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# Modeling on Weighted Utilizations of Network Dimensioning Problems

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***Abstract*****—** We propose two mathematical models with weighted utility functions for the fair bandwidth allocation and QoS routing in communication networks which offer multiple services for several classes of users. The formulation and numerical experiments are carried out in a general utility-maximizing framework. In this work, instead of being fixed, the weight for each utility function is taken as a free variable. The objective of this paper is to find the structure of optimal weights that maximize the weighted sum of utilities of the bandwidth allocation for each class. We solve it by proposing two models in terms of fairness. Model I and II are constructed to compare different choices for optimal weights. For Model I, the structure of optimal weights form a vector which consists of one for a class and zero otherwise. For Model II, the form of optimal weights is that each weight of utility function is equally assigned. The results are proved and illustrated by software GAMS numerically.
***Keywords*** **—** weighted utility functions, numerical experiments, optimal weights.

## 1. INTRODUCTION

Many approaches have been proposed for the fair resource allocation problem where Quality of Service (QoS) routing in communication networks offering multiple services for users. Maher et al. (2005) mentioned that fair resource allocation problems are concerned with the allocation of limited bandwidth among competing activities so as to achieve the best overall performances of the system but providing fair treatment of all classes of competitors. The objective of these optimization problems is to determine the amount of bandwidth for each class to maximize the sum of the users' satisfaction. The optimal solution could satisfy users' preferences with respect to throughput and fairness (see Kelly et al. (1998), Kelly (2003), Ogryczak (2003), and Pióro (2002)).

Wang and Luh (2006a) proposed a precomputation-based maximizing model for the network dimensioning problem. Assume there are $m$ classes in different QoS requirements. The formulation and analysis is carried out in a general utility-maximizing framework. It precomputes bandwidth allocation (rate vector) and end-to-end paths with QoS guarantees, in terms of utility functions. They presented a routing database, identifying an optimal path upon each connection request. The purpose of their paper was to choose the optimal solutions in order to provide a set of solutions satisfying user' preferences with fairness. Numerical results showed sensitivity of utility functions by changing several values of parameters, including the weight of utility function for each class. But it is wondering that if the weight for each class is taken as a free variable, instead of a fixed number. This is because the decision maker is always interested in obtaining the optimal weights in this kind of problem. Hence, trying to get the optimal solution with optimal weights in the model is our objective of this paper.

Consider a directed network topology $G=(V,E)$, as shown in Figure 1, where $V$ and $E$ denote the set of nodes and the set of links in the network respectively. Suppose we are given the maximal possible capacity, $U_e$, of each link $e$. Given the purchasing cost of bandwidth, $k_e$, and the cost of delay, $\mathbf{l}_e$, for each link $e$. In this network, there are $m$ different classes of connections which have their own QoS minimal requirement, $b_i$, and maximal end-to-end delay, $D_i$. Denote the total number of connections, for each class $i$, by $J_i$. Let $\mathrm{K}_i$, for each class $i$, be an index set consisting of $J_i$ connections, that is, $\mathrm{K}_i=\{1,\mathbf{K},J_i\}$. Every connection, in each class, is allocated with the same bandwidth $q_i$ and must satisfies the same QoS minimal requirement. All connections are delivered between the same source and destination nodes in this network. Under a limited available budget, $B$, we want to allocate the bandwidth in order to provide each class with maximal possible QoS. The purpose of this work is to maximize the weighted sum of utility functions of the bandwidth for each class.

We will adopt software GAMS (see Rosenthal (1998) and Sahinidis and Tawarmalani (2004)) to compute the optimal weighted sum of utility functions for each class $i$, $i=1,\mathbf{K},m$. This study is carried out by the models, named Models I and II,

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which are defined in Section 3. When it yields the first result, we carry on changing the parameter $J_i$ to observe the variations of $q_i, w_i$ and total utilization value. Subsequently, we keep on changing the parameter $B$ and other parameters to observe their variations. By the numerical results in Model I, it shows that $w_k$ is equal to 1 for some $k$, the others are equal to zero whatever parameters change. In Model II, the result that the form of optimal weights is a vector $(w_1, w_2, \mathbf{K}, w_m)$ with $w_i = 1/m$, for each $i = 1,2,\mathbf{K}, m$.

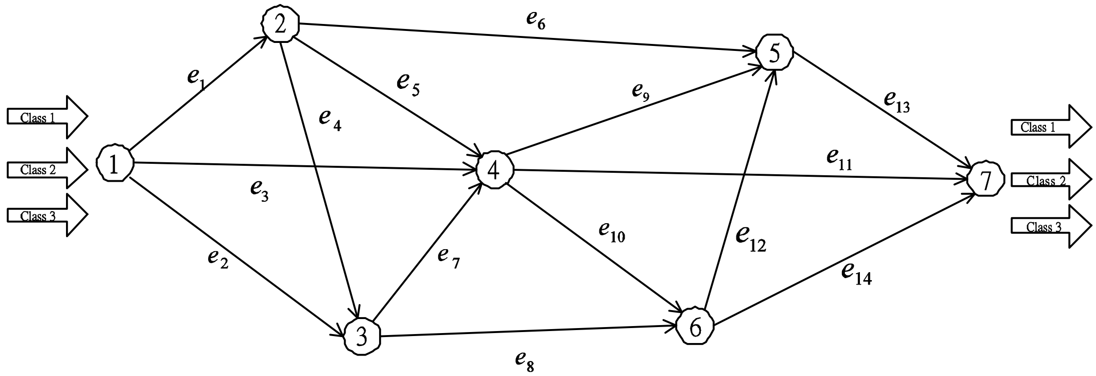


Figure 1: Network Topology for an Illustrative Example

## 2. A NETWORK OPTIMIZATION MODEL

Following the study in Wang and Luh (2006b), we assume that the decision maker specifies requirements in aspiration and reservation levels by introducing desired and required values for several outcomes, depending on the specified aspiration and reservation levels, $a_i$ and $r_i$, respectively. Further, assumes that an utility function of $q_i$ can be viewed as an extension of the fuzzy membership function in terms of a strictly monotonic and concave utility function (see Ogryczak et al. (2003), Stockman (1999), Wang and Luh (2006b), etc.)

