_i - 1 + \frac{1}{D} \rceil$.
$\lceil k_2 r_i - 1 + \frac{1}{D} \rceil = k_2 r_i$ if $k_2 r_i$ is an integer.
$k_2 r_i = \lfloor k_2 r_i \rfloor + (k_2 r_i)$ if $k_2 r_i$ is not an integer. Since $(k_2 r_i) \leq 1 - \frac{1}{D}$, we have, $\lceil k_2 r_i - 1 + \frac{1}{D} \rceil < k_2 r_i$.
Therefore, $\lceil k_2 r_i - 1 + \frac{1}{D} \rceil \leq k_2 r_i$. Thus,
$\sum_{i=1}^{n} (\lceil k_2 r_i - \sqrt{B} \rceil - \lfloor (k_1 - 1) r_i + \sqrt{B} \rfloor) \leq k_2 - k_1 + 1$.

Therefore, the V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$ yields a perfect matching within these bounds.

It is clear that the lower and the upper bounds for the weighted case of the Problem S are $w_{\min}(1 - r_{\max})^2$ and $w_{\min}(1 - \frac{1}{D})^2$ and respectively, where $w_{\min} = \min\{w_i\}$ and $w_{\max} = \max\{w_i\}$, $i = 1, \ldots, n$.

An order-preserving perfect matching in G formed by any instance of the Problem S is analogous to a feasible solution.

Theorem 3.
Any instance of the Problem S has a feasible sequence if and only if, the V_1-convex bipartite graph formed by the instance has an order-preserving perfect matching.

Proof:
Suppose that s be a feasible sequence of any instance $(d_1, \ldots, d_n; D)$ of the problem S. Feasibility implies every (i, j), $i = 1, \ldots, n$; $j = 1, \ldots, d_i$ assigns a unique time unit k, $k = 1, \ldots D$. Sequence s is a bijection $(i, j) \to k$, where $(i, j) \in V_2$ and $k \in V_1$, $i = 1, \ldots, n$; $j = 1, \ldots, d_i$ that creates a perfect matching in the V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$. Corollary 1 incurs the perfect matching to be order-preserving.

Conversely, suppose that the V_1-convex bipartite graph $G = (V_1 \cup V_2, E)$ formed by the instance $(d_1, \ldots, d_n; D)$ of the problem S has an order-preserving perfect matching $E^* \subseteq E$. Edges $\{(i, j_1), k\}, \{(i, j_2), k\} \notin E^*$ as it violates the matching property and there remains no time unit k^* in V_1 unmatched as the matching is perfect. Hence there exists a bisection $(i, j) \to k$, where $(i, j) \in V_2$ and $k \in V_1$, $i = 1, \ldots, n$; $j = 1, \ldots, d_i$. Since the perfect matching is order-preserving, the bijection gives a feasible sequence s.

3.2 The bisection search

Observe that, the upper bound implies there always exists an optimal sequence for the Problem S when the deviation for every product is no more than one unit.

A bisection search algorithm to find an optimal sequence for the problem must run in the interval $[(1 - r_{\max})^2, (1 - \frac{1}{D})^2]$. Let the optimal value be B^*, Then, $B^* = (j - kr_i)^2$. So, $D^2 B^* = (Dj - kd_i)^2$ is an integer in the interval $[(D-d_{\max})^2, (D-1)^2]$. Since $E(i,j)$ and $L(i,j)$ can be calculated in $O(D)$ time, an optimal sequence can be obtained in $O(D\log D^2)$, i.e., $O(D\log D)$ time. Thus,

Theorem 4.

A bisection search algorithm that runs in the interval $[(1-r_{\max})^2, (1-\frac{1}{D})^2]$ can find an optimal sequence of the Problem S in $O(D\log D)$ time.

Corollary 2.

For the weighted Problem S, a bisection search can find an optimal sequence in $O(D\log(D^2 f w_{\max}))$ time.

