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A Queuing Model for an Automated Workstation Receiving Jobs from an Automated Workstation

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Abstract — In this paper we develop queuing model results for a single "automated" workstation that receives jobs from another automated workstation. An automated workstation is a server with deterministic processing times that experiences random operating times between failures, and then subsequent random repair times. We develop analytical expressions for the queue size distribution, the average number in system and the variance of the number in system using a discrete model of this queuing system.

Keywords —Queuing, automated workstation, production systems, markov chain aggregation.

1. INTRODUCTION

In the analysis of production systems, the presence of "automated" workstations is very common. An automated workstation is a server with deterministic processing times that experiences random operating times between failures, and then subsequent random repair times. In this paper we develop analytical expressions for the queue size distribution, the average number in system, and the variance of the number in system for a single automated workstation that receives input from another automated workstation.

The actual workstation that is considered is assumed to have fixed job processing times and exponentially distributed operating times between failures, and exponentially distributed repair times. These have been shown to be reasonable assumptions in practice (Inman 1999, Dallery and Gershwin 1992). Much is known about such automated workstations operating in isolation. In Kim and Alden (1997) a mixed discrete/continuous probability mass/density function for the time to produce a fixed number of jobs on such a workstation is derived (which also includes the special case of a single job). For a discrete model of such a workstation a formula for the variance of the number of jobs produced in a fixed time period is derived in Gershwin (1992). In Kim and Alden (1997) and Hopp and Spearman (2001) formulas can be found for the mean and variance of the time a job spends in such a workstation that includes process and repair time.

One additional step in the analysis of automated workstations is the analysis of a workstation and its input buffer. In this research we develop an analytical model for a single automated workstation that receives its input from an upstream automated workstation and compare this against commonly known G/G/1 approximations with respect to estimating the average number of jobs in the system. Analytical expressions for the distribution of the number of jobs in the system (the workstation and its input buffer) are also derived. In a related paper (Nagarajan and Kim 2006), "linking equations" have been developed so that the results developed here can be applied to the *analytical* analysis of a series of automated workstations using a two-workstation decomposition approximation. This paper presents the details of the modeling approach and derivation of the results for an automated workstation queuing system utilized in Nagarajan and Kim (2006). Simplified formulas as well as the result for the variance of the number in system are new.

A large amount research addressing the modeling of serial automated production systems focuses on modeling

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two automated workstations in series with finite buffer capacities between workstations. The goal of this prior research has focused on estimating the throughput of such systems, and includes continuous and discrete approximations of automated workstations. These models have served as the foundation for models developed to analyze a longer series of automated workstations (with finite buffers) using numerical two-workstation decomposition approximations. A review of such models can be found in Dallery and Gershwin (1992).

There has been less research directed at developing explicit analytical models of an automated workstation with an infinite input buffer, with a focus on predicting the work-in-process or time-in-system. In Altiok (1997) M/G/1 queuing results were applied to a system consisting of a single (possibly automated) workstation with Poisson arrivals. The distribution of the number in system for a workstation with Poisson arrivals, exponentially distributed time between failures (both operating time and elapsed time), and phase-type repair time distributions is examined in Altiok (1997) using continuous time Markov chain analysis. Another analysis approach for a single automated workstation with an infinite capacity input buffer is to apply existing general queuing approximation methods. Such methods are employed in Hopp and Spearman (2001). In Hopp and Spearman (2001) two-moment G/G/1 queuing model approximations are applied. G/G/1 approximations have also been used as part of software packages that estimate the performance of networks of workstations with infinite buffer capacities (Whitt, 1983). There are multiple two-moment G/G/1 approximation models in use, as well as "linking equations" that estimate the parameters of the input process to a workstation as a function of the prior workstations parameters. Various G/G/1 approximations and linking equations are summarized in Shanthikumar and Buzacott (1980) and Buzzacott and Shanthikumar (1993). Because these models do not explicitly model the operation of an automated workstation it is expected that they may not always perform accurately, in particular when the coefficient of variation (CV) of the interarrival and/or service process is high (Buzzacott and Shanthikumar, 1993).