$$f_i(q_i) = \log_{d_i} \frac{q_i}{r_i}, \tag{1}$$

where $d_i = a_i/r_i$. Formally, we define $f_i(\cdot)$ over the range $[0,\infty)$, with $f_i(0) = -\infty$ and $f_i'(0) = \infty$. It is a strictly increasing function of $q_i$, having value 1 if $q_i = a_i$, and value 0 if $q_i = r_i$. The utility function can map the different values onto a normalized scale of the decision maker's satisfaction. Moreover, the logarithmic utility function will be intimately associated with the concept of proportional fairness (see Kelly (2003), Ogryczak et al. (2003), and Pióro et al. (2002).)

**Proposition 1.** The utility function $f_i(q_i)$ is continuous, increasing, and concave.

The proof was given in Wang and Luh (2006a). We will formulate the mathematical model of the fair bandwidth allocation by using the utility function.

For each connection $j$ of class $i$, we denote the routing path connecting the source node 1 and destination node 7 by $p_{i,j}$. To determine whether link $e$ is chosen we define the binary decision variable

$$c_{i,j}(e) = \begin{cases} 1, & \text{if link } e \in p_{i,j} \\ 0, & \text{if link } e \notin p_{i,j}. \end{cases} \tag{2}$$

Given the total available budget $B$ and the marginal cost $k_e$ of bandwidths for each link $e \in E$, we want to allocate the bandwidths in order to provide each class with maximal possible QoS. Denoted by $K_i$ a set of connections in class $i$. Suppose there is the number of connections in $K_i$ is $J_i$, i.e., $|K_i| = J_i$. Then these decision variables must be nonnegative:

$$q_i \geq 0, \forall j \in K_i, \text{for } i = 1, \mathbf{K}, m. \tag{3}$$

Let $q_{i,j}$ be the bandwidth allocated to the connection $j$ of class $i$ respectively. Suppose that every connection in the same class uses the same bandwidth and has the same bandwidth requirement, so we have $q_{i,1} = q_{i,2} = \mathbf{L} = q_{i,J_i}$. Thus, the constraint follows

$$q_i \geq b_i, \tag{4}$$

where $b_i$ is the bandwidth requirement for class $i$. It shows that every connection in the same class uses the same bandwidth and has the same bandwidth requirement.

Due to the limited budget on network planning, We have the budget constraint on the network:

$$\sum_{e\in E}\sum_{i\in I}\sum_{j\in K_i} k_e \cdot q_i \cdot c_{i,j}(e) \leq B \tag{5}$$

and

$$\sum_{i\in I}\sum_{j\in K_i} q_i c_{i,j}(e) \leq U_e, \ \forall e \in E, \tag{6}$$

where $U_e$ is the maximal capacity of each link $e$. The above constraint says that the aggregate bandwidth of all connections at any link does not exceed the capacity. Moreover, for each class $i$, since every connection has the maximal end-to-end delay constraint, we have the end-to-end delay constraint:

$$\sum_{e\in E} \mathbf{l}_e c_{i,j}(e) \leq D_i, \forall i, j, \tag{7}$$

where $\mathbf{l}_e$ is a mean delay allocated to each link $e$ and $D_i$ is maximal end-to-end delay allocated to each class $i$.

Let $E_o \subseteq E$ be the subset of links connected with the source node $o$, then we have

$$\sum_{e\in E_o} c_{i,j}(e) = 1, \forall i, j. \tag{8}$$

Let $E_d \subseteq E$ be the subset of links connected with the destination node $d$, then we have

$$\sum_{e\in E_d} c_{i,j}(e) = 1, \forall i, j. \tag{9}$$

Let $E_n^{in} \subseteq E$ be the subset of links flowed into the node $n$ and $E_n^{out} \subseteq E$ be the set of links flowed out of the node $n$, then we have

$$\sum_{e\in E_n^{in}} c_{i,j}(e) = \sum_{e\in E_n^{out}} c_{i,j}(e), \forall i, j. \tag{10}$$

Constraints (8), (9) and (10) express the node conservation relations indicating that flow in equals flow out for every connection $j$ in class $i$.

In Wang and Luh (2006a), the precomputation-based maximization model with its constraints can be formulated as follows:

$$\text{maximize } \sum_{i\in I} w_i f_i(q_i)$$

$$\text{subject to } \sum_{e\in E}\sum_{i\in I}\sum_{j\in K_i} k_e \cdot q_i \cdot c_{i,j}(e) \leq B$$

$$\sum_{i\in I}\sum_{j\in K_i} q_i c_{i,j}(e) \leq U_e, \forall e \in E$$

$$\sum_{e\in E} \mathbf{l}_e c_{i,j}(e) \leq D_i, \forall j \in K_i, i \in I$$

$$q_i \geq b_i, \forall i \in I$$

$$\sum_{e \in E_o} c_{i,j}(e) = 1, \forall\ j \in K_i, i \in I$$

$$\sum_{e \in E_n^{in}} c_{i,j}(e) = \sum_{e \in E_n^{out}} c_{i,j}(e), \forall n \in V,\ j \in K_i, i \in I$$

$$\sum_{e \in E_d} c_{i,j}(e) = 1, \forall\ j \in K_i, i \in I$$

$$q_i \geq 0, \forall i \in I$$

$$c_{i,j}(e) = 0\ or\ 1, \forall e \in E,\ j \in K_i, i \in I.$$

We apply the above model for computation by software GAMS to obtain the numerical results.