Proof:

An integer f exists such that $D^2 f \bar{B}$, where $\bar{B} = w_i(j-kr_i)^2$, is an integer in $[w_{\min} f(D-d_{\max})^2, w_{\max} f(D-1)^2]$ and $w_i f$ is an integer for all products i, $i = 1,\ldots,n$. A bisection search that runs in this interval finds an optimal sequence in $O(D\log(D^2 f w_{\max}))$ time. Thus, an optimal sequence can be obtained in $O(D\log(D^2 f w_{\max}))$ time.

3.3 Small deviations

As usual, the ideal position $\left\lceil \frac{2j-1}{2r_i} \right\rceil$ is the ceiling function of the point where $(j-kr_i)^2$ and $(j-1-kr_i)^2$ intersect. A sequence of an instance $(d_1,\ldots,d_n;D)$ is optimal if (i,j) be sequenced at $\left\lceil \frac{2j-1}{2r_i} \right\rceil$, $i=1,\ldots,n$; $j=1,\ldots,d_i$ and the ideal positions are pair wise different. But, it is not true in general. An instance $(d_1,\ldots,d_n;D)$ with $d_i = 2^{i-1}$, $i=1,\ldots,n$, $n \ge 2$, has a sequence with all the copies sequenced at the pair wise different ideal positions. Further, the sequence has a small bound $B < \frac{1}{4}$.

Theorem 5.

The instance $(d_1,\ldots,d_n;D)$ with $d_i = 2^{i-1}$, $i=1,\ldots,n$, $n \ge 2$ of the Problem S has a feasible sequence with a bound $B < \frac{1}{4}$.

Proof:

Consider a bound $B = (\frac{D-1}{2D})^2 < \frac{1}{4}$. Let the copy (i,j) be sequenced at the ideal position $2^{n-i}(2j-1)$.

The copies do not compete for the position. Let $\frac{2j-1}{2^i} = \frac{2j'-1}{2^{i'}}$ for some positions. Then since both $(2j-1)$ and $(2j'-1)$ are odd, neither 2^i divides $(2j-1)$ nor $2^{i''}$ divides $(2j'-1)$. This implies $i = i'$ and $j = j'$.
Furthermore,

$$\frac{j-\sqrt{B}}{r_i} = \frac{j-\frac{2^{n-1}-1}{2^n-1}}{\frac{2^{i-1}}{2^n-1}} \le 2^{n-i}(2j-1) \le 2^{n-i}(2j-1)+1-\frac{2j}{2^i} = \frac{j-1+\sqrt{B}}{r_i}+1 .$$

Since $2^{n-i}(2j-1)$ is an integer, $E(i,j) = \left\lceil \frac{j-\sqrt{B}}{r_i} \right\rceil \le 2^{n-i}(2j-1) \le \left\lfloor \frac{j-1+\sqrt{B}}{r_i}+1 \right\rfloor = L(i,j)$.

Thus, the instance has a feasible sequence.

There exists no feasible instance $(d_1,\ldots,d_n;D)$, $i=1,\ldots,n$, $n \ge 2$ for the Problem S with $B < \frac{1}{9}$.

Corollary 3.

No instance $(d_1,...,d_n;D)$, $n \geq 2$ of the Problem S is feasible for $B < \frac{1}{9}$.

Proof:

The bound $(1\text{-}r_{\max})^2$ implies that C. For feasible sequence,
$\frac{j-\sqrt{B}}{r_i} \leq \frac{j-1+\sqrt{B}}{r_i} + 1$ implies $1 - r_i \leq 2\sqrt{B}$ for all $i = 1,...,n$, and then $1 - r_{\min} \leq 2\sqrt{B}$.
This follows that $\sum_{i'=1}^{n} r_{i'} \leq 2\sqrt{B}\ \ r_{\ i'} \neq r_{\min}$, which implies $r_{\max} \leq \sum_{i'=1}^{n} r_{i'} \leq 2\sqrt{B}$.
Then $1 - r_{\max} \geq 1 - 2\sqrt{B}$. Thus, $\frac{1}{3} \leq \sqrt{B}$.