The remainder of this paper is organized as follows. In section 2 we present a Markov chain model for an automated workstation with infinite input buffer capacity receiving its jobs from another automated workstation (assumed to always have work). In section 3, the expressions for queue size distribution, the average number in system, and the variance of the number in system are derived. In section 4, the accuracy of the model is examined by comparisons to simulations and existing G/G/1 approximations.

2. MARKOV CHAIN MODEL OF TWO AUTOMATED WORKSTATIONS IN SERIES

We develop a Markov chain model of an automated workstation with infinite input buffer space, receiving jobs from another automated workstation with an infinite supply of unprocessed jobs. By assuming that the first workstation always has jobs to process, the output process from the first workstation represents output from an automated workstation with no influence of a random arrival process. Modeling the output process in the presence of variable input is addressed in Nagarajan and Kim (2006), where a series of automated workstations is analyzed.

The Markov chain model is a discrete time model where the fixed workstation processing time t, serves as the discrete time unit (it is assumed that both workstations produce at the same speed when up). By the discrete nature of the model, the operating times between failures, and repair times will follow geometric distributions as approximations to exponential distributions. In most automated workstations, this type of discrete approximation is sufficiently accurate since the fixed processing times are normally much smaller than the time between failures and repair times.

The mechanics of the discrete time Markov chain are as follows:

- State transitions occur at the end of each time step.
- Any workstation that is down at the beginning of a time step may be repaired even if the workstation is empty at the beginning of the time step.
- A workstation that is up and not empty at the beginning of a time step will complete its job even if it moves to a down state at the end of the time step.
- Any jobs completed at the end of a time step are moved out of the workstations and new jobs are moved into the workstations even if a workstation moves to a down state. Note that a job may be moved out of both workstations, and a job moved into both workstations at the end of a time step.
- Workstations that are up at the beginning of a time step but idle because they are starved, cannot change to a down state at the end of a time step.

We assume that the long-run processing capacity of the first workstation is strictly less than that of the second workstation. This ensures that the number of jobs in the second workstations input buffer will not steadily increase over time. The objective of the model is to analyze the behavior of the second workstation and its input buffer, which we will refer to as the "system".

We let the state of the Markov chain at time unit n, $X_n=(x_1,x_2,N)$, where x_i = status of workstation i, $i=1,2$

and $x_i \in \{0,1\}$. $x_i = 0$ if the workstation is down a the beginning of a time step, and $x_i = 1$ if the workstation is up a the beginning of a time step. N is the number of jobs in workstation 2 plus the number in its input buffer. We let $p = \left[p_{kj} \right]$ denote the transition probability matrix for this Markov chain.

If workstation i is up and operating (an unprocessed job is in the workstation) at the beginning of the time step, it remains up during the time step and may transition to a down state with probability f_i at the end of the time step. The job being processed in this cycle will be completed and moved out of the workstation. If workstation i is down and under repair at the beginning of the time step, it remains down during the time step and is repaired with probability r_i at the end of the time step. If an unprocessed job was present in this workstation at the beginning of the time step, the job remains unprocessed and stays in the workstation until the workstation is repaired before being processed.

A state transition diagram of the Markov chain model is shown in Figure 1.

