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Monotone Properties of Optimal Maintenance Policy for Two-State Partially Observable Markov Decision Process Model with Multiple Observations

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Abstract—In this paper, we consider a system that is operated periodically and whose state is either good or bad. When the system stays in the good state, it moves to the bad state with a probability. However, the system in the bad state cannot return to the good state. At each time epoch, we may select one of the following five actions: operation with a monitor, operation without a monitor, inspection, repair, and replacement. After operation, the true state is inferred and we can determine the probability that the system is in the bad state. Thus, the accuracy of the inferred result depends on whether the system is monitored or not. We express the model as a partially observable Markov process and derive the total expected discounted cost for an unbounded horizon. For this model, several structural properties of an optimal maintenance policy are investigated.
Keywords —Partially observable Markov process, TP_2, Monotone property, Concave function, Jensen's inequality

1. INTRODUCTION

Any system is generally operated under various environments and constraints. Hence, most systems cannot remain in good operational condition because of deterioration with elapse of time and thus eventually fail without maintenance. In addition, since many systems have recently become larger and more complex, their failure can cause very serious problems, such as loss of life and economic disaster. To avoid these problems, preventive maintenance plays an important role. If we can determine the deterioration level of the system in some way or its failure is crucial, condition-based maintenance (CBM) would be adopted. Thus, it is important to decide when and how to perform maintenance action, such as inspection, repair, or replacement. Previously, many mathematical models were proposed and analyzed to solve maintenance problems.

If it is appropriate to assume that the deterioration level of a system corresponds to a finite non-negative integer, we may express the deterioration of the system as the Markov process. This model is called the Markovian deteriorating system. Derman (1963) has considered a discrete-time and discrete-state Markovian deteriorating system and derived conditions sufficient for obtaining an optimal control limit policy. The model assumes that replacement is the only maintenance action. Douer and Yechiali (1994) have introduced the idea of uncertain repair, which indicates that the result of repair is uncertain and shown that a generalized control limit policy holds under reasonable conditions. Tamura (2007) has analyzed a Markovian deteriorating system with uncertain repair and investigated the necessity of inspection after completion of repair. These studies consider that one can determine the true state of the system completely at any given time. However, when it is costly to identify the true state of a system through inspection, we should not inspect the system from the viewpoint of efficiency. Instead, some information on the deterioration of the system may be obtained. For example, consider a production process that produces items and observe the quality characteristic of the produced item. The degree of deterioration of the production process reflects the quality characteristic of the produced item. That is, the observed value is a function

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of the true state. Then, we infer the deterioration level, which cannot coincide with the true state. In other words, we select an optimal action based on incomplete information. For the analysis of the problem, partially observable Markov decision process (POMDP) models are suitable. The paper published by Monahan (1982) is an excellent review of this area. Ross (1971) and White (1979) have studied the POMDP models and derived conditions sufficient for obtaining the optimal maintenance policies with monotone structures. Ohnish et al. (1986) have extended these models by introducing the concept of monitoring and obtained more generalized results on the optimal maintenance policy. Ivy and Pollock (2005), and Maehara and Suzuki (2005) have analyzed a POMDP model in which the idea of imperfection and/or multiplicity of repair is introduced. Grosfeld-Nir (2007) has focused on a two-state POMDP in which the observed value is continuous. Also, Grosfeld-Nir (1996) has derived simple equations used to find the control limit under the assumptions that the observed value follows a uniform distribution. Thus, these previous studies have assumed only one procedure for the acquisition of incomplete information on the deterioration of the system. In some cases, however, this assumption is not appropriate. For example, to infer the deterioration level of a rotary machine, we would observe the velocity or acceleration of oscillation. Therefore, it is important to consider a model in which several procedures for acquiring incomplete information are available and the accuracy of inference varies with the procedure.

In this paper, we consider a POMDP model to express the behavior of a system that is operated periodically and whose state can be either good or bad. After each time epoch, one of the following actions may be taken: wait without a monitor, wait with a monitor, inspection, repair, and replacement.

When wait with or without a monitor is conducted, the true state is inferred and incomplete information is obtained. Through inspection, however, we can determine the true state of the system. For this model, we show that an optimal maintenance policy has monotone properties under several conditions.

2. MODEL DESCRIPTION

Consider a system that is periodically operated and whose state is either good or bad at any time. Hereafter, we assume that good and bad states correspond to states *0* and state *1*, respectively. At each time epoch, we may select one of the following actions: wait without a monitor, wait with a monitor, inspection, repair, and replacement. If wait without a monitor is selected, we work the system for one period and do not monitor it after working. Then, the system in state *0* moves to state *1* with the probability p at the next time and the system in state *1* continues to stay there. The working cost u_i is incurred for the system in state i. If wait with a monitor is selected, differently from wait without a monitor, we monitor the system after working and obtain an outcome on its deterioration level with the monitoring cost c_m. The outcome of the monitor is represented by a random variable, which follows the probability density function given by $f_i(y)$, when the system is in state i. The probability that the system is in the bad state is obtained by Bayes' theorem. If inspection is selected, we inspect the system and determine its true state after working with the inspection cost c_d. If repair is selected, the system in state i returns to the good state with the probability q_i at the next time and, the repair cost r_i is incurred. This indicates that the result of repair is uncertain and that the system in state *0* moves to the bad state with a certain probability. After the completion of replacement, however, the system returns to state *0* without fail. Thus, the replacement cost c_i is incurred for the system in state i.

The costs and probabilities satisfy the following assumptions.

Assumption 1 $u_1 \geq u_0$, $r_1 \geq r_0$, $c_1 \geq c_0$

Assumption 2 $u_1 - u_0 \geq r_1 - r_0$, $u_1 - u_0 \geq c_1 - c_0$, $r_1 - r_0 \geq c_1 - c_0$

Assumption 3 $c_d \geq c_m$

Assumption 4 $q_0 \geq q_1$

Assumption 5 $\mathrm{P} - \mathrm{Q} \in SI$, where

$$P = \begin{pmatrix} 1-p & p \\ 0 & 1 \end{pmatrix}, \; Q = \begin{pmatrix} q_0 & 1-q_0 \\ q_1 & 1-q_1 \end{pmatrix}$$

and *SI* indicates Stochastic Increasing.

