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An Integrated Inventory Model for Three-tier Supply Chain Systems

Ming-Jong Yao*

Department of Transportation Technology and Management, National Chiao Tung University,

1001 University Road, 30010 Hsinchu, Taiwan

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Abstract---In this paper, we consider an integrated inventory model with one vendor and multiple retailers. In this three-tier supply chain system, the vendor purchases raw material produces into finished products, and delivers finished products to the retailers. We assume that the production (at the vendor) and the replenishment (at all the retailers) of the finished products share the common cycle time (T), but the replenishment cycle of raw material for the vendor is an integer multiple of T. The focus of this study is to solve the optimal common cycle to minimize the average joint total costs ($AJTC$) for the vendor in the whole supply chain system. To solve this problem, we derive the expression for the $AJTC$, and analyze the theoretical properties of the optimal $AJTC$ curve. We show that the optimal $AJTC$ curve is piece-wise convex with respect to T, and the junction points on the optimal $AJTC$ curve can be easily located by a closed-form formula. By utilizing our theoretical properties, we propose an efficient search algorithm for solving this problem. Our random experiments demonstrate that our search algorithm effectively obtains the optimal solution, and interestingly, the vendor could gain more cost saving when more retailers join the three-tier supply chain system.

Keywords — Supply chain, common replenishment epoch, search algorithm, replenishment

1. INTRODUCTION

Recently, information technology strongly enhances the inventories coordination across the entire supply chain. In late 1980's, EDI systems improve vendor-retailer integration and result in better streamline in supply chain systems. An excellent example shown in Sehr (1989) is Levi Strauss, an apparel vendor, who employs LeviLink (an EDI system) to link with its vendors to share the information on inventory and to quick response to the customers' demand change. Udo (1993), Gottardi and Bolisani (1996) and Lambert, Stock and Ellram (1998) all emphasize that inventory information sharing between vendors and retailers leads to successful cases of inventory management.

The information sharing in supply chain systems leads a trend for the researchers to study the cooperation between the vendors and the buyers. It has been advocating that collaboration is an important way for creating win-win relationships among the members in supply chains. Researchers have been addressing lots of efforts on developing efficient strategies for the inventories coordination across the entire supply chain. Research efforts have been addressed to coordinate the inventory policies of the members in the supply chain to reduce the joint inventory costs. One may refer to Banerjee (1986), Goyal (1988), Das and Goyal (1991), Banerjee and Kim (1995) and Goyal and Gupta (1989) for the integrated inventory models. Also, Banerjee and Banerjee (1992) and Thomas and Griffin (1996) provide reviews on the related research works on the integrated inventory models.

The one warehouse multi-retailer (OWMR) lot-sizing problem is one of the most representative inventory models that integrate partners in supply chain systems. The OWMR concerns with the determination of lot sizes and schedule of n retailers replenished from the central warehouse. Many researchers have been addressing their

* Corresponding author's email: myaoie@gmail.com

efforts to solve the optimal solution for the OWMR. Schwarz (1973) derived the necessary conditions for an optimal policy and some analytical bounds under stationary-nested policy. He also proposed a heuristic that usually solves a near-optimal solution for the OWMR lot-sizing problem. Schwarz and Schrage (1975) focused on solving the optimal lot sizes of a single product in multi-echelon assembly systems under stationary-nested policy. Graves and Schwarz (1977) investigated the characteristics of optimal continuous review policies for arborescent systems under stationary-nested policy. Maxwell and Muckstadt (1985) proposed a heuristic for complex multi-stage, multi-product systems under stationary nested policy. Lu and Posner (1994) solved the OWMR lot-sizing problem under so-called *integer-ratio* policy which restricts each retailer orders at an integer or reciprocal of an integer multiple of the warehouse order interval. Mitchell (1987) extended Roundy's (1985) results to allow backlogging and introduce a class of policies, called *nearly-integer-ratio* policies which is different from the class of integer-ratio policies by not requiring stationarity of orders placed by retailers. Later, Anily and Federgruen (1990, 1991), and Hall (1991) added the vehicle routing costs in the OWMR systems. We note that the OWMR closely relates to this study. But it possesses some characteristics that make its decision-making scenario significantly different from that in this study. First, the OWMR does not take the inventory of raw material into accounts. Also, the warehouse does not produce finished items so that the capacity of the production facility would not be considered in the OWMR.

All of the papers reviewed above are for the inventory control problems where the vendor/distributor and the buyers/retailers play the key roles in the two-echelon supply chains. For the recent extensions on the two-echelon inventory models, one may refer to Chen, Federgruen and Zheng (2001).

On the other hand, some researchers have been devoted to inventory models that include raw material inventory in their formulation. Most of these articles consider the integration between the raw material requirement and the production batch size for a single product. One may refer to Kim and Chandra (1987), Roan, et al. (2000) and Sarker and Khan (1999, 2001) for reference. We note that Sarker and Khan's (2001) paper reviews two delivery policies of raw material in integrated production/inventory system, *viz.*, “lot for lot” and “multiple lot for a lot” policies. Here, we adopt the “multiple lot for a lot” policy to formulate our mathematical model in this study. On the other hand, some researchers take account the raw material inventory in the economic lot scheduling problem (ELSP), which is closely related to our problem in this study. For instance, Hwang and Moon's (1991) model considers the special case with only two products, but the raw material is deteriorating. Gallego and Joneja (1994) formulate the mathematical model for the ELSP and consider various issues associated with the management of raw materials for production. Sarker and Newton (2002) proposes a genetic algorithm to solve the ELSP with raw material in which the production system has a limited storage space and the transportation fleet capacity is of known capacity. Interested readers may also refer to an excellent review on this category of problems in Goyal and Deshmukh's (1992) paper. We note that these studies did not take account the distribution aspect (to the retailers) in their models.

Yang and Wee (2003) propose a model that is very similar to ours in this study. They consider a supply chain with one vendor and multiple buyers, and the raw material and the finished product are deteriorating. The key difference from our model is that they employ the concept of JIT lot-splitting from raw material supply to production and from production to distribution in their formulation.

Recently, Munson and Rosenblatt (2001) formulate a mathematical model for a three-tier supply chain system. In their study, there is one raw material supplier, one manufacturer and only one vendor in the supply chain. The vendor makes the inventory decision according to the EOQ rule. Then, the manufacture determines the integrated inventory policy follows this assumption. Interestingly, they derive their solution approach by exploring the optimality structure of the optimal cost function with respect to the ordering quantity from the vendor.

In this paper, we extend Munson and Rosenblatt's (2001) study to another case in which there is one raw material supplier, one manufacturer and *multiple retailers* in a three-tier supply chain system. Similar to Munson and Rosenblatt's methodology, we derive theoretical results on the curve of the optimal objective function value and propose an effective search algorithm.