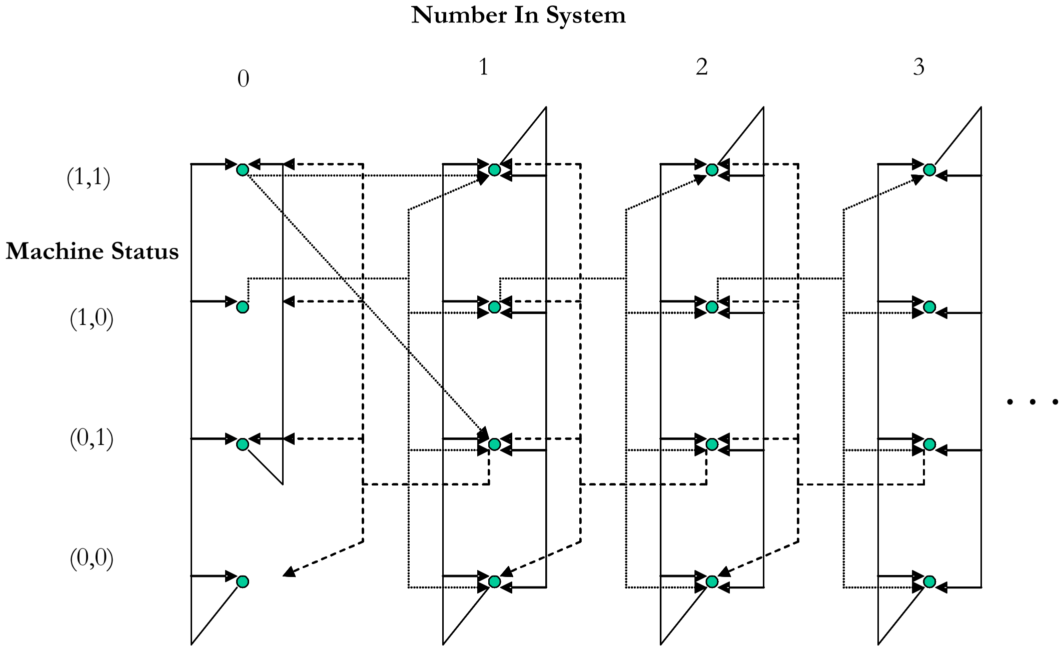

Figure 1. State transition diagram for the Markov chain model. State = (x_1, x_2, N) *where N is a column label and* (x_1, x_2) *are row label. Different marked transitions (e.g., dashed) represent transition from states with the same* (x_1, x_2) *values.*

The transition probabilities between any two states in figure 1 are functions of the workstation status at time *n* and *n+1*. For example, $p_{(1,1,2),(1,0,2)}$ the transition probability from state $(1,1,2)$ to $(1,0,2)$ equals $(1-f_1)f_2$. When there are no customers in the system (i.e., workstation 2 is starved), the transition probabilities reflect the assumption that workstation 2 cannot fail if it is starved.

Since the actual workstations are assumed to have exponentially distributed operating times between failures, and repair times, the probabilities f_i and r_i are computed as a function of the fixed processing time *t*, $MTBF_i$, and $MTTR_i$, $MTBF_i$ is the mean operating time between failures for workstation *i,* and $MTTR_i$ is the mean repair time for workstation *i* (both parameters of exponential distributions). The probabilities f_i and r_i are computed such that the mean time to process a job, and the variance of the time to complete a job (which includes workstation downtime) in the discrete model match that for the actual workstation. Without loss of generality let $t = 1$, and let $T_i =$ the total time spent by a job in workstation *i*. In Kim and Alden (1997) and Hopp and Spearman (2001) it is shown that,

$$E[T_i] = \frac{MTBF_i + MTTR_i}{MTBF_i} \tag{1}$$

$$Var[T_i] = \frac{2 * MTTR_i^{\ 2}}{MTBF_i} \tag{2}$$

For the discrete model it is straightforward to show that

$$. E[T_i] = 1 + \frac{f_i}{r_i}$$

To derive $Var[T_i]$ as a function of f_i and r_i, let $I = 1$ if a job just moves into a workstation that has just failed, and $I = 0$ otherwise. By conditioning on the indicator variable I we get (Nagarajan and Kim 2006),

$$E[T_i^2] = E[T_i^2 \mid I = 1] * p(I = 1) + E[T_i^2 \mid I = 0] * p(I = 0)$$

$$E[T_i^2] = \left(\sum_{k=1}^{\infty} (k+1)^2 (1-r_i)^{k-1} r_i \right) * f_i + (1)^2 * (1 - f_i) = \frac{r_i^2 + f_i(2 + r_i)}{r_i^2}.$$