Let $\mathbf{x} = \{(q_i, c_{i,j}(e)) \mid \forall j \in K_i, \text{for} i = 1, \mathbf{K}, m, \forall e \in E\} \in R^n$ denote the vector of decision variables and $Q^* = \{\mathbf{x} \mid \mathbf{x}\,\text{satisfies constraints}\,(2)-(10)\}$ denote the feasible set. We consider a resource allocation problem defined as an optimization problem with $m$ objective functions $f_i(q_i)$:

$$\max\{\mathbf{f(q)} : x \in Q^*\}, \tag{11}$$

where $\mathbf{f(q)} = (f_1(q_1), f_2(q_2), \mathbf{K}, f_m(q_m))$ is a vector-function of bandwidth allocation $q = (q_1, q_2, \mathbf{K}, q_m)$. In Ogryczak et al. (2003), it has been shown that (11) of multiple criteria may give Pareto optimal solutions if and only if it is a fair solution of the resource allocation problem, where a fair solution shall be defined with Model II in the next section.

## 3. TWO MODELS WITH WEIGHTED UTILITY FUNCTIONS

### 3.1 Model I

Now, the following equations are obtained for the queueing system, using supplementary variable technique: Let $w_i$ be each weight of $f_i(q_i), \forall i = 1, \mathbf{K}, m$, for a strictly concave, increasing function $f_i : R \to R$, the function $\sum_{i=1}^{m} w_i f_i(q_i)$ is a strictly monotonic and strictly Schur-concave function in Ogryczak et al. (2003). The function $f_i(q_i)$ is also continuous, increasing, and concave, so is $\sum_{i=1}^{m} w_i f_i(q_i)$. In the following, we construct a model, Model I, to solve a problem of nonlinear objective function, such that

$$\begin{aligned} \text{(ModelI)} \quad \text{maximize} \quad & \sum_{i=1}^{m} w_i f_i(q_i) \\ \text{subject to} \quad & \sum_{i=1}^{m} w_i = 1 \\ & w_i \geq 0 \\ & \mathbf{x} \in Q^*. \end{aligned} \tag{12}$$

We use software GAMS to solve this kind of network problems. In general, it takes almost less than several minutes to get a numerical result. Besides, we also use it to solve the two weighted models: Model I and II. The numerical results of (12) are given in Section 4 for demonstration.

### 3.2 Model II

Consider the Ordered Weighted Averaging Method provided in Ogryczak et al. (2003). First, we define the ordering map

$$\hat{\mathbf{\Psi}} : R^m \to R^m.$$

Assume that

$$\hat{\mathbf{\Psi}}\mathbf{(f(q))} = (\hat{\Psi}_1(\mathbf{f}(\mathbf{q})), \hat{\Psi}_2(\mathbf{f}(\mathbf{q})), \mathbf{K}, \hat{\Psi}_m(\mathbf{f}(\mathbf{q}))), \tag{13}$$

where $\hat{\Psi}_1(\mathbf{f}(\mathbf{q})) \le \hat{\Psi}_2(\mathbf{f}(\mathbf{q})) \le \dots \le \hat{\Psi}_m(\mathbf{f}(\mathbf{q}))$ and there exists a permutation $t$ of set $S = \{1,2,\dots,m\}$ such that $\hat{\Psi}_k(\mathbf{f}(\mathbf{q})) = f_{t(k)}(\mathbf{q})$ for $k = 1,\dots,m$. Then we define the cumulative ordering map $\mathbf{Y}(\mathbf{f(q)}) = (\mathbf{y}_1(\mathbf{f(q)}), \dots, \mathbf{y}_m(\mathbf{f(q)}))$ as $\mathbf{y}_k(\mathbf{f}(\mathbf{q})) = \sum_{i=1}^{k}\hat{\Psi}_i(\mathbf{f}(\mathbf{q}))$, for $i = 1,2,\dots,m$. In the following, we adopt an effective modeling technique for quantities $\mathbf{y}_k(\mathbf{f}(\mathbf{q}))$ with arbitrary $i$. In Ogryczak et al. (2003), for a given outcome vector $\mathbf{f}(\mathbf{q})$ the quantity $\mathbf{y}_k(\mathbf{f}(\mathbf{q}))$ may be found by solving the following linear program:

$$\begin{aligned} \mathbf{y}_k(\mathbf{f}(\mathbf{q})) = \quad & \text{maximize} && kt_k - \sum_{i=1}^{m} d_i \\ & \text{subject to} && t_k - f_i(q_i) \le d_i,\ i = 1,\dots,m \\ & && d_i \ge 0,\ i = 1,\dots,m, \end{aligned} \tag{14}$$

where $t_k$ is an unrestricted variable when nonnegative variables $d_i$ represent their downside deviations from the value of $t_k$ for several values $f_i(q_i)$. Taking an example, the simplest outcome may be defined by the following optimization:

$$\mathbf{y}_1(\mathbf{f}(\mathbf{q})) = \max\{t_1 : t_1 \le f_i(q_i) \text{ for } i = 1,\dots,m\},$$

where $t_1$ is an unrestricted variable. Formula (14) provides us with a computational formulation for the worst conditional mean $M_{\frac{k}{m}}(\mathbf{f}(\mathbf{q}))$ defined as the mean outcome for the $k$ worst-off services, i.e.,

$$M_{\frac{k}{m}}(\mathbf{f}(\mathbf{q})) = \frac{1}{k}\mathbf{y}_k(\mathbf{f}(\mathbf{q})), \text{ for } k = 1,\dots,m.$$

For $k = 1$, $M_{\frac{1}{m}}(\mathbf{f}(\mathbf{q})) = \mathbf{y}_1(\mathbf{f}(\mathbf{q})) = \hat{\Psi}_1(\mathbf{f}(\mathbf{q}))$ represents the minimum outcome and for $k = m$, $M_{\frac{m}{m}}(\mathbf{f}(\mathbf{q})) = \frac{1}{m}\mathbf{y}_m(\mathbf{f}(\mathbf{q})) = \frac{1}{m}\sum_{k=1}^{m}\hat{\Psi}_k(\mathbf{f}(\mathbf{q})) = \frac{1}{m}\sum_{i=1}^{m} f_i(q_i)$ represents the mean outcome.