Assumption 1 indicates that it is more costly to work the system, repair, or replace it with deterioration. Assumption 2 indicates that, as the system deteriorates, the merit of repair or replacement becomes larger than that of working, and the merit of replacement becomes larger than that of repair. Assumption 3 indicates that the cost for inspection is larger than that for monitoring because, unlike monitoring, inspection reveals the true state of the system. Assumption 4 expresses that it is more unlikely for the system to return to the good state with deterioration. Assumption 5 indicates that, with deterioration, the system worked for one period is more likely to move to a worse state compared with that immediately after completion of repair. Therefore, these assumptions are valid from the realistic viewpoint. Assumptions 4 and 5 are called the stochastic order relations (see Kijima (1997)).

3. FORMULATION

Let $f(x,y)$ be the probability density function of the outcome of the monitor given that the state probability is x. The state probability indicates the probability that the true state of the system is the bad state. Thus,

$$f(x,y) = (1-x) f_0(y) + x f_1(y) \tag{1}$$

holds. Moreover, we denote the next state probability when the current state probability is x by $h(x,y)$. Thus,

$$h(x,y) = \frac{x f_1(y) + p(1-x) f_0(y)}{f(x,y)} \tag{2}$$

is obtained.

Let p $(0 < p < 1)$ be the discount factor. We let $V(x)$ denote the total expected discounted cost for an unbounded horizon when the probability that the system stays in the bad state is x. In addition, $W(x)$, $W_M(x)$, $I(x)$, $R_R(x)$, and $R_T(x)$ indicate that wait with a monitor, wait without a monitor, inspection, repair, and replacement are taken when the state probability is x, respectively. Thus, we obtain

$$W(x) = (u_1 - u_0)x + u_0 + \alpha V\left[(1-p)x + p\right], \tag{3}$$

$$W_M(x) = c_m + (u_1 - u_0)x + u_0 + \alpha \int_{-\infty}^{\infty} V\left[h(x,y)\right] f(x,y)dy, \tag{4}$$

$$I(x) = c_d + (u_1 - u_0)x + u_0 + \alpha\left[(1-x)(1-p)V(0) + \{x + (1-x)p\}V(1)\right], \tag{5}$$

$$R_R(x) = (r_1 - r_0)x + r_0 + \alpha\left[\{(q_1 - q_0)x + q_0\}V(0) + \{1 - q_0 + (q_1 - q_0)x\}V(1)\right], \tag{6}$$

$$R_T(x) = (c_1 - c_0)x + c_0 + \alpha V(0). \tag{7}$$

Hence, $V(x)$ is expressed as

$$V(x) = \min\left[W(x), W_M(x), I(x), R_R(x), R_T(x)\right]. \tag{8}$$

To determine the optimal maintenance policy, we should solve the recursive equation Eq.(8). If the optimal maintenance policy has a specific structure, it enables us to reduce the computational time for determining it.

4. STRUCTURAL PROPERTIES OF OPTIMAL MAINTENANCE POLICY

In this section, we give some results on the optimal maintenance policy that minimizes the total expected discounted cost for an unbounded horizon. As a preliminary, the properties of the functions are presented.

Lemma 1 $V(x)$ is concave and increasing in x.

Lemma 2 $W(x)$-$I(x)$, $W_M(x)$-$I(x)$, $W(x)$-$R_R(x)$, and $W_M(x)$-$R_R(x)$, are concave in x.

Lemma 3 $W(x)$-$R_T(x)$, $W_M(x)$-$R_T(x)$, $I(x)$-$R_T(x)$, $R_R(x)$-$R_T(x)$, and $I(x)$-$R_R(x)$ are concave and increasing in x.

Lemma 4 There exists a real number such that if $c_m \le \lambda$, then $W(x)$-$W_M(x)$ changes its sign twice at most. Also, if sign changes occur, then the first change is from minus to plus and the second one is from plus to minus.

By using lemmas 2, 3, and 4, we can derive a monotone property of the optimal maintenance policy. However, we cannot determine the real number from the parameters given in advance, such as the costs and probabilities. This indicates that lemma 4 is not valuable for obtaining the optimal maintenance policy.

Consequently, we assume that $f_i(y)$ is the uniform distribution as follows.

Assumption 6 The outcome Y follows the uniform distribution such that

$$f_i(y)=\begin{cases}1 & y\in K_i\cup K,\\ 0 & y\notin K_i\cup K,\end{cases} \tag{9}$$

where the intervals K_1, K and K_0 are disjoint and are next to each other. In addition,

$$\int_{y\in K_0} f_0(y)dy=\int_{y\in K_1} f_1(y)dy=1-\gamma .$$

Thus, the following result is obtained.

Lemma 5 $W(x)$-$W_M(x)$ is concave in x.

In contrast to lemma 4, lemma 5 is valuable for deriving the optimal maintenance policy. Hereafter, we analyze the model under assumptions 1, 2, 4, and 5.

Let $D(x)$ be the optimal action at the state probability x. For simplicity, if $D(x)=A$ for $x < x_k$ and $D(x)=B$ for, then $x\geq x_k$, we write $A\xrightarrow{x_k}B$, where the suffix indicates that the k-th optimal action is selected below the k-th control limit.

Then, by using the above two lemmas, we can derive several structural properties of the optimal policy.