$$\Rightarrow Var[T_i] = \frac{f_i(2 - r_i - f_i)}{r_i^2}.$$

f_i and r_i as functions of $MTBF_i$ and $MTTR_i$ can then be expressed as:

$$f_i = \frac{2MTTR_i}{MTBF_i + MTTR_i + 2 * MTBF_i * MTTR_i}$$

$$r_i = \frac{2MTBF_i}{MTBF_i + MTTR_i + 2 * MTBF_i * MTTR_i}$$

3. DERIVATIONS OF THE AVERAGE NUMBER IN SYSTEM AND THE DISTRIBUTION OF THE NUMBER IN SYSTEM

To derive analytical expressions for the average number in system and the distribution of the number of jobs in the system, we take advantage of the transition structure of the Markov chain, and Markov chain aggregation/disaggregation results (Feinberg and Chiu, 1987, Kim and Smith, 1995). Figure 1 was drawn in such a way that the number in system defines a natural partitioning of the system states. Following the terminology defined in Kim and Smith (1995), a set of four states in the Markov chain that represent the same number in system will form a *macrostate*. A Markov chain is in a particular macrostate whenever it is in any state contained in the macrostate. The transitions from macrostate to macrostate also constitute a Markov chain (Kim and Smith, 1995). The solution to this *macrostate Markov chain* represents the solution to the queuing model since the macrostates represent the number in system, and the macrostate Markov chain steady state probabilities will equal the sum of the steady state probabilities of all states contained in the macrostate (Kim and Smith, 1995). A diagram of the macrostate Markov chain is shown in figure 2.

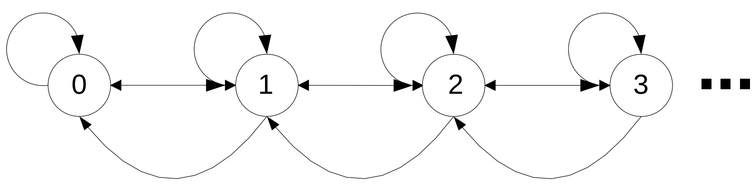

Figure 2. Macrostate Markov chain model.

3.1 Microstate Markov Chain Transition Probabilities

One method to compute the transition probabilities of the macrostate Markov chain (transition probability matrix denoted by **P**), is to examine the states within each individual macrostate in isolation where all transitions after a macrostate is left are ignored. The states within a macrostate are referred to as *microstates*, and the process realized by viewing the microstates in isolation constitutes a Markov chain (Kim and Smith, 1995). These Markov chains are referred to as *microstate Markov chains*. If the steady state probabilities of the microstate Markov chain are known, then they can be used to calculate the macrostate Markov chain transition probabilities (Kim and Smith, 1995). To find the transition probabilities of the microstate Markov chains (denoted by p^N for the microstate chain associated with N customers in system) we take advantage of the transition structure.

Consider the microstate chain that corresponds to zero customers/jobs in the system. All transitions leaving the set of states contained in this chain must eventually return to these states (due to ergodicity), and re-enter the set of states from only a single state. Thus the transition probabilities for those transitions leaving the microstate Markov chain are known. This is shown in Figure 3.