For modeling various fair preferences, one may use some combinations of the cumulative ordered outcomes $\mathbf{y}_k(\mathbf{f(q)})$. In specific, for $v_k \ge 0$, the weighted sum is

$$\sum_{k=1}^{m} v_k \mathbf{y}_k(\mathbf{f(q)}). \tag{15}$$

Note that, due to the definition of map $\mathbf{y}_k$, the above function can be expressed in the form with ordered weights $\hat{w}_k = \sum_{j=k}^{m} v_j$ $(k = 1,\dots,m)$ allocated to coordinates of the ordered outcome vector. When substituting $v_k$ with $\hat{w}_k$ where $\hat{w}_h$ is an ordered weight, (15) becomes $\sum_{h=1}^{m}\hat{w}_k\hat{\Psi}_k(\mathbf{f(q)})$, where $\sum_{k=1}^{m}\hat{w}_k = \sum_{i=1}^{m} w_i = 1$ and $\hat{w}_k \ge 0$, $\forall k = 1,\dots,m$.

Applying the Ordered Weighted Averaging Method to problem (11), we get

$$\max\{\sum_{k=1}^{m}\hat{w}_k\hat{\Psi}_k(\mathbf{f(q)}) : \mathbf{x} \in Q^*\}, \tag{16}$$

where (16) becomes $\sum_{i=1}^{m} w_i f_i(q_i)$. If ordered weights $\hat{w}_k$ are decreasing and nonnegative, that is $\hat{w}_1 \ge \hat{w}_2 \ge \dots \ge \hat{w}_{m-1} \ge \hat{w}_m \ge 0$, then each optimal solution of the problem (16) is a **fair solution** of (11). Actually, formulas (14) and (15) allow us to formulate the following mathematical programming of the original multiple criteria problem:

$$
\begin{aligned}
\text{(Model II)}\quad \text{maximize}\quad & \sum_{k=1}^{m} v_k \mathbf{y}_k \\
\text{subject to}\quad & \mathbf{y}_k = kt_k - \sum_{i=1}^{m} d_{ki}, \forall k = 1, \mathbf{K}, m \\
& t_k - d_{ki} \le f_i(q_i), \forall i, k = 1, \mathbf{K}, m \\
& d_{ki} \ge 0, \forall i, k = 1, \mathbf{K}, m \\
& t_k \text{ unrestricted}, \forall k = 1, \mathbf{K}, m \\
& \sum_{k=1}^{m} kv_k = 1 \\
& v_k \ge 0, \forall k = 1, \mathbf{K}, m \\
& \mathbf{x} \in Q,^{*}
\end{aligned}
\tag{17}
$$

where $v_m = w_m$, $v_k = \hat{w}_k - \hat{w}_{k+1}$ for $k = 1, \mathbf{K}, m-1$, $\hat{w}_k \in [0,1]$ for each $k$, and $\sum_{k=1}^{m} \hat{w}_k = 1$. The individual function $\mathbf{y}_k$ is the first $k$ sum of the ordered multiple objective functions $\hat{\mathbf{\Psi}}(\mathbf{f}(\mathbf{q}))$ in the allocation pattern $\mathbf{x} \in Q^*$. In this work, we use software GAMS to maximize the weighted sum of logarithms of the bandwidth for each class $i$, $i = 1, \mathbf{K}, m$. First, we carry on changing the parameter $J_i$ to observe the variations of $q_i, w_i$ and total utilization value. In the next step, we keep on changing the parameter $B$, $a_i$, $r_i$ to observe the variations and see what affects the constraints about $B$, $a_i$, $r_i$. All numerical results are given in Section 4.

## 4. AN ILLUSTRATIVE EXAMPLE AND NUMERICAL RESULTS

### 4.1 An Illustrative Example

Consider a sample network as shown in Fig. 1, where $E = \{e_k, k = 1, 2, \mathbf{K}, 14\}$ denote the set of links in the network respectively. Let node 1 and node 7 be the source and destination respectively. Each connection is delivered from node 1 to node 7. Table 1 shows the capacity $U_e$, mean delay $\mathbf{l}_e$, and the link cost $k_e$ of bandwidth for each link $e \in E$. In Table 2, three different QoS classes are given, where class 1 has the highest priority and class 3 has the lowest priority. We assume every connection in class $i$, for $i = 1,2,3$, has the same aspiration level $a_i$ kbps (i.e. kilobits/sec), reservation level $r_i$ kbps, mean packet size $s_i$ kb, maximal end-to-end delay $D_i$, and bandwidth requirement $b_i$ kbps.

Table 1: Characteristics of Each Link

| Characteristics | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|
| Capacity (kbps) | 230,000 | 350,000 | 100,000 | 250,000 | 210,000 | 220,000 | 200,000 |
| Cost ($) | 5 | 6 | 10 | 5 | 4 | 11 | 6 |
| Delay (sec) | 0.03 | 0.032 | 0.035 | 0.012 | 0.02 | 0.012 | 0.03 |
| | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $e_{12}$ | $e_{13}$ | $e_{14}$ |
| Capacity (kbps) | 300,000 | 210,000 | 270,000 | 150,000 | 180,000 | 30,000 | 350,000 |
| Cost ($) | 8 | 6 | 7 | 12 | 6 | 5 | 6 |
| Delay (sec) | 0.015 | 0.027 | 0.012 | 0.03 | 0.02 | 0.035 | 0.035 |

Table 2: Characteristics of Each QoS Class

| Class $i$ | Bandwidth Requirement $b_i$ (kbps) | Aspiration. Level $a_i$ (kbps) | Reservation. Level $r_i$ (kbps) | Mean Packet Size $s_i$ (kb) | Maximum Delay $D_i$ (sec) |
|---|---|---|---|---|---|
| 1 | 160 | 334 | 167 | 35 | 0.89 |
| 2 | 80 | 166 | 83 | 16.6 | 1.02 |
| 3 | 25 | 56 | 28 | 12.5 | 2.34 |