We outline the rest of this paper as follows. In section 2, we give a brief introduction to the problem formulation and present the mathematical model. Also, we derive some theoretical analysis on the optimality structure of the mathematical model. In section 3, we propose our search algorithm based on our theoretical results. Then, we show that our search algorithm effectively obtains the optimal solution by random experiments in section 4. Finally we address our concluding remarks in section 5.

2. MATHEMATICAL MODEL AND THEORETICAL ANALYSIS

In this section, we first discuss the scenario that the decision-maker faces in section 2.1. Then, we present the formulation of the mathematical model in section 2.2. Also, we conduct full analysis on the curve of the optimal objective function value in sections 2.3 to 2.5.

2.1 The decision-making scenario

In this study, we consider an integrated inventory model for a three-tier supply chain system. A single vendor purchases raw material, produces into finished products, and delivers finished products to multiple retailers in this supply chain.

We assume that the production (at the vendor) and the replenishment (at all the retailers) of the finished products share the Common Replenishment Epoch (CRE), which is denoted by T. We note that CRE has been popularly used in model formulation for deriving coordination mechanism in supply chain systems. And, the vendor may consolidate several retailers' replenishment orders and save the order processing costs by adopting such a CRE mechanism. One may refer to Viswanathan and Piplani's (2001) and Mishra's (2003) papers for more discussion on the implementation of CRE mechanism. On the other hand, in order to save the ordering cost of raw material, the vendor replenishes the raw material in an integer multiple of T, *i.e.*, mT where $m \in N^+$. Furthermore, we assume that raw material shortage is not allowed for the vendor, and no shortage is allowed for the retailers. Also, the retailers are willing to take the vendor's replenishment strategy.

We note that such a decision-making scenario applies to many of the suppliers in retailing business and grocery. In these industries, the retailers grant the selling channel to the vendor by providing floor space or storage racks in the retailers' store and assisting the vendor to sell the products to the customers. The retailers pursue their profit by charging the vendor for the floor space and earning the markup from the selling price of the products. Also, the retailers often authorize the vendors to replenish their products at their will (but, usually with a pre-specified replenishment quantity) in such a case. Note that the CRE mechanism not only is simply to implement, but also, it guarantee a feasible production schedule for the vendor. Therefore, the CRE mechanism could significantly simplify the production scheduling and the logistics in the vendor's production system by using a regular and repetitive replenishment schedule for each retailer.

Before presenting our mathematical model, we define the notation needed later. Let a_r be the ordering cost per raw materials order for the vendor. The carrying cost per unit of raw material per unit time is denoted as h_r. Denote the production rate of the vendor as P, which is a known constant. Let S be the setup cost per production run for the vendor and u be the usage rate of raw materials for producing each unit of the finished product. The carrying cost for each unit of finished product held per unit time is denoted as h_f. The demand rate at retailer i, is denoted as D_i. Each order from retailer i incurs for an order processing cost of a_i. For the vendor, the carrying cost for the finished product sent to retailer i is h_i per unit per unit time held. All of the parameters are constants and known to the decision makers.

2.2 The objective function

In this study, we formulate this integrated inventory model from the vendor's point of view. Our focus is to get the optimal CRE T^* and the optimal multiplier m^* to minimize the average joint total costs for the vendor in the whole supply chain system.

The objective function includes three categories of cost terms: (1) the vendor's average total costs for the finished products sent to all the retailers, (2) the average total costs incurred by the finished products held by the vendor, and (3) the retailer's average purchasing and inventory holding costs for the raw material. These cost terms are derived as follows.

We denote the average total costs for the finished products sent to retailer i as $TC_i(T)$. Notice that the vendor is the owner of the finished products stored at each retailer. Therefore, the vendor takes the inventory holding costs for the finished products stored at each retailer. Since we assume that the vendor delivers the finished products to retailer i after a fixed cycle T, $TC_i(T)$ is given by

$$TC_i(T) = \frac{a_i}{T} + \frac{T}{2}[h_i D_i] \quad \text{(1)}$$

We denote the average total costs incurred by the finished products held by the vendor as $TC_f(T)$. By Figure 1, one may easily observe that the total costs incurred in a replenishment cycle T is $S + \frac{h_f}{2}\left(\sum_{i=1}^{n} D_i^2 / p\right) T^2$. Therefore, $TC_f(T)$ is given by

$$TC_f(T) = \frac{S}{T} + \left(\frac{Th_f}{2p}\right)\sum_{i=1}^{n} D_i^2 \quad \text{(2)}$$

Next, we derive the expression for the vendor's average purchasing and inventory holding costs for the raw material which is denoted as $TC_r(m,T)$. In order to calculate the average inventory holding cost, we need to evaluate the inventory level within a replenishment cycle for the raw material, *i.e.*, mT.

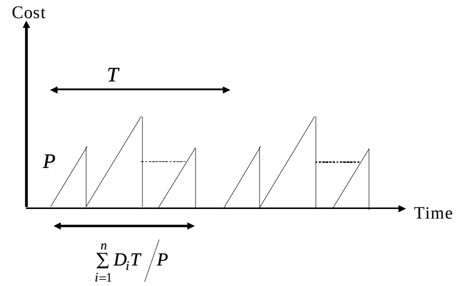

Figure 1. The inventory level of the finished product.

By carefully observing Figure 2, one may find that there are m triangles and m rectangles in the inventory holding area for the raw material. The total holding costs incurred in a cycle of mT are given by

$$h_r\left(\frac{m}{2}\left[\sum_{i=1}^{n} D_i T \Big/ p\right]\left[u\sum_{i=1}^{n} D_i T\right]+\sum_{j=1}^{m-1} j\left[u\sum_{i=1}^{n} D_i T^2\right]\right) \quad \ldots\ldots (3)$$

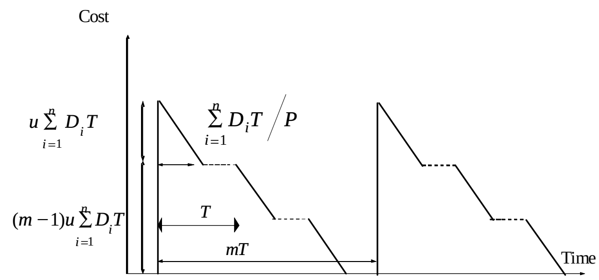

Figure 2. The inventory level of the raw materials

Therefore, one shall have the expression for $TC_r(m,T)$ as follows.

$$TC_r(m,T)=\frac{a_r}{mT}+\frac{uh_r T}{2}\left(\left[\sum_{i=1}^{n} D_i\right]^2 \Big/ p+(m-1)\sum_{i=1}^{n} D_i\right) \quad \ldots\ldots (4)$$