Theorem 1 The optimal maintenance policy must be classified into one of the following six structures:

1. $W\xrightarrow{x_1^*}W_M\xrightarrow{x_2^*}W\xrightarrow{x_3^*}I\xrightarrow{x_4^*}W\xrightarrow{x_5^*}R_R\xrightarrow{x_6^*}W\xrightarrow{x_7^*}R_T$
2. $W\xrightarrow{x_1^*}W_M\xrightarrow{x_2^*}I\xrightarrow{x_3^*}W_M\xrightarrow{x_4^*}W\xrightarrow{x_5^*}R_R\xrightarrow{x_6^*}W\xrightarrow{x_7^*}R_T$
3. $W\xrightarrow{x_1^*}W_M\xrightarrow{x_2^*}I\xrightarrow{x_3^*}W_M\xrightarrow{x_4^*}R_R\xrightarrow{x_5^*}W_M\xrightarrow{x_6^*}W\xrightarrow{x_7^*}R_T$
4. $W\xrightarrow{x_1^*}I\xrightarrow{x_2^*}W\xrightarrow{x_3^*}W_M\xrightarrow{x_4^*}W\xrightarrow{x_5^*}R_R\xrightarrow{x_6^*}W\xrightarrow{x_7^*}R_T$
5. $W\xrightarrow{x_1^*}I\xrightarrow{x_2^*}W\xrightarrow{x_3^*}W_M\xrightarrow{x_4^*}R_R\xrightarrow{x_5^*}W_M\xrightarrow{x_6^*}W\xrightarrow{x_7^*}R_T$
6. $W\xrightarrow{x_1^*}I\xrightarrow{x_2^*}W\xrightarrow{x_3^*}R_R\xrightarrow{x_4^*}W\xrightarrow{x_5^*}W_M\xrightarrow{x_6^*}W\xrightarrow{x_7^*}R_T$

where $0\leq x_1^*\leq x_2^*\leq x_3^*\leq x_4^*\leq x_5^*\leq x_6^*\leq x_7^*\leq 1.$

Theorem 1 reveals that the optimal maintenance policy may be characterized by eight regions at most. Furthermore, some constraints of the parameter can reduce the numbers of regions and structures of the optimal maintenance policy.

Firstly, we focus on the constraint on the repair probabilities.

Lemma 6 If $q_0=q_1$, $W(x)$-$R_R(x)$, $W_M(x)$-$R_R(x)$, $I(x)$-$R_T(x)$, $R_R(x)$-$R_T(x)$, $I(x)$-$R_R(x)$, $W(x)$-$R_R(x)$ and $W_M(x)$-$R_R(x)$ are concave and are increasing in x, $W(x)$-$I(x)$, $W_M(x)$-$I(x)$ and $W(x)$-$W_M(x)$ are concave in x.

According to lemma 6, the following property is derived.

Theorem 2 If $q_0=q_1$, then the optimal maintenance policy must be classified into one of the following three structures:

1. $W\xrightarrow{x_1^*}W_M\xrightarrow{x_2^*}W\xrightarrow{x_3^*}I\xrightarrow{x_4^*}W\xrightarrow{x_5^*}R_R\xrightarrow{x_6^*}R_T$
2. $W\xrightarrow{x_1^*}W_M\xrightarrow{x_2^*}I\xrightarrow{x_3^*}W_M\xrightarrow{x_4^*}W\xrightarrow{x_5^*}R_R\xrightarrow{x_6^*}R_T$
3. $W\xrightarrow{x_1^*}I\xrightarrow{x_2^*}W\xrightarrow{x_3^*}W_M\xrightarrow{x_4^*}W\xrightarrow{x_5^*}R_R\xrightarrow{x_6^*}R_T$

where $0\leq x_1^*\leq x_2^*\leq x_3^*\leq x_4^*\leq x_5^*\leq x_6^*\leq 1.$

Theorem 2 as well as theorem 1, indicates that the optimal maintenance policy is characterized by seven regions at most. If the monitoring cost c_m=0, then there exists the unique optimal maintenance policy and the number of regions is less than that of theorem 2 as follows.

Theorem 3 If $c_m=0$, then there exists the optimal maintenance policy of the following structure,

$$D(x) = \begin{cases} W_M & \text{for } 0 \le x < x_1^*, \\ I & \text{for } x_1^* \le x < x_2^*, \\ W_M & \text{for } x_2^* \le x < x_3^*, \\ R_R & \text{for } x_3^* \le x < x_4^*, \\ W_M & \text{for } x_4^* \le x < x_5^*, \\ R_T & \text{for } x_5^* \le x \le 1, \end{cases}$$

where $0 \le x_1^* \le x_2^* \le x_3^* \le x_4^* \le x_5^* \le 1.$

Theorem 3 coincides with the basic structure of the optimal maintenance policy in the previous studies. In other words, the above property includes the optimal maintenance policy for the previous two-state models. Furthermore, from theorems 2 and 3, we can derive the following result.

Corollary 1 If c_m=0 and q_0=q_1, then there exists the optimal maintenance policy of the following structure,

$$D(x) = \begin{cases} W_M & \text{for } 0 \le x < x_1^*, \\ I & \text{for } x_1^* \le x < x_2^*, \\ W_M & \text{for } x_2^* \le x < x_3^*, \\ R_R & \text{for } x_3^* \le x < x_4^*, \\ R_T & \text{for } x_4^* \le x \le 1, \end{cases}$$

where $0 \le x_1^* \le x_2^* \le x_3^* \le x_4^* \le 1.$

Corollary 1 reveals that the minimum number of regions of the optimal maintenance policy is five.

5. NUMERICAL EXAMPLES

In the former section, we have shown the structural properties of the optimal maintenance policy that can be used to solve the recursive equation Eq.(8). This section provides several numerical results that can be used to investigate further the quantitative characteristics of the optimal maintenance policy.

Firstly, we set the following parameters for numerical analysis.

$u_0 = 10.0$, $u_1 = 20.0$, $r_0 = 21.0$, $r_1 = 30.8$, $c_0 = c_1 = 35.4$, $c_m = 0.473$, $c_d = 1.256$,
$\beta = 0.80$, $p = 0.30$, $\gamma = 0.55$, $q_0 = 0.90$, $q_1 = 0.66$.

These parameters satisfy assumptions 1, 2, 4 and 5. Now, the purpose is to investigate the widths of the respective regions where the identical action is selected as an optimal action. Table 1 shows the optimal maintenance policy under the above parameters.