Suppose, for each link $e$, we have a mean delay $\mathbf{l}_e$ related to the link's speed, propagation delay, and maximal transfer unit. The maximal possible link capacity is $U_e$ on each link $e \in E$, and the link cost is $k_e$ of using one unit bandwidth. A connection $j$ in each class $i$ should be routed through a path $p_{i,j}$ between node 1 and node 7. Under a limited available budget $B$, we plan to allocate the bandwidth in order to provide each class with maximal possible QoS and determine the optimal path from node 1 to node 7 under guaranteed service. Decision variables are listed as follows: $q_i$ denotes the bandwidth allocated to each connection in class $i$, and $c_{i,j}(e)$ is a binary variable which determines whether the link $e$ is chosen for connection $j$ in class $i$. The purpose of this work is to present an mathematical model that provides the decision maker to explore a set of solutions satisfying users' preferences with fairness.

Suppose the number of connections in each class $i$ is $J_i$ for $i = 1,2,3$. Under the total available budget $B = \$1{,}000{,}000$, we want to allocate the bandwidths in order to provide each class with maximal possible QoS defined via the utility function in Luh and Wang (2004). For each class $i$, we consider the objective function $f_i$ as below: $f_1(q_1) = \log_2(q_1/167)$, $f_2(q_2) = \log_2(q_2/83)$, $f_3(q_3) = \log_2(q_3/28)$. Suppose $w_i$, for $i = 1,2,3$, is the weight assigned to each objective function $f_i(q_i)$ and $\sum_{i=1}^{3} w_i = 1$.

### 4.2 Numerical Results of Model I

We want to know weights $w_i$ where $w_i \in [0,1]$, the total sum of optimal utilization, and the selected path $e$ how to change as $J_i$, $B$, $a_i$ and $r_i$ changing. In Table 2, three different QoS classes are given, where class 1 has the highest priority and class 3 has the lowest priority. We change the numbers of connections in each class, and the numerical results are shown in Fig. 2, Fig. 3, and Fig. 4. Next, given $J_1 = J_2 = J_3 = 100$, we change the total budget $B$, and the numerical results are shown in Fig. 5. When $B \le 500{,}000$, we get infeasible solutions.

By numerical results in Figures 2-5, we have a proposition of structures of optimal weights as follow:

**Proposition 2.** For Optimization Model I, there exists a unit vector, such that the total weighted utilization function value in Model I is maximized.

**Proof.**
For each $\mathbf{x} \in Q^*$, there exists $k \in \{1,2,\mathbf{K},m\}$, such that $f_k(q_k) \ge f_i(q_i)$, for each $i = 1,2,\mathbf{K},m$.

Because of $0 \le w_i \le 1$ and $f_i(q_i) \ge 0$, for each $i = 1,2,\mathbf{K},m$, we have $\sum_{i=1}^{m} w_i f_i(q_i) \le \sum_{i=1}^{m} w_i f_k(q_k) \le f_k(q_k) \cdot \sum_{i=1}^{m} w_i \le f_k(q_k)$.

It implies that $w_k = 1$ and $w_j = 0$, for all $j \in \{1,2,\mathbf{K},m\} - \{k\}$.

### 4.3 Numerical Results of Model II

We want to know weights $w_i$ where $w_i \in [0,1]$, the total sum of optimal utilization, and the selected path $e$ how to change as $J_i$, $B$, $a_i$ and $r_i$ changing. We change the numbers of connections in each class, and the numerical results are shown in Fig. 6, Fig. 7, and Fig. 8. Next, given $J_1 = J_2 = J_3 = 100$, we change the total budget $B$, and the numerical results are shown in Fig. 9. When $B \le 500{,}000$, we get infeasible solutions.

From (13), it is easy to determine the permutation $t$ when $\hat{w}_k$ is given, $\forall k$. Then the optimal weights $w_i$, $\forall i$, are obtained. First, we use $q_i$ to obtain the value $f_i(q_i)$. Because there exists a permutation $t$ of set $S = \{1,2,3\}$ such that $\hat{\Psi}_k(\mathbf{f}(\mathbf{q})) = f_{t(k)}(\mathbf{q})$ for $k = 1,2,3$, we have $\hat{\Psi}_k(\mathbf{f}(\mathbf{q}))$. For example, $\hat{\Psi}_1(\mathbf{f}(\mathbf{q})) = f_3(q_3)$, $\hat{\Psi}_2(\mathbf{f}(\mathbf{q})) = f_1(q_1)$, and $\hat{\Psi}_3(\mathbf{f}(\mathbf{q})) = f_2(q_2)$, and we know $\hat{w}_1 = w_3$, $\hat{w}_2 = w_1$, $\hat{w}_3 = w_2$. Thus the relation between $\hat{w}_i$ and $w_i$ is obtained.

By numerical results in Figures 6-9, we have a proposition of structures of optimal weights as follow:

**Proposition 3.** For Optimization Model II, the form of optimal weights is a vector $(w_1, w_2, \mathbf{K}, w_m)$ with $w_i = \frac{1}{m}$, for each $i = 1,2,\mathbf{K},m$.

**Proof.**

Because of $v_m = \hat{w}_m$ and $v_k = \hat{w}_k - \hat{w}_{k+1}$, for each $k = 1,2,\mathbf{K}, m-1$, and $v_k \geq 0$, for all $k = 1,2,\mathbf{K}, m$, it yields $\hat{w}_1 \geq \hat{w}_2 \geq \mathbf{K} \geq \hat{w}_m \geq 0$.