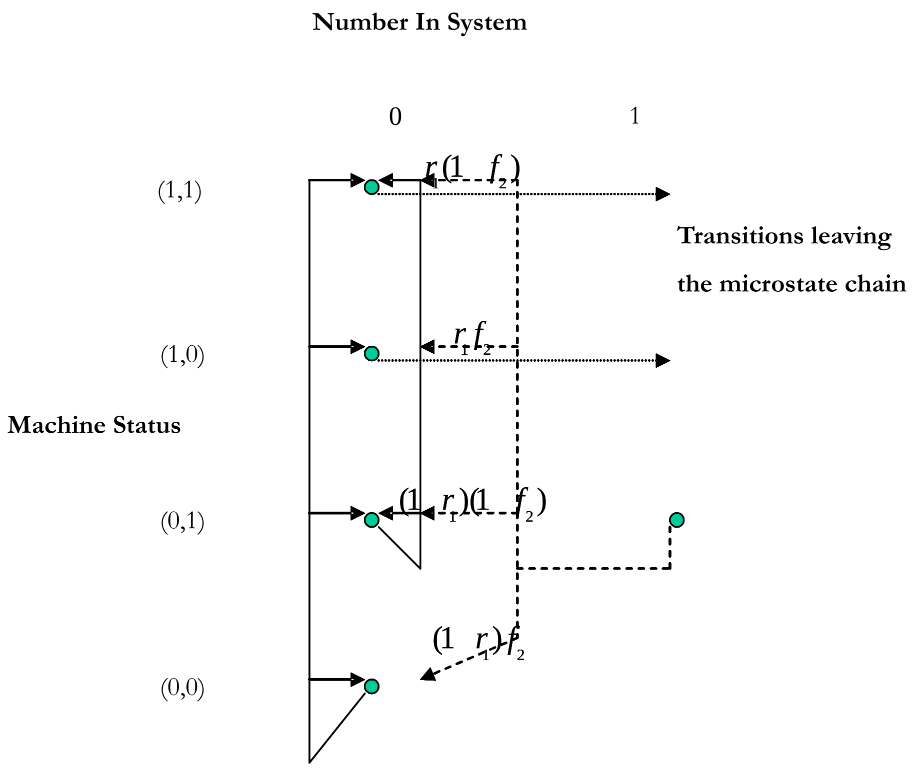

Figure 3. Finding the microstate Markov chain transition probabilities.

The transition probability matrix for the microstate Markov chain for zero customers/jobs in the system is,

$$p^0 = \begin{array}{c} \\ (1,1) \\ (1,0) \\ (0,1) \\ (0,0) \end{array} \begin{array}{c} \begin{array}{cccc} (1,1) & (1,0) & (0,1) & (0,0) \end{array} \\ \begin{bmatrix} r_1(1-f_2) & r_1 f_2 & (1-r_1)(1-f_2) & (1-r_1) f_2 \\ r_1(1-f_2) & r_1 f_2 & (1-r_1)(1-f_2) & (1-r_1) f_2 \\ r_1 & 0 & (1-r_1) & 0 \\ r_1 r_2 & r_1(1-r_2) & (1-r_1) r_2 & (1-r_1)(1-r_2) \end{bmatrix} \end{array} .$$

We next address the microstate Markov chain transition probabilities when there are two or more customers in the system. In figure 1 it can be seen that all transitions that increase the number in system occur when workstation 1 is up and workstation 2 is down. Similarly all transitions that decrease the number in system occur when workstation 1 is down and workstation 2 is up. Furthermore the Markov chain structure (transitions into, within, leaving) for the microstate Markov chains for a number in system of two or greater is the same. Therefore $p^I = p^J$ for I, J ≥ 2. Using similar reasoning as used to find p^0, when transitions leave the states in a microstate chain to the 'left', they return to the microstate chain via a single state. Also when transitions leave the states in a microstate chain to the 'right', they also return to the microstate chain via a single state. Therefore we get,

$$p^N = \begin{matrix} & (1,1) & (1,0) & (0,1) & (0,0) \\ (1,1) & (1-f_1)(1-f_2) & (1-f_1)f_2 & f_1(1-f_2) & f_1 f_2 \\ (1,0) & r_1(1-f_2) & r_1 f_2 & (1-r_1)(1-f_2) & (1-r_1)f_2 \\ (0,1) & (1-f_1)r_2 & (1-f_1)(1-r_2) & f_1 r_2 & f_1(1-r_2) \\ (0,0) & r_1 r_2 & r_1(1-r_2) & (1-r_1)r_2 & (1-r_1)(1r f_2) \end{matrix} \quad \text{for } N \geq 2.$$