Table 1. Numerical results

$D(x)$	$[x_i^*, x_{i+1}^*)$
W	[0.000, 0.651)
W_M	[0.651, 0.666)
I	[0.666, 0.670)
W_M	[0.670, 0.873)
W	[0.873, 0.890)
R_R	[0.890, 0.936)
W	[0.936, 0.939]
R_T	[0.939, 1.000]

Table 1 indicates that theorem 1 holds. Intuitively, we might infer that the widths of the regions where the identical action is optimal decrease as the state probability increases. For example, when the three regions of $D(x)=W$ are compared, region [0.000,0.651) is larger than region [0.873,0.890) and region [0.936,0.939) is the smallest among them. In the case of $D(x)=W_M$, however, this property does not hold. Therefore, we find that the verification of our inference requires additional assumptions.

Secondly, we obtain some numerical results that can be used to investigate how the constraint on the parameters affects the widths of the respective regions. Consequently, the repair probabilities q_0 and q_1 are assumed be identical. Table 2 provides the numerical results for three cases below.

Table 2. Optimal Maintenance Policies under different repair probabilities

$q_0=q_1=0.78$		$q_0=q_1=0.69$		$q_0=q_1=0.66$	
$D(x)$	$[x_i^*, x_{i+1}^*)$	$D(x)$	$[x_i^*, x_{i+1}^*)$	$D(x)$	$[x_i^*, x_{i+1}^*)$
W	[0.000, 0.444)	W	[0.000, 0.667)	W	[0.000, 0.650)
I	[0.444, 0.742)	W_M	[0.667, 0.830)	I	[0.650, 0.760)
W_M	[0.742, 0.758)	W	[0.830, 0.905)	W_M	[0.760, 0.887)
W	[0.758, 0.829)	R_R	[0.905, 0.963)	W	[0.887, 0.940)
R_R	[0.829, 1.000]	R_T	[0.963, 1.000]	R_T	[0.940, 1.000]

The numerical results reveal that the numbers of regions are identical and the orders of the optimal actions are different. Since the system is operated upon the selection of the action W, W_M, or I, we conjecture that the increase in repair probability raises the control limit for maintenance for the action R_R or R_T. However, Table 2 shows that $D(x)=R_R$ over $x=0.829$ for $q_i=0.78$, $D(x)=R_R$ or R_T over $x=0.905$ for $q_i=0.69$, and $D(x)=R_T$ over $x=0.940$ for $q_i=0.66$, where $i=0$ and 1. Therefore, we find that the control limit for the action R_R or R_T does not necessarily depend on the repair probability only.

6. CONCLUSIONS

We construct a two-state POMDP model with two types of observation, by which incomplete information can be obtained. This model is difficult to analyze because the value function consists of two concave functions. For this model, in this paper, we investigate the structural properties of the optimal maintenance policy.

By introducing the assumption that the outcome of monitoring follows a uniform distribution, we have shown that the optimal maintenance policy may be characterized by eight regions at most and it must be classified into one of the specific six structures. This seems to be complex because the optimal maintenance policy in the previous studies can be unique. However, the result would be very remarkable since the model assumes the two procedures for obtaining incomplete information differently from the past ones.

Furthermore, we have shown that some constraints of the parameters, such as the costs and the probabilities, reduce the numbers of regions and structures of the optimal maintenance policy. Thus, we ascertain that the elimination of one concave function can derive the uniqueness of the optimal maintenance policy whose structure is commonly observed in the previous studies.

Our future work is to transform the model into a system consisting of multistate more than three states under the same assumptions used in this study. Also, it is important to relax the assumption on the uniform distribution and make assumptions that reduce the numbers of structures of the optimal maintenance policy further.

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APPENDIX

A. Proof of Lemmas

A.1 Proof of Lemma 1

We write the n-period discounted cost as $V^n(x)$. From assumption 1, $V^1(x)$ is concave and increasing in x since Eqs.(3)-(7) are linear and increasing in x. If we assume that $V^{n-1}(x)$ is concave and increasing in x, for $x_\lambda \equiv (1-\lambda)x + \lambda x' \ (0 \le \lambda \le 1)$,

$$\begin{aligned}
&V^{n-1}\left[\frac{x_\lambda f_1(y) + p(1-x_\lambda) f_0(y)}{f(x_\lambda, y)}\right] f(x_\lambda, y) \\
&= V^{n-1}\left[\frac{(1-\lambda) f(x,y)}{f(x_\lambda, y)} \frac{x f_1(y) + p(1-x) f_0(y)}{f(x,y)} + \frac{\lambda f(x',y)}{f(x_\lambda, y)} \frac{x' f_1(y) + p(1-x') f_0(y)}{f(x',y)}\right] f(x_\lambda, y) \\
&\ge (1-\lambda) f(x,y) V^{n-1}\left[\frac{x f_1(y) + p(1-x') f_0(y)}{f(x,y)}\right] + \lambda f(x',y) V^{n-1}\left[\frac{x' f_1(y) + p(1-x') f_0(y)}{f(x',y)}\right]
\end{aligned}$$

holds, and $V^n(x) \ge V^n(x')$ for $x \ge x'$ because $h(x,y)$ is increasing in x. Since $V^n(x)$ converges on $V(x)$ uniformly, the lemma holds.

A.2 Proof of Lemma 2

From Eqs.(3)-(7), $W(x)$-$I(x)$, $W_M(x)$-$I(x)$, $W(x)$-$R_R(x)$ and $W_M(x)$-$R_T(x)$ can be expressed by linear and concave functions since theorem 1 holds. Therefore, the above four functions are concave in x.

A.3 Proof of Lemma 3

From Eqs.(3)-(7), $W(x)$-$R_T(x)$, $W_M(x)$-$R_T(x)$, $I(x)$-$R_T(x)$ and $R_R(x)$-$R_T(x)$ can be expressed by linear and concave functions as lemma 2. Then, according to assumptions 2 and 4, and theorem 1, these functions are increasing in x. Hence, the lemma holds.