Since it holds $\hat{\Psi}_1(\mathbf{f}(\mathbf{q})) \leq \hat{\Psi}_2(\mathbf{f}(\mathbf{q})) \leq \mathbf{K} \leq \hat{\Psi}_m(\mathbf{f}(\mathbf{q}))$, and $\sum_{k=1}^{m} v_k \mathbf{y}_k(\mathbf{f}(\mathbf{q})) = \sum_{k=1}^{m} \hat{w}_k \hat{\Psi}_k(\mathbf{f}(\mathbf{q}))$, for all $\mathbf{q} \in Q^*$,

we have $\hat{w}_1 \leq \hat{w}_2 \leq \mathbf{K} \leq \hat{w}_m$.

Because of $\sum_{k=1}^{m} \hat{w}_k = 1$, it gives $\hat{w}_1 = \hat{w}_2 = \mathbf{K} = \hat{w}_m = \dfrac{1}{m}$, and $w_1 = w_2 = \mathbf{K} = w_m = \dfrac{1}{m}$.

## 5. CONCLUSIONS

In the paper, we present an approach for weighted utilizations of the fair resource allocation problem that considers multiple services for users under budget constraints while the weighted utilization functions may be interpreted as a measure of QoS on telecommunication networks. We use the solver BARON in software GAMS to solve Model I and II to compare different choices for optimal weights. We list all the numerical results of Model I and II and compare the structure of optimal weights between them. Numerical results show that the optimal weights have simple structures in accord with two models, respectively.

## 6. ACKNOWLEDGEMENTS

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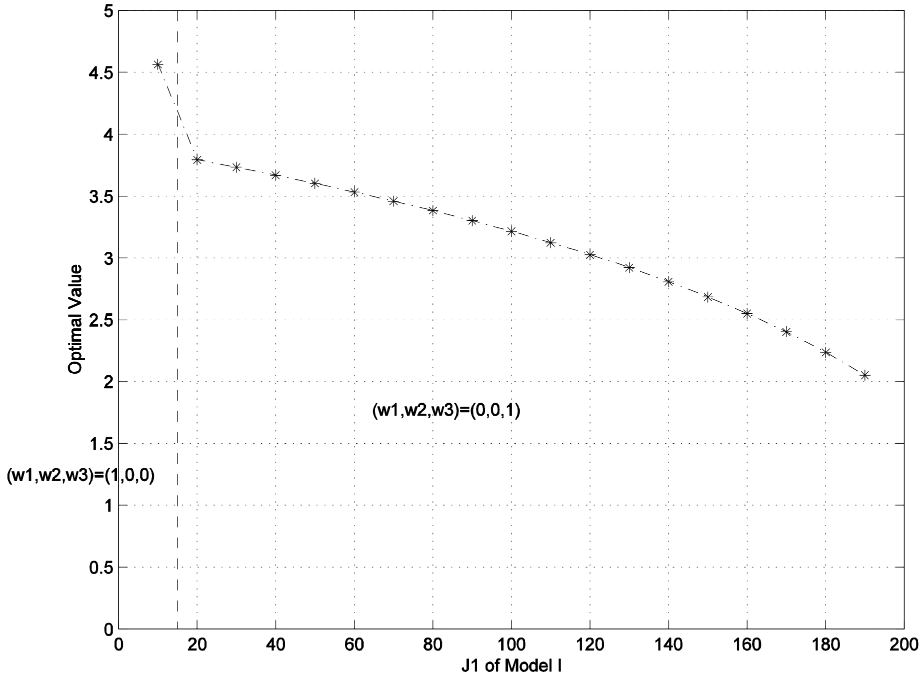