By eqs. (1), (2) and (4), we have the objective function, denoted by $\Gamma(m,T)$, as follows.

$$\Gamma(m,T)=TC_f(T)+TC_r(m,T)+\sum_{i=1}^{n} TC_i(T) \quad \ldots\ldots (5)$$

Therefore, our focus is to solve the problem (P) in (6) as follows.

$$\text{(P)} \quad \min_{m\in\aleph^+, T\in R^+} \Gamma(m,T)=\frac{1}{T}\left[S+\frac{a_r}{m}+\sum_{i=1}^{n} a_i\right]+\frac{T}{2}\left[(m-1)uh_r\sum_{i=1}^{n} D_i\right] +\frac{T}{2}\left[\frac{1}{p}\left[uh_r(\sum_{i=1}^{n} D_i)^2+h_f\sum_{i=1}^{n} D_i^2\right]+\sum_{i=1}^{n} h_i D_i\right]. \quad \ldots\ldots (6)$$

where $\Gamma(m,T)$ is the average joint total costs (*AJTC*) for the vendor in the whole three-tier supply chain system.

2.3 Characterization of the optimality structure

In order to investigate the theoretical properties on the curve of the optimal objective function value, we first solve the optimal value of $m^*(T')$ and compute the optimal objective function value for each given value of $T = T'$ on the T-axis. (That is, $m^*(T')= \arg\min_{m\in N^+}\{\Gamma(m(T'),T')\}$.) Let $\underline{\Gamma}(T)$ be the optimal average joint total costs function value with respect to T. (In other words, $\underline{\Gamma}(T)\equiv\{\Gamma(m^*(T),T)\,|\,T\in R^+\}$ is a function of T.) Then, we may plot $\underline{\Gamma}(T)$ using a small-step ΔT as shown in Figure 3.

Importantly, Figure 3 shows two interesting observations on the $\underline{\Gamma}(T)$ function:

1. $\underline{\Gamma}(T)$ is piece-wise convex with respect to T.
2. Let m_L^* and m_R^*, respectively, are the optimal multipliers of the left-side and right-side convex curves with regard to a junction point in the plot of the $\underline{\Gamma}(T)$ function. Then, $m_L^* = m_R^* + 1$.

The first observation motivates us to obtain the optimal solution for $\underline{\Gamma}(T)$ at any given value of T by some close-form calculation. The second leads us to the idea to change the optimal multiplier m at the "junction point" of two neighboring convex curves in the proposed algorithm. In the following discussion, we will have further analysis on these two observations and will formally prove them as the theoretical properties on the $\underline{\Gamma}(T)$ function.

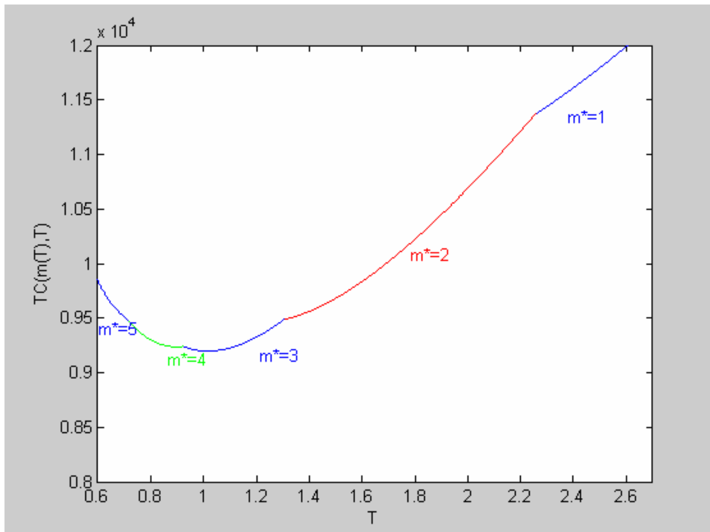

Figure 3. The curve of the $\underline{\Gamma}(T)$ function

Before having further discussions on the first observation, we present an important property of the function $\underline{\Gamma}(T)$ in Lemma 1.

Lemma 1 Assume that $T_a < T_b$ and $m^*(T_a) = m^*(T_b)$. Then, $m^*(T') = m^*(T_a)$ for all $T' \in (T_a, T_b)$.

Proof： Please refer to Appendix A.1. ■

Next, we assert the optimality structure of the function $\underline{\Gamma}(T)$ in Proposition 1.

Proposition 1 $\underline{\Gamma}(T)$ is piece-wise convex with respect to T.

Proof： By eq. (6), we may compute the second derivative (with respect to T) for $\Gamma(m,T)$ by $\partial^2\Gamma(m,T)/\partial T^2 = \frac{a_r}{mT^3}$. Therefore, given any positive integer $\overline{m}$, $\partial^2\Gamma(\overline{m},T)/\partial T^2 > 0$ for all $T > 0$. So, we conclude that $\Gamma(\overline{m},T)$ is convex with respect to T. Let $s(\overline{m})$ be the set of T such that $m^*(T) = \overline{m}$, *i.e.*,

$$s(\overline{m}) \equiv \{T \mid m^*(T) = \overline{m}, T > 0\} \qquad (7)$$

By Lemma 1, $s(\overline{m})$ must be an interval on the T-axis. Also, different values of $\overline{m}$ form non-overlapping intervals on the T-axis. Therefore, $\underline{\Gamma}(T)$ is piece-wise convex since $\Gamma(\overline{m},T)$ is convex on its support set $s(\overline{m})$. ■

2.4 Junction points

Next, we introduce the "junction points" on the $\underline{\Gamma}(T)$ curve, which is a piece-wise convex function with respect to T. We define a junction point for $\underline{\Gamma}(T)$ as a particular value of T where two consecutive convex curves $\Gamma(m,T)$ and $\Gamma(m+1,T)$ concatenate. These junction points determines at "what value of T" where one should change the value of m so as to obtain the optimal value for the $\underline{\Gamma}(T)$ function.