A.4 Proof of Lemma 4

From Eqs.(3) and (4),

$$W(x) - W_M(x) = \alpha\left\{V\left[(1-p)x + p\right] - \int_{-\infty}^{\infty} V\left[h(x,y)\right] f(x,y) dy\right\} - c_m. \tag{10}$$

According to Jensen's inequality,

$$V\left[(1-p)x + p\right] \ge \int_{-\infty}^{\infty} V\left[h(x,y)\right] f(x,y) dy$$

since theorem 1 holds, and these functions $W(x)$ and $W_M(x)$ intersect at two points $x=0$ and $x=1$ from Eqs.(1) and

(2). Hence, Eq.(10) has a maximal value at the point x' where $0 < x' < 1$. If the maximal value is greater than c_m, the lemma holds.

A.5 Proof of Lemma 5

If assumption 6 holds,

$$h(x,y) = \begin{cases} p & y \in K_0, \\ p(1-x)+x & y \in K, \\ 1 & y \in K_1, \end{cases}$$

and

$$W_M(x) = u_0(1-x) + u_1 x + c_m + \beta(1-\gamma)\{(1-x)V(p) + xV(1)\} + \gamma\beta V[p(1-x)+x].$$

Then

$$W(x) - W_M(x) = \beta(1-\gamma)V[p(1-x)+x] - c_m - \beta(1-\gamma)[V(p) - x\{V(p)+V(1)\}],$$

and this function is concave in x. Therefore, lemma 5 holds.

A.6 Proof of Lemma 5

Since $q_0 = q_1$,

$$R_R(x) = (r_1 - r_0)x + r_0 + \alpha q_0 V(0) + (1-q_0)V(1).$$

Hence, we find that $W(x)$-$R_R(x)$ and $W_M(x)$-$R_R(x)$ are linear in x. This completes the proof.

B. Proof of Theorems

B.1 Proof of Theorem 1

Lemma 3 reveals that there exist real numbers $\tilde{x}_i\ (i = 1,\cdots,5)$ such that

$$W(x) \le R_T(x) \ \ \text{for } 0 \le x < \tilde{x}_1 \ \ \text{and} \ \ W(x) > R_T(x) \ \ \text{for } \tilde{x}_1 \le x \le 1, \tag{11}$$
$$W_M(x) \le R_T(x) \ \ \text{for } 0 \le x < \tilde{x}_2 \ \ \text{and} \ \ W_M(x) > R_T(x) \ \ \text{for } \tilde{x}_2 \le x \le 1, \tag{12}$$
$$I(x) \le R_T(x) \ \ \text{for } 0 \le x < \tilde{x}_3 \ \ \text{and} \ \ I(x) > R_T(x) \ \ \text{for } \tilde{x}_3 \le x \le 1, \tag{13}$$
$$R_R(x) \le R_T(x) \ \ \text{for } 0 \le x < \tilde{x}_4 \ \ \text{and} \ \ R_R(x) > R_T(x) \ \ \text{for } \tilde{x}_4 \le x \le 1, \tag{14}$$
$$I(x) \le R_R(x) \ \ \text{for } 0 \le x < \tilde{x}_5 \ \ \text{and} \ \ I(x) > R_R(x) \ \ \text{for } \tilde{x}_5 \le x \le 1. \tag{15}$$

Also, lemma 2 reveals that there exist real numbers such that $\tilde{x}_i\ (i = 6,\cdots,15)$

$$W(x) \le I(x),\ 0 \le x < \tilde{x}_6,\ W(x) > I(x),\ \tilde{x}_6 \le x < \tilde{x}_7,\ W(x) \le I(x),\ \tilde{x}_7 \le x \le 1,$$
$$W(x) \le R_R(x),\ 0 \le x < \tilde{x}_8,\ W(x) > R_R(x),\ \tilde{x}_8 \le x < \tilde{x}_9,\ W(x) \le R_R(x),\ \tilde{x}_9 \le x \le 1,$$
$$W_M(x) \le I(x),\ 0 \le x < \tilde{x}_{10},\ W_M(x) > I(x),\ \tilde{x}_{10} \le x < \tilde{x}_{11},\ W_M(x) \le I(x),\ \tilde{x}_{11} \le x \le 1,$$
$$W_M(x) \le R_R(x),\ 0 \le x < \tilde{x}_{12},\ W_M(x) > R_R(x),\ \tilde{x}_{12} \le x < \tilde{x}_{13},\ W_M(x) \le R_R(x),\ \tilde{x}_{13} \le x \le 1,$$
$$W(x) \le W_M(x),\ 0 \le x < \tilde{x}_{14},\ W(x) > W_M(x),\ \tilde{x}_{14} \le x < \tilde{x}_{15},\ W(x) \le W_M(x),\ \tilde{x}_{15} \le x \le 1,$$

where $\tilde{x}_6 \le \tilde{x}_7$, $\tilde{x}_8 \le \tilde{x}_9$, $\tilde{x}_{10} \le \tilde{x}_{11}$, $\tilde{x}_{12} \le \tilde{x}_{13}$ *and* $\tilde{x}_{14} \le \tilde{x}_{15}$.

Now, suppose $\bar{x} \equiv \max[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4]$. Eqs.(11)-(14) show that $D(x)$=R_T for $\bar{x} \le x \le 1$ since $\tilde{x}_5 \le \tilde{x}_4$ always holds from Eqs.(13)-(15).

Consequently, we temporarily ignore $R_T(x)$ and focus on $W(x)$, $I(x)$, and $R_R(x)$. The number of orders of the intersections $\tilde{x}_5$, $\tilde{x}_6$, $\tilde{x}_7$, $\tilde{x}_8$, and $\tilde{x}_9$ can be four since the other orders are inconsistent with lemmas 2 and 3. We give the possible four orders and the minimum functions in the respective regions as follows.