Figure 2: Change in the Number of Connections in Class 1 ( $J_1$ ) for Model I

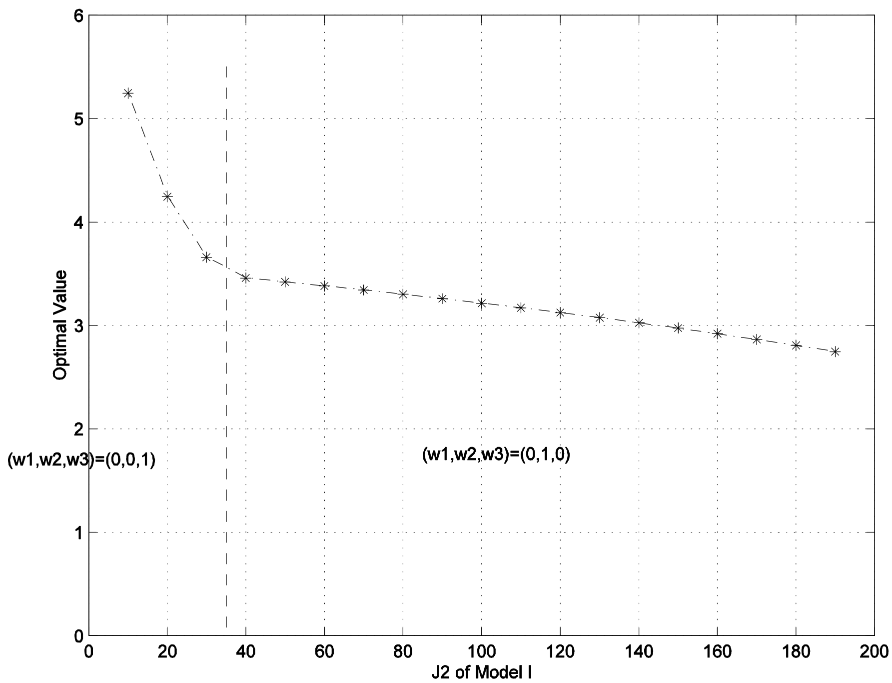


Figure 3: Change in the Number of Connections in Class 2 ( $J_2$ )for Model I

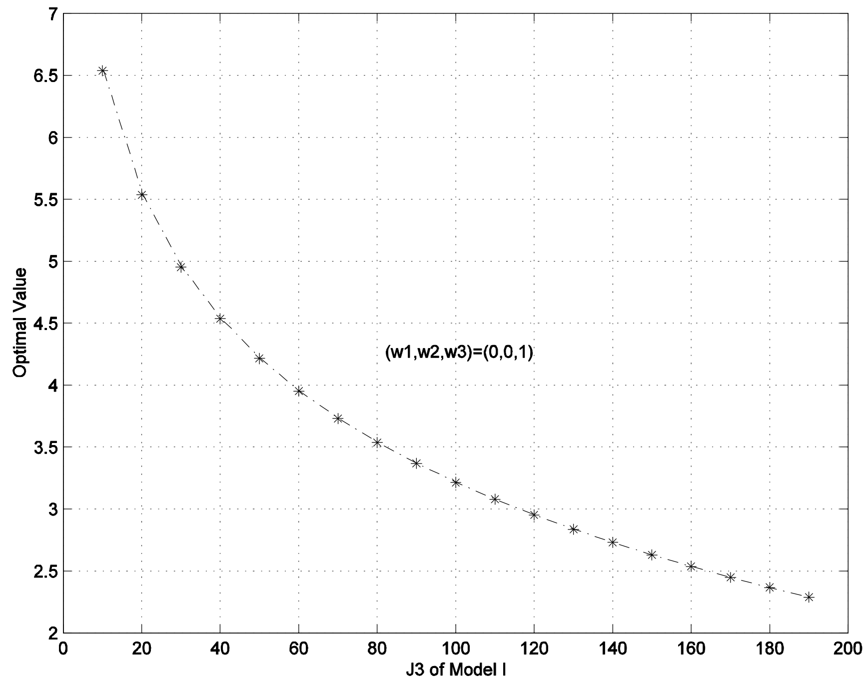


Figure 4: Change in the Number of Connections in Class 3 ( $J_3$ ) for Model I

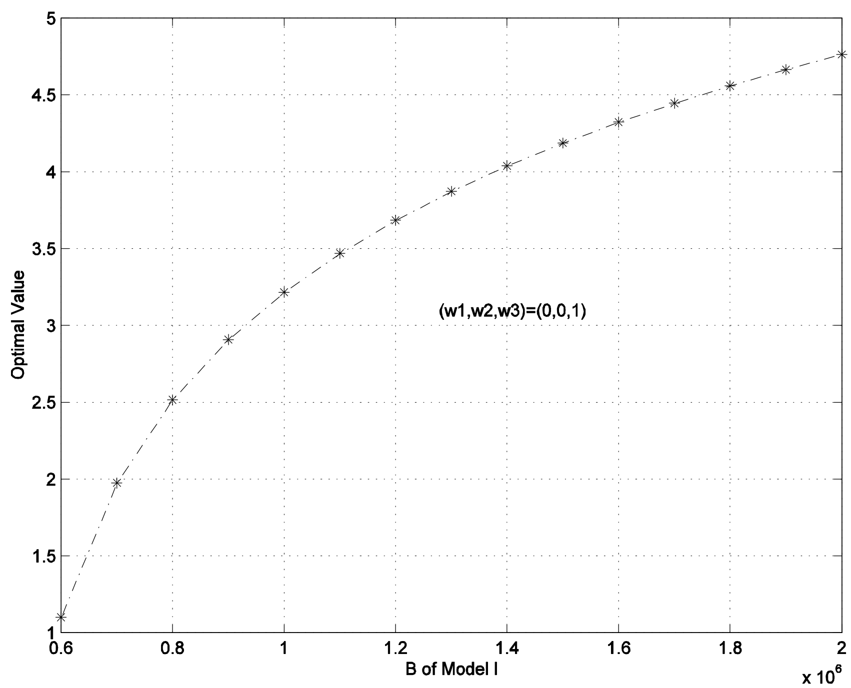


Figure 5: Change in the Total Budget ( $B$ ) for Model I

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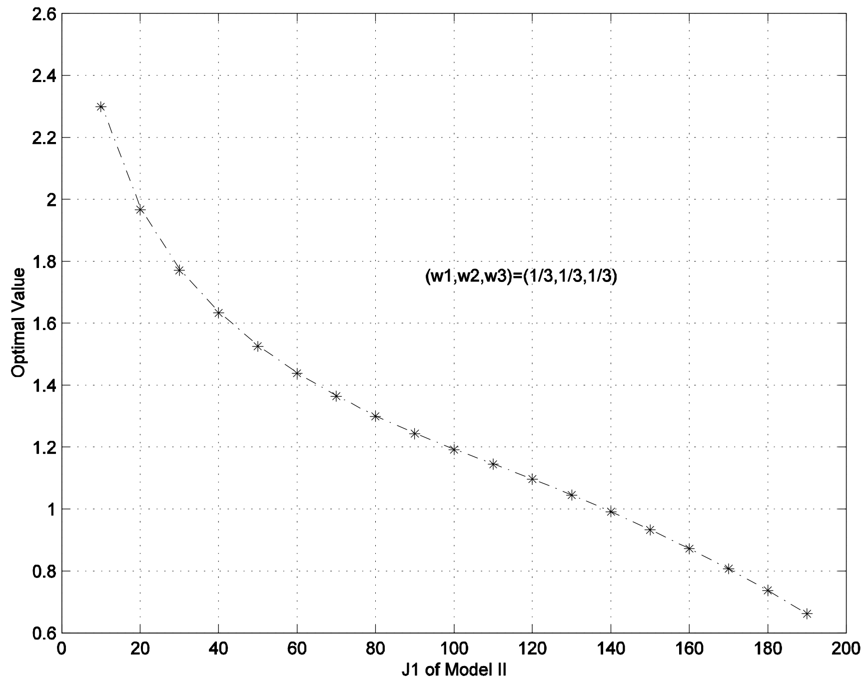