We first derive a closed-form for the location of the junction points. We define the difference function $\Delta(m+1,m,T)$ by

$$\Delta(m+1,m,T)=\Gamma(m+1,T)-\Gamma(m,T)=-\frac{a_r}{m(m+1)T}+\tfrac{T}{2}\left(uh_r\sum_{i=1}^{n}D_i\right) \qquad (8)$$

We note that $\Delta(m+1,m,T)$ is the cost difference between using m and m+1 as its multiplier. Since the first derivative of the function $\Delta(m+1,m,T)$ is always positive for all $T>0$, $\Delta(m+1,m,T)$ is an increasing function with respect to *T*. Suppose that the search algorithm proceeds from an upper bound toward smaller values of *T*, we evaluate $\Delta(m+1,m,T)$ from positive values, to zero and finally, to negative values. Let *w* be the point where $\Delta(m+1,m,T)$ reaches zero. Assume that *m* is the optimal multiplier for $T>w$. This scheme implies that one should keep using *m* until it meets *w*. From the point *w* onwards, the value of $\underline{\Gamma}(T)$ can be improved by using *m*+1 as its optimal multiplier. We note that *w* is the point where two neighboring convex curves $\Gamma(m+1,T)$ and $\Gamma(m,T)$ meet. Importantly, such a junction point *w* not only provides us with the information on at "what value of *T*" where one should change the value of *m* so as to secure the optimal value for the $\underline{\Gamma}(T)$ function.

By the rationale discussed above, we derive a closed form to locate the junction points by letting $\Delta(m+1,m,T)=0$ as follows.

$$J(m)=\sqrt{2a_r\Big/\left[m(m+1)uh_r\textstyle\sum_{i=1}^{n}D_i\right]} \qquad (9)$$

Note that $J(m=i)$ indicates the location of the i^{th} junction points. By (9), the following inequality (10) holds

$$J(v)<\ldots<J(m)<\ldots<J(2)<J(1) \qquad (10)$$

where *v* is an (unknown) upper bound on the value of *m*.

Theorem 1 is an immediate result from (9) and (10), and it provides strengthen foundation for our search scheme.

Theorem 1 Suppose that m_L^* and m_R^* are the optimal multipliers of the left-side and right-side convex curves with regard to a junction point *w* of the $\underline{\Gamma}(T)$ function, then $m_L^*=m_R^*+1$.

The following corollary is also a by-product of (9) and (10), and it provides an easier way to obtain the optimal multiplier $m^*(T)$ for any given $T>0$.

Corollary 1 For any given $T>0$, the optimal multiplier $m^*(T)$ is given by

$$m^*(T)=\left\lceil -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+8a_r\Big/\left[T^2uh_r\textstyle\sum_{i=1}^{n}D_i\right]}\right\rceil \qquad (11)$$

Proof: Please refer to Appendix A.2. ■

2.5 Local optimum

Recall that $\underline{\Gamma}(T)$ is piece-wise convex with respect to *T* (as shown in Proposition 1). It is important for us to locate the local minima for the $\underline{\Gamma}(T)$ function since they are the candidates for the optimal solution. Let $\breve{T}_k$ be the local minimum candidate for the $\underline{\Gamma}(T)$ function given $m^*(T)=k$ for $k\in N^+$ where

$$\breve{T}_k=\sqrt{2\left[S+\frac{a_r}{k}+\sum_{i=1}^{n}a_i\right]\Bigg/\left[(k-1)uh_r\sum_{i=1}^{n}D_i+\frac{1}{p}\left[uh_r(\sum_{i=1}^{n}D_i)^2+h_f\sum_{i=1}^{n}D_i^2\right]+\sum_{i=1}^{n}h_iD\right]} \qquad (12)$$

and $\breve{T}_k$ is derived from solving the equation by setting the first derivative of $\Gamma(m,T)$ equal to zero. Therefore, one may use the following rule to check the existence of a local minimum for the $\underline{\Gamma}(T)$ function: if either of the following conditions holds, (1) $\breve{T}_1\geq J(1)$ and (2) $\breve{T}_k\in[J(k+1),J(k))$, then $\breve{T}_k$ exists as a local minimum for the $\underline{\Gamma}(T)$ function.

3. THE OPTIMAL SEARCH ALGORITHM

In this section, we present a search scheme which obtains the optimal solution for the problem (P) in (6).

One may refer to (6) for the complicated expression of $\Gamma(m,T)$. We note that $\Gamma(m,T)$ is not trivial to solve since it is actually a *nonlinear* integer programming problem. On the other hand, recall that the $\underline{\Gamma}(T)$ function is piece-wise convex with respect to *T*. Also, our theoretical results on the junction points for the $\underline{\Gamma}(T)$ curve encourage us to solve the optimal solution by searching along the *T*-axis.

To design our search algorithm, we need to define the search range by setting a lower and an upper bound on the *T*-axis, denoted by T_{low} and T_{up}, respectively. In order to efficiently find T_{low} and T_{up}, we refer to Wildeman, Frenk and Dekker's (1997) approach to obtain the search range for the joint replenishment problem (JRP). We note that they obtain the search range $[T_{low}, T_{up}]$ by solving a relaxed version of the JRP in their paper.

Before our derivation, we first separate the relaxed problem for $\Gamma(m,T)$ into two parts, *i.e.*, $\Gamma(m,T) = q(T) + \Phi(m,T)$ where $\Phi(m,T)$ collects all the terms containing both the decision variables m and T in $\Gamma(m,T)$. The expressions for $q(T)$ and $\Phi(m,T)$ are given in equations (13) and (14), respectively.

$$q(T) = \frac{T}{2}\left(\frac{uh_r}{p}\left(\sum_{i=1}^{n} D_i\right)^2 - uh_r \sum_{i=1}^{n} D_i\right) + TC_f(T) + \sum_{i=1}^{n} TC_i(T) \quad \text{(13)}$$

$$\Phi(m,T) = \frac{a_r}{mT} + \frac{T}{2} muh_r \sum_{i=1}^{n} D_i, \quad m \in N^+ \quad \text{(14)}$$

Similar to Wildeman, Frenk and Dekker's approach, we first transform $\Gamma(m,T)$ from a *nonlinear* integer program to a convex programming problem by relaxing the constraint $m \in N^+$ by $m \geq 1, m \in R$. We denote the relaxed problem for $\Gamma(m,T)$ by $\Gamma_R(m,T)$. Therefore, we have

$$\Gamma_R(m,T) = q(T) + \Phi_R(m,T) \quad \text{(15)}$$

where $\Phi_R(m,T) = \{\Phi(m,T) : m \geq 1, m \in R\}$ and $\Gamma_R(m,T)$ are continuous relaxations for $\Phi(m,T)$ and $\Gamma(m,T)$, respectively.