Order 1 $0 \le \tilde{x}_6 \le \tilde{x}_7 \le \tilde{x}_5 \le \tilde{x}_8 \le \tilde{x}_9 \le 1.$

$\min[W(x), I(x), R_R(x)] = W(x), 0 \le x < \tilde{x}_6$, $\min[W(x), I(x), R_R(x)] = I(x), \tilde{x}_6 \le x < \tilde{x}_7$,

$\min[W(x), I(x), R_R(x)] = W(x), \tilde{x}_7 \le x < \tilde{x}_8$, $\min[W(x), I(x), R_R(x)] = R_R(x), \tilde{x}_8 \le x < \tilde{x}_9$,

$\min[W(x), I(x), R_R(x)] = W(x), \tilde{x}_9 \le x \le 1.$

Order 2 $0 \le \tilde{x}_6 \le \tilde{x}_8 \le \tilde{x}_5 \le \tilde{x}_7 \le \tilde{x}_9 \le 1.$

$\min[W(x), I(x), R_R(x)] = W(x), 0 \le x < \tilde{x}_6$, $\min[W(x), I(x), R_R(x)] = I(x), \tilde{x}_6 \le x < \tilde{x}_5$,

$\min[W(x), I(x), R_R(x)] = R_R(x), \tilde{x}_5 \le x < \tilde{x}_9$, $\min[W(x), I(x), R_R(x)] = W(x), \tilde{x}_9 \le x \le 1.$

Order 3 $0 \le \tilde{x}_6 \le \tilde{x}_8 \le \tilde{x}_9 \le \tilde{x}_7 \le \tilde{x}_5 \le 1.$

$\min[W(x), I(x), R_R(x)] = W(x), 0 \le x < \tilde{x}_6$, $\min[W(x), I(x), R_R(x)] = I(x), \tilde{x}_6 \le x < \tilde{x}_7$,

$\min[W(x), I(x), R_R(x)] = W(x), \tilde{x}_7 \le x \le 1.$

Order 4 $0 \le \tilde{x}_5 \le \tilde{x}_8 \le \tilde{x}_6 \le \tilde{x}_7 \le \tilde{x}_9 \le 1.$

$\min[W(x), I(x), R_R(x)] = W(x), 0 \le x < \tilde{x}_8$, $\min[W(x), I(x), R_R(x)] = R_R(x), \tilde{x}_8 \le x < \tilde{x}_9$,

$\min[W(x), I(x), R_R(x)] = W(x), \tilde{x}_9 \le x \le 1.$

The above result reveals that order 1 includes orders 2, 3 and 4. If we replace *W(x)* with $W_M(x)$, then the same orders of minimum functions as the above four hold.

Hence, we can summarize the order of minimum functions with an increase in state probability below.

$$\min[W(x), I(x), R_R(x), R_T(x)] = \begin{cases} W(x), & 0 \le x < x_1^\bullet, \\ I(x), & x_1^\bullet \le x < x_2^\bullet, \\ W(x), & x_2^\bullet \le x < x_3^\bullet, \\ R_R(x), & x_3^\bullet \le x < x_4^\bullet, \\ W(x), & x_4^\bullet \le x < \overline{x}, \\ R_T(x), & \overline{x} \le x \le 1, \end{cases} \tag{16}$$

where $0 \le x_1^\bullet \le x_2^\bullet \le x_3^\bullet \le x_4^\bullet \le \overline{x} \le 1.$

We can ignore the intersections in Eq.(16) because the purpose is to establish the structure of the optimal maintenance policy for the proposed model.

Next, we investigate the order of minimum functions including the function $W_M(x)$ in detail. *W(x)* and $W_M(x)$ intersect at two points $\tilde{x}_{14}$ and $\tilde{x}_{15}$. These are located in two regions of $[0, x_1^\bullet), [x_1^\bullet, x_2^\bullet), [x_2^\bullet, x_3^\bullet), [x_3^\bullet, x_4^\bullet)$ and $[x_4^\bullet, \overline{x})$ at most from Eq.(16) since $D(x)=R_T$ for $x \in [\overline{x}, 1]$ from Eqs.(11) and (12). The total number of orders is 15, because if the two points are located in two different regions, then there are ${}_5C_2$ combinations, and if the two points are located in one region, then there are ${}_5C_1$ combinations.

Now, we enumerate the total orders without contradiction based on lemmas 2 and 3 below. The orders of minimum functions are described and the intersections are omitted for the respective cases because of the former reason.

- In the case of $\tilde{x}_{14} \in [0, x_1^\bullet)$,
  - ✓ if $\tilde{x}_{15} \in [0, x_1^\bullet)$, then the order of minimum functions is
    $W(x) \to W_M(x) \to W(x) \to I(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x).$
  - ✓ if $\tilde{x}_{15} \in [x_1^\bullet, x_2^\bullet)$, then the order of minimum functions is
    $W(x) \to W_M(x) \to I(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x).$
  - ✓ if $\tilde{x}_{15} \in [x_2^\bullet, x_3^\bullet)$, then the order of minimum functions is
    $W(x) \to W_M(x) \to I(x) \to W_M(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x).$
  - ✓ if $\tilde{x}_{15} \in [x_3^\bullet, x_4^\bullet)$, then the order of minimum functions is
    $W(x) \to W_M(x) \to I(x) \to W_M(x) \to R_R(x) \to W(x) \to R_T(x).$

✓ if $\tilde{x}_{15} \in \left[x_4^{\bullet}, \overline{x}\right)$, then the order of minimum functions is
$W(x) \to W_M(x) \to I(x) \to W_M(x) \to R_R(x) \to W_M(x) \to W(x) \to R_T(x)$.

· In the case of $\tilde{x}_{14} \in \left[x_1^{\bullet}, x_2^{\bullet}\right)$,

✓ if $\tilde{x}_{15} \in \left[x_1^{\bullet}, x_2^{\bullet}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x)$.

✓ if $\tilde{x}_{15} \in \left[x_2^{\bullet}, x_3^{\bullet}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W_M(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x)$.

✓ if $\tilde{x}_{15} \in \left[x_3^{\bullet}, x_4^{\bullet}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W_M(x) \to R_R(x) \to W(x) \to R_T(x)$.