Figure 6: Change in the Number of Connections in Class 1 ( $J_1$ ) for Model II

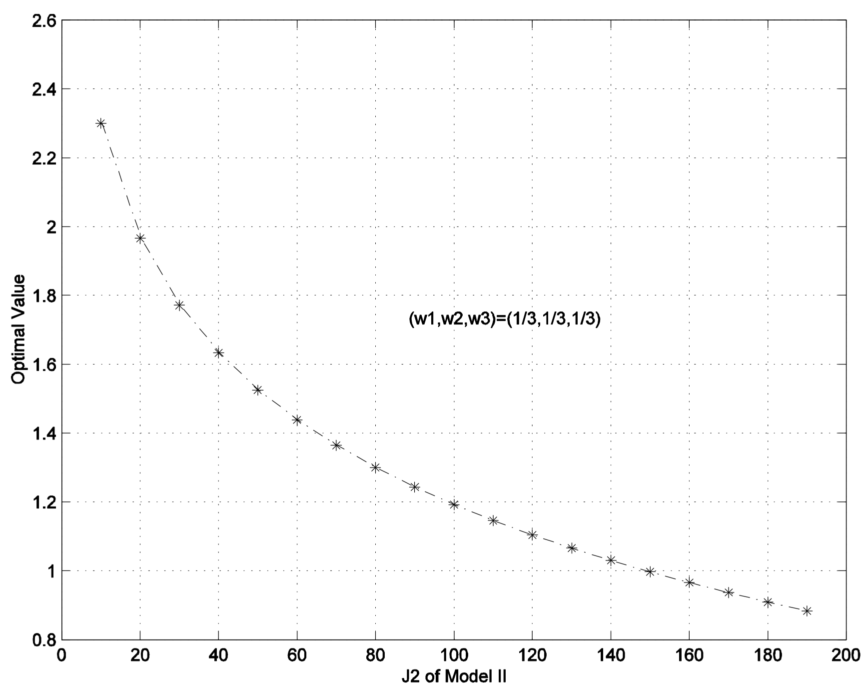


Figure 7: Change in the Number of Connections in Class 2 ( $J_2$ ) for Model II

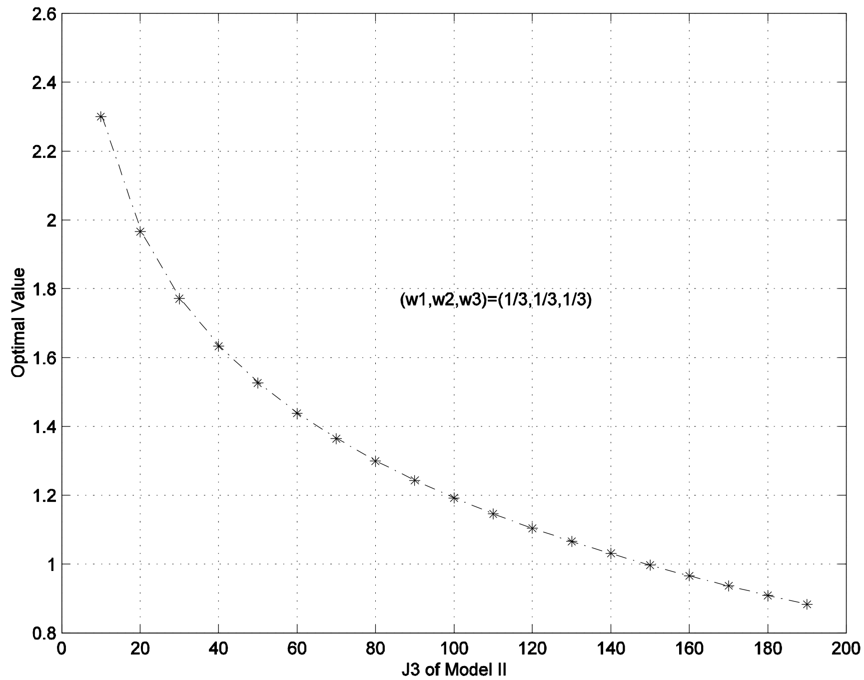


Figure 8: Change in the Number of Connections in Class 3 ( $J_3$ ) for Model II

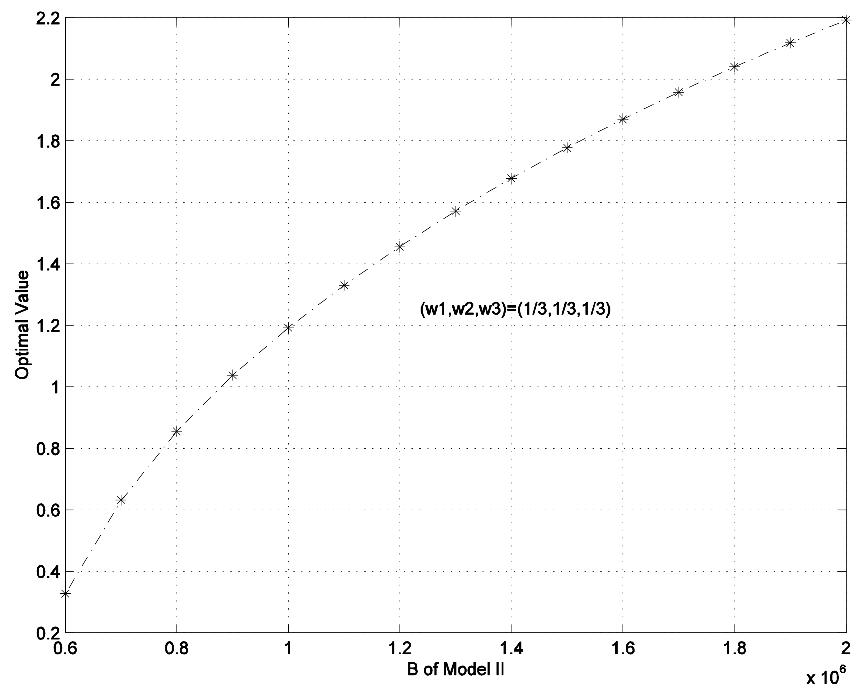


Figure 9: Change in the Total Budget ( $B$ ) for Model II