The sequence is optimal since it is obtained sequencing the copies at the ideal position without competition. Here, the small deviation instance $d_i = 2^{i-1}$ can be sequenced in the ideal positions without competition with $B < \frac{1}{4}$ whereas the bound is $B < \frac{1}{2}$ in the case of the Problem A.

3.4 The cyclic sequences

The results pertaining to cyclic relation obtained for the Problem A are valid for the Problem S also.

Theorem 6.

If $u = \gcd(d_1,...,d_n) > 1$, then the Problem S, with given $B < 1$, consists of repetitions of the optimal sequence.

Proof:

For a feasible sequence, copy (i, j) must be scheduled in $[E(i, j), L(i, j)]$, for $i = 1,...,n$; $j = 1,...,d_i$. Let $u = \gcd(d_1,...,d_n) > 1$. Here, u is a factor of d_n and D with $d_n = uv_i$, $i = 1,...,n$; $D = uv$, $v = \sum_{i=1}^{n} v_i$; and $r_i = \frac{v_i}{v}$.
Since it holds $E(i,(e-1)v_i) + 1 = \left\lfloor \frac{(e-1)v_i+1-\sqrt{B}}{r_i} + 1 \right\rfloor \leq (e-1)v$ and $L(i, ev_i) = \left\lfloor \frac{ev_i - 1 + \sqrt{B}}{r_i} + 1 \right\rfloor \leq ev$.
we can write $(e-1)v < E(i,(e-1)v_i + 1) \leq L(i,(e-1)v_i + 1) \leq E(i, ev_i) \leq L(i, ev_i) \leq ev$. This implies that v_i copies $(e-1)v_i + 1,..., ev_i$ of product i occupy positions in $[(e-1)v + 1, ev]$. Thus, the sequence is periodic. The e^{th} period of copies of product i is labelled as $(e-1)v_i + f$, where $e = 1,...,u$; $f = 1,...,v_i$.

The linear relations $E(i, ev_i + f) = E(i, j) + ev$, for, and $L(i, ev_i + f) = L(i, j) + ev$ for $f = 1,...,v_i$, imply that each period consists of v units of products and all units in $(e+1)^{st}$ period must be sequenced in the same order after sequencing the v units in the e^{th} period.

An optimal sequence can be determined for the first period. Then an optimal sequence consisting of u repetitions of this sequence exists for the entire problem.

Remark that cyclic sequence analogously exists for the weighted problem with appropriate weights.

4.. CONCLUSION

Steiner and Yeomans (1993) obtain bounds for the Problem A and solve it pseudo-polynomially applying binary search technique on the feasible solutions obtained as the order-preserving perfect matching on the V_1-convex bipartite graph. Brauner and Crama (2004) present a set of algebraic necessary and sufficient conditions for the existence of a schedule with a given objective value.

The Problem S has not been considered yet like the other problems. The method of Kubiak and Sethi (1994) with specific norm, see Dhamala and Kubiak (2005), would be an approach for the bottleneck PRVP. Another approach is finding the solution via the solution of the Problem A, Lebacque et al. (2007). However, explicit bounds for a feasible solution of the Problem S were awaited; see Dhamala and Kubiak (2005).

Here, we give bounds explicitly for the first time and solve the Problem S with time complexity $O(D\log D)$, using the bounds, by necessary modifications on the solution method applied for the Problem A. Thus, the solution, independent of the objectives, of this problem is obtained. Moreover, we show that the cyclic sequences are optimal in this case, too. There exists a set of problem instances with the optimal value less than $\frac{1}{4}$. Furthermore, we show there exists no feasible instance with the objective value less than $\frac{1}{9}$.