Let $\underline{\Phi}_R(T) = \inf\{\Phi(m,T) : m \geq 1, m \in R\}$ and $\underline{\Gamma}_R(T) = \inf\{\Gamma(m,T) : m \geq 1, m \in R\}$. Then, by the definition of (15), we may link the relationship between $\underline{\Phi}_R(T)$ and $\underline{\Gamma}_R(T)$ by eq. (16) as follows.

$$\underline{\Gamma}_R(T) = q(T) + \underline{\Phi}_R(T) \quad \text{(16)}$$

Interestingly, one may easily explore the optimality structure of $\underline{\Phi}_R(T)$ as follows.

$$\underline{\Phi}_R(T) = \begin{cases} \sqrt{2a_r uh_r \sum_{i=1}^{n} D_i}, & \text{if } T \leq x^*. \\ \frac{a_r}{T} + \frac{T}{2} uh_r \sum_{i=1}^{n} D_i, & \text{if } T > x^*. \end{cases} \quad \text{(17)}$$

where $x^* = \sqrt{2a_r / [uh_r \sum_{i=1}^{n} D_i]}$, and x^* corresponds to the local minimum for the function $\Phi(m=1,T)$. By (17), one has the second-order derivative for $\underline{\Phi}_R(T)$ by

$$\frac{d^2}{dT^2}\underline{\Phi}_R(T) = \begin{cases} 0, & \text{if } T \leq x^*. \\ 2a_r / T^3, & \text{if } T > x^*. \end{cases} \quad \text{(18)}$$

Therefore, $\underline{\Phi}_R(T)$ is convex with respect to T for $T>0$ by (18). On the other hand, it can be easily shown that $\frac{\partial^2 q(T)}{\partial T^2} > 0, \forall T > 0$, *i.e.*, $q(T)$ is also convex with respect to T. Thus, $\underline{\Gamma}_R(T)$ is obviously a convex function. Denote the optimal solution of $\underline{\Gamma}_R(T)$ by T_R. The following proposition indicates the location of T_R.

Proposition 2 One may locate T_R by

$$T_R = \begin{cases} T_1^R, & \text{if } q'(x^*) \geq 0. \\ T_2^R, & \text{if } q'(x^*) < 0. \end{cases} \quad \text{(19)}$$

where $T_1^R = \sqrt{2(S + \sum_{i=1}^{n} a_i) \Big/ \left[\sum_{i=1}^{n} h_i D_i + \frac{uh_r}{p}\left(\sum_{i=1}^{n} D_i\right)^2 - uh_r \sum_{i=1}^{n} D_i + \frac{h_f}{p}\left(\sum_{i=1}^{n} D_i^2\right)\right]}$, and

$$T_2^R = \sqrt{2(S + \sum_{i=1}^{n} a_i + a_r) \Big/ \left[\sum_{i=1}^{n} h_i D_i + \frac{uh_r}{p}\left(\sum_{i=1}^{n} D_i\right)^2 + \frac{h_f}{p}\left(\sum_{i=1}^{n} D_i^2\right)\right]}.$$

Proof: Please refer to Appendix A.3.

Denote T^* as the optimal replenishment cycle time for the function $\underline{\Gamma}(T)$. In the following proposition, we will show that a lower bound and an upper bound on the search range can be obtained by the values of T where the objective value of $\underline{\Gamma}_R(T)$ equals $\underline{\Gamma}(T_R)$.

Proposition 3 Let T_{low} be the smallest and T_{up} be the largest T for which the function $\underline{\Gamma}_R(T)$ is equal to $\underline{\Gamma}(T_R)$. Then $T_{low} \leq T^* \leq T_{up}$.

Proof: Since the $\underline{\Gamma}_R(T)$ function is strictly convex, we have the result that $T_{low} \leq T^* \leq T_{up}$. Consequently, for values of $T < T_{low}$ the $\underline{\Gamma}_R(T)$ is larger than $\underline{\Gamma}(T_R)$. Since the $\underline{\Gamma}_R(T)$ is a relaxation of $\underline{\Gamma}(T)$, $\underline{\Gamma}(T) > \underline{\Gamma}_R(T_R)$ for $T < T_{low}$, T_{low} is a lower bound on T^*.

The proof for that $T^* \leq T_{up}$ can be done similarly. ■

It is shown how the bounds T_{low} and T_{up} are secured in Figure 4. We note that the bounds T_{low} and T_{up} can be obtained by any line search algorithm (*e.g.*, bisection search, or quadratic fit search, see Bazarra, et al., 1993).

Now, we are ready to enunciate the search algorithm.

The search algorithm

1. Secure T_R by Proposition 2. Then, obtain the bounds T_{low} and T_{up} by some line search algorithm, *e.g.*, bisection search.

2. Use (9) to get all the junction points in $[T_{low}, T_{up}]$. Start the search from the largest junction point, say $J(k)$, by setting $T^* = J(k)$, $m^* = k-1$ and $\Gamma^* = \infty$ (*i.e.*, the optimal objective value at $T^* = J(k)$). Then, move to the next junction point, and go to Step 3.

3. At each junction point $J(k)$, do the following items:

(a) If all the junction points in $[T_{low}, T_{up}]$ are examined, go to Step 4; otherwise, go to Step 3(b).

(b) Compute $\Gamma(k-1, J(k))$ and check: if $\Gamma(k-1, J(k)) < \Gamma^*$, then $\Gamma^* = \Gamma(k-1, J(k))$, $T^* = J(k)$ and $m^* = k-1$.

(c) Check the existence of a local minimum by Corollary 2. If the local minimum $\breve{T}_{k-1}$ exists, then compute $\Gamma(k-1, \breve{T}_{k-1})$ and check: if $\Gamma(k-1, \breve{T}_{k-1}) < \Gamma^*$, then $\Gamma^* = \Gamma(k-1, \breve{T}_{k-1})$, $T^* = \breve{T}_{k-1}$ and $m^* = k-1$.

4. Report the optimal solution: the optimal replenishment cycle T^*, the optimal multiplier m^*, and the optimal objective value Γ^*. Stop.

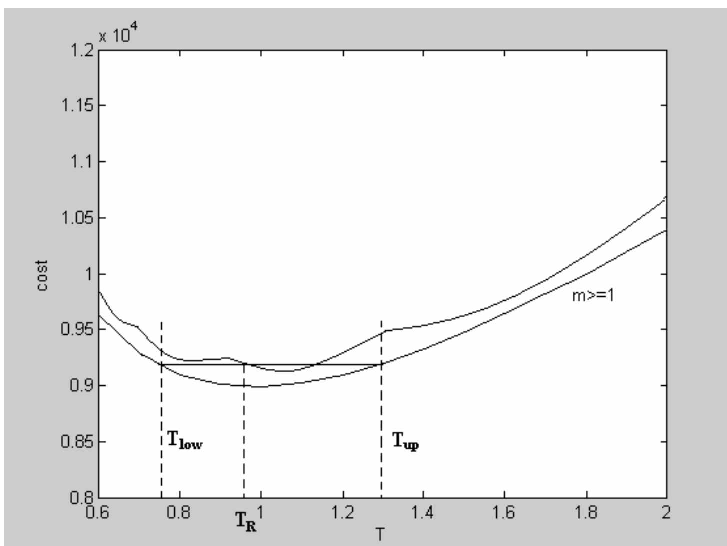

Figure 4. The search range of $[T_{low}, T_{up}]$ in the search algorithm

4. NUMERICAL EXPERIMENTS

In this section, we present a numerical example to demonstrate the implementation of the proposed search algorithm. Also, we show that our search algorithm effectively obtains the optimal solution by random experiments.