✓ if $\tilde{x}_{15} \in \left[x_4^{\bullet}, \overline{x}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W_M(x) \to R_R(x) \to W_M(x) \to W(x) \to R_T(x)$.

· In the case of $\tilde{x}_{14} \in \left[x_2^{\bullet}, x_3^{\bullet}\right)$,

✓ if $\tilde{x}_{15} \in \left[x_2^{\bullet}, x_3^{\bullet}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W(x) \to W_M(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x)$.

✓ if $\tilde{x}_{15} \in \left[x_3^{\bullet}, x_4^{\bullet}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W(x) \to W_M(x) \to R_R(x) \to W_M \to W(x) \to R_T(x)$.

✓ if $\tilde{x}_{15} \in \left[x_4^{\bullet}, \overline{x}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W(x) \to W_M(x) \to R_R(x) \to W_M(x) \to W(x) \to R_T(x)$.

· In the case of $\tilde{x}_{14} \in \left[x_3^{\bullet}, x_4^{\bullet}\right)$,

✓ if $\tilde{x}_{15} \in \left[x_3^{\bullet}, x_4^{\bullet}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x)$.

✓ if $\tilde{x}_{15} \in \left[x_4^{\bullet}, \overline{x}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W(x) \to R_R(x) \to W_M(x) \to W(x) \to R_T(x)$.

· In the case of $\tilde{x}_{14} \in \left[x_4^{\bullet}, \overline{x}\right)$,

✓ if $\tilde{x}_{15} \in \left[x_4^{\bullet}, \overline{x}\right)$, then the order of minimum functions is
$W(x) \to I(x) \to W(x) \to R_R(x) \to W(x) \to W_M(x) \to W(x) \to R_T(x)$.

Therefore, the orders of minimum functions are summarized as follows.

1. For $\tilde{x}_{14} \in \left[0, x_1^{\bullet}\right)$, and $\tilde{x}_{15} \in \left[0, x_1^{\bullet}\right)$,
$W(x) \to W_M(x) \to W(x) \to I(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x)$.
2. For $\tilde{x}_{14} \in \left[0, x_1^{\bullet}\right)$, and $\tilde{x}_{15} \in \left[x_2^{\bullet}, x_3^{\bullet}\right)$,
$W(x) \to W_M(x) \to I(x) \to W_M(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x)$.
3. For $\tilde{x}_{14} \in \left[0, x_1^{\bullet}\right)$, and $\tilde{x}_{15} \in \left[x_4^{\bullet}, \overline{x}\right)$,
$W(x) \to W_M(x) \to I(x) \to W_M(x) \to R_R(x) \to W_M(x) \to W(x) \to R_T(x)$.
4. For $\tilde{x}_{14} \in \left[x_2^{\bullet}, x_3^{\bullet}\right)$, and $\tilde{x}_{15} \in \left[x_2^{\bullet}, x_3^{\bullet}\right)$,
$W(x) \to I(x) \to W(x) \to W_M(x) \to W(x) \to R_R(x) \to W(x) \to R_T(x)$.
5. For $\tilde{x}_{14} \in \left[x_2^{\bullet}, x_3^{\bullet}\right)$, and $\tilde{x}_{15} \in \left[x_3^{\bullet}, \overline{x}\right)$,
$W(x) \to I(x) \to W(x) \to W_M(x) \to R_R(x) \to W_M(x) \to W(x) \to R_T(x)$.
6. For $\tilde{x}_{14} \in \left[x_4^{\bullet}, \overline{x}\right)$, and $\tilde{x}_{15} \in \left[x_4^{\bullet}, \overline{x}\right)$,
$W(x) \to I(x) \to W(x) \to R_R(x) \to W(x) \to W_M(x) \to W(x) \to R_T(x)$.

Then, the number of regions of the optimal maintenance policy is eight at most. This completes the proof.

B.1 Proof of Theorem 2

We show theorem 3 by using an argument similar to the proof of theorem 1. The difference between the two proofs is the use of lemma 6.

Lemmas 3 and 6 indicate that there exist real numbers $\hat{x}_i\ (i=1,\cdots,7)$ such that

$W(x) \le R_T(x)$ for $0 \le x < \hat{x}_1$, $W(x) > R_T(x)$ for $\hat{x}_1 \le x \le 1$, (17)

$W_M(x) \le R_T(x)$ for $0 \le x < \hat{x}_2$, $W_M(x) > R_T(x)$ for $\hat{x}_2 \le x \le 1$, (18)

$I(x) \le R_T(x)$ for $0 \le x < \hat{x}_3$, $I(x) > R_T(x)$ for $\hat{x}_3 \le x \le 1$, (19)

$R_R(x) \le R_T(x)$ for $0 \le x < \hat{x}_4$, $R_R(x) > R_T(x)$ for $\hat{x}_4 \le x \le 1$, (20)

$I(x) \le R_R(x)$ for $0 \le x < \hat{x}_5$, $I(x) > R_R(x)$ for $\hat{x}_5 \le x \le 1$, (21)

$W(x) \le R_R(x)$ for $0 \le x < \hat{x}_6$, $W(x) > R_R(x)$ for $\hat{x}_6 \le x \le 1$, (22)

$W_M(x) \le R_R(x)$ for $0 \le x < \hat{x}_7$, $W_M(x) > R_R(x)$ for $\hat{x}_7 \le x \le 1$, (23)

Also, lemma 2 indicates that there exist real numbers $\hat{x}_8$, $\hat{x}_9$, and $\hat{x}_{10}$ such that

$W(x) \le I(x),\ 0 \le x < \hat{x}_8$, $W(x) > I(x),\ \hat{x}_8 \le x < \hat{x}_9$, $W(x) \le I(x),\ \hat{x}_9 \le x \le 1$, (24)

$W_M(x) \le I(x),\ 0 \le x < \hat{x}_{10}$, $W_M(x) > I(x),\ \hat{x}_{10} \le x < \hat{x}_{11}$, $W_M(x) \le I(x),\ \hat{x}_{11} \le x \le 1$, (25)

$W(x) \le W_M(x),\ 0 \le x < \hat{x}_{12}$, $W(x) > W_M(x),\ \hat{x}_{12} \le x < \hat{x}_{13}$, $W(x) \le W_M(x),\ \hat{x}_{13} \le x \le 1$, (26)

where $\hat{x}_8 \le \hat{x}_9$, $\hat{x}_{10} \le \hat{x}_{11}$, and $\hat{x}_{12} \le \hat{x}_{13}$.