The results of this paper could be extended to more general convex, symmetric and nonnegative functions. Moreover, the relation between optimal sequences of the Problem A and the Problem S or the problem with more general objective functions would be interesting for further research.

Acknowledgements:

The authors are indebted to the referees for their helpful comments for the improvement of the paper. The first author would like to acknowledge his visit to Chonbuk National University, Division of Electronics and Information Engineering. The third author acknowledges the support by F01-2008-000-10021-0, KOSEF, Korea.

REFERENCES

1. Brauner, N. and Crama, Y. (2004). The maximum deviation just-in-time scheduling problem. *Discrete Applied Mathematics*, 134, 25-50.
2. Brauner, N., Jost, V. and Kubiak, W. (2004). On symmetric Fraenkel's and small deviations conjecture. *Les Cahiers du Laboratoire Leibniz*-IMAG, 54, Grenoble, France.
3. Corominas, A. and Moreno, N. (2003). On the relations between optimal solutions for different types of min-sum balanced JIT optimisation problems. INFOR, 41, 4, 333-339.
4. Dhamala, T. N. and Kubiak, W. (2005). A brief survey of just-in-time sequencing for mixed-model systems. *International Journal of Operational Research*, 2, 2, 38-47.
5. Glover, F (1967). Maximum matchings in a convex bipartite graph. *Naval Research Logistics Quarterly*, 4, 313-316.
6. Jost, V. (2003). Deux problems d'approximation Diophantine: Le patage proportionnel en nombers entries et Les
7. pavages equilibres de Z. DEA ROCO, *Laboratoire Leibniz*-IMAG.
8. Kubiak, W. (1993). Minimizing variation of production rates in just-in-time systems: A survey. *European Journal of Operational Research*, 66, 259-271.
9. Kubiak, W. (2003a). On small deviation conjecture. *Bulletin of the Polish Academy of Sciences*, 51, 189-203.
10. Kubiak, W. (2003b). Cyclic just-in-time sequences are optimal. *Journal of Global Optimization*, 27, 333-347.
11. Kubiak, W. and Sethi, S. (1994). Optimal just-in-time schedules for flexible transfer lines. *The International Journal of Flexible Manufacturing Systems*, 6, 137-154.
12. Kubiak, W., Steiner G. and Yeomans, J.S. (1997). Optimal level schedules for mixed-model multi-level just-in-time assembly systems. *Annals of Operations Research*, 69, 241-259.
13. Lebacque, V. Jost, V. and Brauner, N. (2007). Simultaneous optimization of classical objectives in JIT scheduling. *European Journal of Operational Research*, 182, 29-39.
14. Miltenburg, J. (1989). Level schedules for mixed-model assembly lines in just-in-time production systems. *Management Science*, 35, 2, 192-207.
15. Miltenburg, J. and Sinnamon, G. (1989). Scheduling mixed-model multilevel just-in-time production systems. *International Journal of Production Research,* 27, 9, 1487-1509.
16. Moreno, N. and Corominas, A. (2006). Solving the minmax product rate variation problem (PRVP) as a bottleneck assignment problem. *Computers and Operations Research*, 33, 928-939.
17. Steiner, G. and Yeomans, J. (1993). Level schedules for mixed-model, just-in-time processes. *Management Science*, 36, 6, 728-735.
18. Steiner G. and Yeomans, J. (1994). A bicriterion objective for levelling the schedule of a mixed-model, JIT assembly process. *Mathematical & Compututer Modelling*, 20, 2, 123-134.
19. Steiner, G. and Yeomans, J. (1996). Optimal level schedules in mixed-model multi-level JIT assembly systems with pegging. *European Journal of Operational Research*, 95, 38-52.
20. Still, J. (1979). A class of new methods for congressional apportionment. *SIAM Journal of Applied Mathematics*, 37, 401- 418.
21. Tijdeman, R. (1980). The Chairman Assignment Problem. *Discrete Mathematics*, 32, 323-330.