To find p^I requires knowledge of the steady state probabilities of p^0 since transitions leaving the microstate chain (for one in the system) to the 'left' in figure 1 may return to the microstate chain via two different states (as shown in figures 1 and 3). Let $\pi^N = [\pi^N_{1,1}, \pi^N_{1,0}, \pi^N_{0,1}, \pi^N_{0,0}]$ represent the steady state probabilities of the microstate Markov chain for *N* in the system, where $\pi^N_{i,j}$ is the steady state probability for the microstate representing workstation 1 in state *i*, and workstation 2 in state *j*. Let $c = \pi^0_{1,1} \big/ (\pi^0_{1,1} + \pi^0_{1,0})$.

Then,

$$\mathrm{p}^1 = \begin{matrix} & (1,1) & (1,0) & (0,1) & (1,1) \\ (1,1) & (1-f_1)(1-f_2) & (1-f_1)f_2 & f_1(1-f_2) & f_1 f_2 \\ (1,0) & r_1(1-f_2) & r_1 f_2 & (1-r_1)(1-f_2) & (1-r_1)f_2 \\ (0,1) & c(1-f_1)+(1-c)(1-f_1)r_2 & (1-c)(1-f_1)(1-r_2) & cf_1+(1-c)f_1 r_2 & (1-c)f_1(1-r_2) \\ (0,0) & r_1 r_2 & r_1(1-r_2) & (1-r_1)r_2 & (1-r_1)(1-r_2) \end{matrix}.$$

3.2. Microstate Markov Chain Solutions

It is possible to derive manageable analytical solutions for the microstate Markov chains as functions of the workstation failure and repair probabilities, since they are only four-state Markov chains. The solutions for the microstate chains satisfy $\pi^N = \pi^N p^N$, $\pi^N > 0$, $\pi^N_{1,1} + \pi^N_{1,0} + \pi^N_{0,1} + \pi^N_{0,0} = 1$, and thus represent the unique steady state solution to the microstate Markov chains. Although we are primarily interested in the steady state probabilities for those microstates that have transitions out of the set of microstates, all microstate steady state probabilities have been derived. These steady state solutions are presented next.

Let $a = f_1 f_2 - f_1 - f_2$ *and* $b = r_1 r_2 - r_1 - r_2$

Microstate Markov chain steady state probabilities for $N = 0$.

$$\pi^0{}_{1,1} = \frac{f_2 r_1^2}{b} + r_1$$

$$\pi^0{}_{1,0} = \frac{-f_2 r_1^2}{b}$$

$$\pi^0{}_{0,1} = \frac{-(f_2 r_1^2 - f_2 r_1)}{b} + (1 - r_1)$$

$$\pi^0{}_{0,0} = \frac{f_2 r_1^2 - f_2 r_1}{b}$$

Microstate Markov chain steady state probabilities for $N = 1$.

$$\pi^1_{1,1} = \frac{((a+1)r_1(r_2-1) + bf_2 r_1 + f_2 r_1 r_2 + f_1 r_2 - r_2)(b + f_2 r_1)}{2bf_2 r_2(r_1-1) + b^2 + f_2^2(r_1-1)(b + r_1 r_2) + f_1 f_2(b(3r_2 + f_2 - 1) + ar_2 + 2f_2 r_1 r_2 + f_1 r_2^2)}$$

$$\pi^1_{1,0} = \frac{f_2(b+f_1r_2)^2}{2bf_2r_2(r_1-1)+b^2+f_2^2(r_1-1)(b+r_1r_2)+f_1f_2(b(3r_2+f_2-1)+ar_2+2f_2r_1r_2+f_1r_2^2)}$$

$$\pi^1_{0,1} = \frac{-b(b+f_2r_1)(a+f_2r_1)}{2bf_2r_2(r_1-1)+b^2+f_2^2(r_1-1)(b+r_1r_2)+f_1f_2(b(3r_2+f_2-1)+ar_2+2f_2r_1r_2+f_1r_2^2)}$$

$$\pi^1_{0,0} = \frac{f_2(b+f_1r_2)(a+f_2r_1)}{2bf_2r_2(r_1-1)+b^2+f_2^2(r_1-1)(b+r_1r_2)+f_1f_2(b(3r_2+f_2-1)+ar_2+2f_2r_1r_2+f_1r_2^2)}$$