4.1 A Demonstrative Example

First, we use a simple example with one vendor and three retailers to demonstrate the search process of the proposed algorithm. The data set for this example is shown in Table 1 as follows.

We summarize the search process as follows.

1. By Proposition 2, we secure T_R by T_R =4.4639. Then, obtain the bounds T_{low} and T_{up} by some line search algorithm, *e.g.*, bisection search. We have T_{low} = 3.591 and T_{up} = 5.1277 for this example.

2. We locate the largest junction point *J*(1)=4.3301, and compute the optimal objective value at *J*(1) by $\Gamma(m=1, J(1)) = \$1{,}145.4$. Set $T^* = J(1)$, $m^* = 1$ and $\Gamma^* = \Gamma(1, J(1))$. Since $\breve{T}_1 \in [J(1), \infty)$, $\breve{T}_1$ exists as a local minimum. We evaluate $\Gamma(1,\breve{T}_1)$=\$1,142.7 which is less than Γ^*. Therefore, we replace the optimal solution with that at $\breve{T}_1$ by setting $T^* = \breve{T}_1$, and $\Gamma^* = \Gamma(1,\breve{T}_1)$.

3. Then, we move to the next junction point *J*(2) = 2.5. Since *J*(2) < T_{low}, it is the last junction point that we need to examine. Now, we locate $\breve{T}_2$, *i.e.*, the local minimum for *m*=2, by $\breve{T}_2$ = 3.986. Since $\breve{T}_2$ locates between its neighboring junction points, *i.e.*, $\breve{T}_2 \in [J(2), J(1))$, it exists as a local minimum. We evaluate $\Gamma(2,\breve{T}_2)$ =\$1,141.5 which is less than Γ^*. Therefore, we replace the optimal solution with that at $\breve{T}_2$ by setting $T^* = \breve{T}_2$, $m^* = 2$, and $\Gamma^* = \Gamma(2,\breve{T}_2)$.

4. Since *J*(2) is the last junction point that we need to examine, we stop the search algorithm. We shall report the optimal solution by $T^* = \breve{T}_2$, $m^* = 2$, and $\Gamma^* = \Gamma(2,\breve{T}_2)$ =\$1,141.5.

Table 1. The data set of the demonstrative example.

Retailers			
Parameters	1	2	3
Ordering cost：a_i	700	400	500
Holding cost：h_i	0.05	0.04	0.06
Demand rate (annual)：D_i	950	700	850
Vendor			
Raw material ordering cost：a_r	750		
Raw material holding cost：h_r	0.02		
Finished items holding cost：h_f	0.07		
Setup cost：S	300		
Production rate：P	2700		
usage rate of raw materials：u	0.8		

4.2 Random experiments

In this section, we perform some experiments using randomly generated examples to analyze the characteristics of the optimal solutions for different problem settings. For example, we would like to observe how different number of retailers and different utilization rates of the vendor's production system may affect the run time and solution quality of the proposed algorithm.

First, we discuss how to randomly generate the instances in our numerical experiments. Table 2 presents the mean and range, *i.e.*, the two necessary parameters of any uniform distribution, for all of the parameters of the experimental problems. Then, all the parameters for an experimental problem are uniformly generated from $[mean - range/2, mean + range/2]$. Since the production is a value-added process, we assume that the inventory holding cost rate for the raw material shall be no larger than that for the finished product, *i.e.*, $h_r \le h_f$.

We divide our experiments into 16 settings by combing different number of retailers and different levels of. Here, we designate four levels of $P\big/\sum_{i=1}^{n} D_i$ at 1.1, 1.2, 1.3 and 1.4 (that correspond to *utilization rates* of the vendor's production system at 0.91, 0.83, 0.77, and 0.71, respectively). We note that $P\big/\sum_{i=1}^{n} D_i$ must be greater than 1 to guarantee a feasible production schedule for the vendor. Also, the number of retailers in the supply chain systems is set to be 10, 15, 20 and 25, respectively.

For each combination of $P\big/\sum_{i=1}^{n} D_i$ and the number of retailers, we randomly generate 25 instances. After solving each instance, we collect the run time and the error estimate of the proposed algorithm. We define the error estimate by $\left(\Gamma^* - \underline{\underline{\Gamma}}\right)/\underline{\underline{\Gamma}}$ where $\underline{\underline{\Gamma}}$, expressed in (31), is a lower bound on the optimal objective function value of the problem (P) in (6). (One may refer to Appendix A.4 for the derivation of $\underline{\underline{\Gamma}}$.)

To examine the effectiveness of the proposed search algorithm, we first review the run time data in Table 3. The average run time of the 400 instances is around only 0.693 seconds. Also, though the average run time does increase with the number of retailers, its growth rate is not significant. Therefore, our run time data in Table 3 verify that our search algorithm effectively obtains the optimal solution for the instances in our numerical experiments.

Table 2. The ranges for the parameters used in the random experiments

	Mean	Range	
Ordering cost of the retailers	800	1600	[0,1600]
Holding cost of the retailers	0.05	0.1	[0,0.1]
Demand rate of the retailers	1000	1000	[500,1500]
Raw materials ordering cost of the vendor	600	1200	[0,1200]
Raw materials holding cost of the vendor	0.035	0.07	[0,0.07]
Finished items holding cost of the vendor	$h_f = \max\{h_i\} + 0.02$		
Setup cost of the vendor	500	1000	[0,1000]
Production rate of the vendor	Depend on the sum of retailers' demand		
Usage rate raw materials of the vendor	0.9	0.6	[0.6,1.2]

Next, we review the solution quality of the proposed algorithm by examining the data of the error estimates. The average value of the error estimates for the 400 instances in Table 3 is around 3.565% that could be reasonable for most of the decision makers in the real world. According to our experiments, the average error estimate does not significantly vary with the utilization rate of the vendor's production system. But, interestingly, the average error estimate decreases as the number of the retailers increases. It implies that the vendor could gain more cost saving when more retailers join the three-tier supply chain system in the scenario of this study.