Now, suppose $\bar{\bar{x}} \equiv \max[\hat{x}_5, \hat{x}_6, \hat{x}_7]$. Eqs.(21), (22) and (23) show that $D(x)=R_R$ for $\bar{\bar{x}} \le x < \bar{x}$. As explained previously in the proof of theorem 1, $D(x)=R_T$ for $\bar{x} \le x \le 1$ holds where $\bar{\bar{x}} \le \bar{x}$.

Hence, when we focus on the functions $W(x)$, $I(x)$, $R_R(x)$ and $R_T(x)$, the order of minimum functions is given as

$$\min[W(x), I(x), R_R(x), R_T(x)] = \begin{cases} W(x), & 0 \le x < x_1^{\circ}, \\ I(x), & x_1^{\circ} \le x < x_2^{\circ}, \\ W(x), & x_2^{\circ} \le x < \bar{\bar{x}}, \\ R_R(x), & \bar{\bar{x}} \le x < \bar{x}, \\ R_T(x), & \bar{x} \le x \le 1. \end{cases} \tag{27}$$

If we replace $W(x)$ with $W_M(x)$, then the same order of minimum functions as Eq.(27) holds.

Consecutively, we investigate the order of minimum functions including the function $W_M(x)$. Eqs.(26) and (27) show that the intersects $\hat{x}_{12}$ and $\hat{x}_{13}$ are located in two regions of $[0, x_1^{\circ}), [x_1^{\circ}, x_2^{\circ}), [x_2^{\circ}, \bar{\bar{x}}), [\bar{\bar{x}}, \bar{x})$ and $[\bar{x}, 1]$ at most. However, the two intersections cannot be located in regions $[\bar{\bar{x}}, \bar{x})$ and/or $[\bar{x}, 1]$ from lemma 6. Thus, we pay attention to the other three regions and enumerate the total orders that are consistent with lemma 6. The total number of orders is six because there are ${}_3C_2$ combinations when the two intersections are located in two different regions and there are ${}_3C_1$ combinations when the two intersections are located in one region. Accordingly, we obtain the orders of minimum functions with an increase in state probability as follows.

- In the case of $\hat{x}_{12} \in [0, x_1^{\circ})$,
  - ✓ if $\hat{x}_{13} \in [0, x_1^{\circ})$, then the order of minimum functions is
    $W(x) \to W_M(x) \to W(x) \to I(x) \to W(x) \to R_R(x) \to R_T(x)$.
  - ✓ if $\hat{x}_{13} \in [x_1^{\circ}, x_2^{\circ})$, then the order of minimum functions is
    $W(x) \to W_M(x) \to I(x) \to W_M(x) \to W(x) \to R_R(x) \to R_T(x)$.
  - ✓ if $\hat{x}_{13} \in [x_2^{\circ}, \bar{\bar{x}})$, then the order of minimum functions is
    $W(x) \to W_M(x) \to I(x) \to W_M(x) \to W(x) \to R_R(x) \to R_T(x)$.
- In the case of $\hat{x}_{12} \in [x_1^{\circ}, x_2^{\circ})$,

✓ if $\hat{x}_{13} \in \left[x_1^{\circ}, x_2^{\circ}\right)$, then the order of minimum functions is

$W(x) \to I(x) \to W(x) \to R_R(x) \to R_T(x)$.

✓ if $\hat{x}_{13} \in \left[x_2^{\circ}, \overline{\overline{x}}\right)$, then the order of minimum functions is

$W(x) \to I(x) \to W_M(x) \to W(x) \to R_R(x) \to R_T(x)$.

· In the case of $\hat{x}_{12} \in \left[x_2^{\circ}, \overline{\overline{x}}\right)$,

✓ if $\hat{x}_{13} \in \left[x_2^{\circ}, \overline{\overline{x}}\right)$, then the order of minimum functions is

$W(x) \to I(x) \to W(x) \to W_M(x) \to W(x) \to R_R(x) \to R_T(x)$.

Therefore, the number of orders of minimum functions is three as follows.

1. For $\hat{x}_{12} \in \left[0, x_1^{\circ}\right)$, and $\hat{x}_{13} \in \left[0, x_1^{\circ}\right)$,
   $W(x) \to W_M(x) \to W(x) \to I(x) \to W(x) \to R_R(x) \to R_T(x)$.
2. For $\hat{x}_{12} \in \left[0, x_1^{\circ}\right)$, and $\hat{x}_{13} \in \left[x_1^{\circ}, \overline{\overline{x}}\right)$,
   $W(x) \to W_M(x) \to I(x) \to W_M(x) \to W(x) \to R_R(x) \to R_T(x)$.
3. For $\hat{x}_{12} \in \left[x_2^{\circ}, \overline{\overline{x}}\right)$, and $\hat{x}_{13} \in \left[x_2^{\circ}, \overline{\overline{x}}\right)$,
   $W(x) \to I(x) \to W(x) \to W_M(x) \to W(x) \to R_R(x) \to R_T(x)$.

Then, the number of regions of the optimal maintenance policy is seven at most. This completes the proof.

B.3 Proof of Theorem 3

If c_m=0, according to Jensen's inequality,

$$\int_{-\infty}^{\infty} V\left[h(x,y)\right] f(x,y)\,dy \le V\left[(1-p)x+p\right]$$

and $W(x) \ge W_M(x)$ for $0 \le x \le 1$. This indicates that the action W is not always selected. Hence, theorem 3 holds.