Table 3. The summary of the random experiments

	10 retailers		15 retailers		20 retailers		25 retailers	
$P/\sum_{i=1}^{n} D_i$	Error	Run time (sec.)	Error	Run time (sec.)	Error	Run time (sec.)	Error	Run time (sec.)
1.1	5.00%	0.69	3.82%	0.69	3.32%	0.67	2.65%	0.67
1.2	5.04%	0.61	3.76%	0.69	2.57%	0.63	2.59%	0.69
1.3	5.03%	0.61	4.03%	0.68	3.00%	0.68	2.24%	0.69
1.4	5.38%	0.60	3.74%	0.69	2.79%	0.66	2.08%	0.71
Average	5.11%	0.63	3.84%	0.69	2.92%	0.66	2.39%	0.69

5. CONCLDING REMARKS

In this paper, we consider an integrated inventory model for a three-tier supply chain system with one vendor and multiple retailers. In this supply chain, the vendor purchases raw material, produces into finished products, and delivers finished products to the retailers. We assume that the production (at the vendor) and the replenishment (at all the retailers) of the finished products share the common cycle time (*T*), but the replenishment cycle of raw material for the vendor is an integer multiple of *T*, *i.e.*, mT, $m \in N^+$. The focus of this study is to secure the optimal common cycle T^* and the optimal multiplier m^* to minimize the average joint total costs for the vendor in the whole supply chain system.

To approach this problem, we derive the expression for the average joint total costs, and analyze the theoretical properties of the optimality objective function value with respect to *T*, *i.e.*, $\underline{\Gamma}(T)$. We show that the $\underline{\Gamma}(T)$ function is piece-wise convex with respect to *T*, and the junction points on the $\underline{\Gamma}(T)$ curve can be easily located by a closed-form formula. By utilizing our theoretical properties, we propose an efficient search algorithm to solve this problem. Our random experiments demonstrate that our search algorithm effectively obtains the optimal solution, and interestingly, the vendor could gain more cost saving when more retailers join the three-tier supply chain system.

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APPENDIX

A.1 The proof for Lemma 1

Proof: Let $m_1 < m_2$ where m_1 and m_2 are positive integers. We define a function $\Delta(m_1, m_2, T)$ by $\Delta(m_1,m_2,T) = \Gamma(m_1,T) - \Gamma(m_2,T) = \frac{(m_2-m_1)}{m_1 m_2}\frac{a_r}{T} - \frac{(m_2-m_1)}{2}\left(uh_r \sum_{i=1}^{n} D_i\right)T$. Then, by $\partial^2\Delta(m_1,m_2,T)/\partial T^2 = \frac{2(m_2-m_1)}{m_1 m_2}\frac{a_r}{T^3} > 0$ (for all T>0), we assert that $\Delta(m_1,m_2,T)$ is a convex function with respect to T. Therefore, Case (1) in Figure 5 never exists since it contradicts our assertion that $\Delta(m_1,m_2,T)$ is a convex function. Also, one may locate the intersection point of $\Gamma(m_1,T)$ and $\Gamma(m_2,T)$ at $T_{\text{int}}(m_1,m_2) = \sqrt{2a_r \Big/ m_1 m_2 uh_r \sum_{i=1}^{n} D_i} > 0$ by setting $\Delta(m_1,m_2,T) = 0$ as shown in Case (2) of Figure 5.

Next, we will prove Lemma 1 by contradiction. Assume that there exists a value of $T' \in (T_a, T_b)$ such that $m^*(T') \neq m^*(T_b)$. So, $m^*(T') \neq m^*(T_a)$, and $T_{\text{int}}(m^*(T'), m^*(T_a)) < T' < T_b$. That is, we would have the following result, $\Gamma(m^*(T'),T_b) < \Gamma(m^*(T_b),T_b) = \Gamma(m^*(T_a),T_b)$, which contradicts our assumption that the optimal multiplier at T_b is $m^*(T_b)= m^*(T_a)$. ■

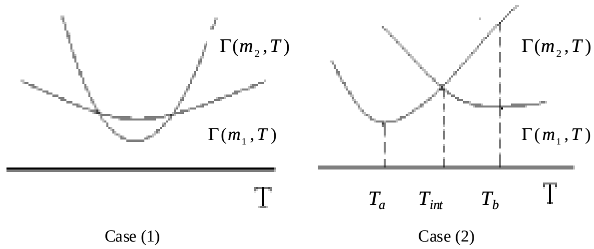

Figure 5. The two possible cases in the $\Delta(m_1,m_2,T)$ function

A.2 The proof for Corollary 1

Proof: By the definition of junction point $J(m)$, $m^*(T)=m$ when $J(m)\le T < J(m-1)$. By the closed form for the junction point, we have

$$\sqrt{2a_r\Big/\left[m(m+1)uh_r\sum_{i=1}^{n}D_i\right]}\le T \qquad (20)$$

and

$$T<\sqrt{2a_r\Big/\left[m(m-1)h_r\sum_{i=1}^{n}D_i\right]} \qquad (21)$$

One may simplify (20) by $m^2+m-2a_r\Big/\left[T^2uh_r\sum_{i=1}^{n}D_i\right]\ge 0$. Since m must be a positive integer, we have

$$m\ge -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+8a_r\Big/\left[T^2uh_r\sum_{i=1}^{n}D_i\right]} \qquad (22)$$

Similarly, one may simplify (21) and get (23) as follows.

$$m<\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+8a_r\Big/\left[T^2uh_r\sum_{i=1}^{n}D_i\right]} \qquad (23)$$

By combining (22) and (23), we have

$$-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+8a_r\Big/\left[T^2uh_r\sum_{i=1}^{n}D_i\right]}\le m<\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1+8a_r\Big/\left[T^2uh_r\sum_{i=1}^{n}D_i\right]} \qquad (24)$$

Since the difference between the two inequalities in (24) is equal to 1, it implies that an integer exists between $J(m)$ and $J(m\text{-}1)$ or both sides of (24) are both integers. In either case, taking the upper-entire of the expression in the right-hand side satisfies (11). ■

A.3 The proof for Proposition 2

Proof: In Proposition 2, we use the value of $q'\left(x^*\right)$ to dichotomize into two possible cases as follows. Recall that $\underline{\Gamma}_R(T)=q(T)+\underline{\Phi}_R(T)$ where $q(T)$ and $\underline{\Phi}_R(T)$ are expressed in Section 3.

Case 1): $q'\left(x^*\right)\ge 0$. By the definition of $\underline{\Phi}_R(T)$, $\frac{d}{dT}\underline{\Phi}_R(T)>0$ for $T>x^*$. Also, by the convexity of $q'(T)$ and (16), it is obvious that $\underline{\Gamma}_R'(T)\ge 0$ for $T>x^*$ when $q'\left(x^*\right)\ge 0$. Therefore, the local minimum must exist for $\underline{\Gamma}_R(T)$ when $T\le x^*$. Recall that $\underline{\Phi}_R(T)=\sqrt{2a_r uh_r\sum_{i=1}^{n}D_i}$ when $T\le x^*$. Hence, we may assert that the minimum value of $\underline{\Gamma}_R(T)$ must exist at

$$T_1^R=\sqrt{2(S+\sum_{i=1}^{n}a_i)\Big/\left[\sum_{i=1}^{n}h_iD_i+\frac{uh_r}{p}(\sum_{i=1}^{n}D_i)^2-uh_r\sum_{i=1}^{n}D_i+\frac{h_f}{p}(\sum_{i=1}^{n}D_i^2)\right]}.$$

Case 2: $q'\left(x^*\right)<0$. By the definition of $\underline{\Phi}_R(T)$, $\frac{d}{dT}\underline{\Phi}_R(T)=0$ for $T\le x^*$. By the convexity of $q(T)$ and (16), $\underline{\Gamma}_R'(T)<0$ for $T\le x^*$ when $q'\left(x^*\right)<0$. Therefore, the local minimum must exist for $\underline{\Gamma}_R(T)$ when $T>x^*$. Recall that $g^*(T)=\dfrac{a_r}{T}+\dfrac{T}{2}uh_r\sum_{i=1}^{n}D_i$ when $T>x^*$. Hence, we may assert that the minimum value of $\underline{\Gamma}_R(T)$ must exist at

$$T_2^R = \sqrt{2(S+\sum_{i=1}^{n} a_i + a_r) \Big/ \left[\sum_{i=1}^{n} h_i D_i + \frac{uh_r}{p}(\sum_{i=1}^{n} D_i)^2 + \frac{h_f}{p}(\sum_{i=1}^{n} D_i^2)\right]}\ .\blacksquare$$

A.4 The derivation of the lower bound on the objective function value

In order to derive the lower bound on the objective function value, we separate it into three parts and derive a lower bound for each part correspondingly.

1. A lower bound on <u>the average total costs for the *n* buyers</u>:

We may express the average total costs for the n buyers by $\sum_{i=1}^{n} TC_i(T) = a_i/T + \frac{T}{2}[h_i D_i]$. Obviously, the EOQ formula (i.e., $T_i^* = \sqrt{2a_i/(h_i D_i)}$) provides an easy lower bound on the total costs for each buyer. Therefore, we have a lower bound on the average total costs for the n buyers by

$$\sum_{i=1}^{n} \underline{TC}_i(T_i^*) = \sum_{i=1}^{n} \sqrt{2a_i h_i D_i} \quad \text{(25)}$$

2. A lower bound on <u>the average total costs for the *vendor*</u>:

By (2), we have the average total costs for the vendor expressed by $TC_f(T) = S/T + (Th_f/2p)\sum_{i=1}^{n} D_i^2$. We may solve its optimal value of T by taking its first derivative with respect to T, and setting it equal to zero by $T_f^* = \sqrt{2pS/h_f \sum_{i=1}^{n} D_i^2}$. Therefore, we have a lower bound on the average total costs for the vendor by

$$\underline{TC}_f(T_f^*) = \sqrt{2pSh_f \sum_{i=1}^{n} D_i^2} \quad \text{(26)}$$

3. A lower bound on <u>the average total costs for the *raw material*</u>:

Recall that we express the average total costs for the raw material as follows.

$$TC_r(m,T) = \frac{a_r}{mT} + \frac{T}{2}\left(\frac{uh_r}{p}\left[\sum_{i=1}^{n} D_i\right]^2 + (m-1)uh_r \sum_{i=1}^{n} D_i\right) \quad \text{(27)}$$

In order to utilize our analysis, we collect the terms in $TC_r(m,T)$ with the variable m in the function $\Phi(m,T) = \frac{a_r}{mT} + \frac{T}{2} muh_r \sum_{i=1}^{n} D_i$, as we did in (14). Also, we define a new function $y(T)$ for those terms in $TC_r(m,T)$ without the variable m as follows.

$$y(T) = \frac{uh_r T}{2p}\left(\sum_{i=1}^{n} D_i\right)^2 - \frac{uh_r T}{2}\sum_{i=1}^{n} D_i \quad \text{(28)}$$

So, we have $TC_r(m,T) = \Phi(m,T) + y(T)$.

To minimize the objective function value of $\Phi(m,T)$, we would have the optimal solution by $x^* = \sqrt{2a_r / uh_r \sum_{i=1}^{n} D_i}$.

Let us define a new function $w(T)$ by $w(T) = \{\Phi(m,T) : m \ge 1\}$. Obviously, the objective function value of $\Phi(m,T)$ is bounded from below by $w(T)$ since $w(T)$ is a continuous relaxation of $\Phi(m,T)$. Furthermore, we have the exact expression for the function $w(T)$ as follows.

$$w(T) = \begin{cases} \sqrt{2a_r uh_r \sum_{i=1}^{n} D_i}, & \text{if } T < x^*. \\ \Phi(1,T), & \text{if } T \ge x^*. \end{cases} \quad \text{(29)}$$

Therefore, we have a lower bound on the average total costs for the raw material by

$$\underline{TC}_r(T) = w(T) + y(T) \quad \text{(30)}$$

By the expression of $w(T)$, we note that there are two possible cases to obtain the minimizer T^* for $\underline{TC}_r(m,T)$, namely, either $T^* < x^*$ or $T^* \ge x^*$. Now, we have further analysis for these two cases as follows.

<u>Case 1</u>: $T^* < x^*$:

By (25), we know that $w'(T) = 0$ in Case 1. Also, we have $y'(T) = \left[\sum_{i=1}^{n} D_i - p\right] uh_r \left(\sum_{i=1}^{n} D_i\right) \Big/ 2p < 0$ since it must hold that $\sum_{i=1}^{n} D_i < p$ for the feasibility of the production capacity utilization. Therefore, we conclude that $\underline{TC}_r'(T) = w'(T) + y'(T) < 0$. Importantly, it implies that there exist no optimal solution in Case 1.

Case 2: $T^* \geq x^*$:

In Case 2, we have $\underline{TC}_r(T) = \frac{uh_r T}{2p}\left(\sum_{i=1}^n D_i\right)^2 - \frac{uh_r T}{2}\sum_{i=1}^n D_i + \frac{a_r}{T} + \frac{uh_r T}{2}\sum_{i=1}^n D_i$. We may easily solve the optimal solution by plugging the minimizer $T_r^* = \sqrt{2a_r p \big/ uh_r\left(\sum_{i=1}^n D_i\right)^2}$ in the function $\underline{TC}_r(T)$.

Summarizing our discussion above, we have a lower bound for the average total costs in the whole supply chain system, denoted as $\underline{\underline{\Gamma}}$, by

$$\underline{\underline{\Gamma}} = \sum_{i=1}^n \underline{TC}_i(T_i^*) + \underline{TC}_f(T_f^*) + TC_r(T_r^*) \quad (31)$$