Microstate Markov chain steady state probabilities for $N \geq 2$.

$$\pi^N{}_{1,1} = \frac{(b+f_1r_2)*(b+f_2r_1)}{(a+b+f_1r_2+f_2r_1)^2}$$

$$\pi^N{}_{1,0} = \frac{(b+f_1r_2)*(a+f_1r_2)}{(a+b+f_1r_2+f_2r_1)^2}$$

$$\pi^N{}_{0,1} = \frac{(a+f_2r_1)*(b+f_2r_1)}{(a+b+f_1r_2+f_2r_1)^2}$$

$$\pi^N{}_{0,0} = \frac{(a+f_2r_1)*(a+f_1r_2)}{(a+b+f_1r_2+f_2r_1)^2}$$

3.3. Macrostate Markov Chain Solution and Queuing Model Results

Let $P = [P_{IJ}]$ be the transition probability matrix for the macrostate Markov chain. The macrostate Markov chain depicted in Figure 2 has the following transition probability matrix structure and values for P_{IJ}. In this section N is always ≥ 2.

$$P = \begin{bmatrix} \pi^0_{00}+\pi^0_{01} & \pi^0_{11}+\pi^0_{10}=r_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \pi^1_{01} & 1-(\pi^1_{10}+\pi^1_{01}) & \pi^1_{10} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \pi^N_{01} & 1-(\pi^N_{10}+\pi^N_{01}) & \pi^N_{10} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \pi^N_{01} & 1-(\pi^N_{10}+\pi^N_{01}) & \pi^N_{10} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \pi^N_{01} & 1-(\pi^N_{10}+\pi^N_{01}) & \pi^N_{10} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

The transition probabilities are computed from the results of the microstate Markov chain analysis. Transition probabilities for a macrostate are computed as weighted averages using the microstate chain steady probabilities as weights. For example, a transition probability for a transition to the 'right' is the weighted sum over the states within the microstate chain, multiplied by the probability of a transition to the right. That is,

$$P_{I,I+1} = \sum_{(i,j)}\sum_{(m,n)} \pi^I_{(i,j)} * p_{(i,j,I),(m,n,I+1)}$$

where $(i,j),(m,n) \in \{(1,1),(1,0),(0,1),(0,0)\}$.

Let $\pi = [\pi_0, \pi_1, \pi_2, \ldots]$ represent the steady state probabilities of P. Because of the simple structure (time reversible) of the macrostate Markov chain, it is straightforward to show that,

$$\pi_0 = K * \frac{\pi_{01}^1}{r_1 \pi_{10}^1} \tag{3}$$

$$\pi_1 = K * \frac{1}{\pi_{10}^1} \tag{4}$$

$$\pi_i = \frac{K}{\pi_{01}^N} \left(\frac{\pi_{10}^N}{\pi_{01}^N} \right)^{i-2} \quad \text{for } i \geq 2 \tag{5}$$

$$\text{where } K = \frac{(\pi_{01}^N - \pi_{10}^N) r_1 \pi_{10}^1}{(\pi_{01}^N - \pi_{10}^N)(\pi_{01}^1 + r_1) + r_1 \pi_{10}^1} \tag{6}$$

K can be expressed as a function of f_i *and* r_i,

$$K = \frac{f_2 r_1 (f_1 r_2 + b)^2 (f_2 r - f_1 r_2)}{(f_2 r_1 + a)(f_1 r_2 + r_1 r_2) + (f_2 r_1 + b)^2} \tag{7}$$

Equations 3-7 represent the probability distribution of the number of jobs in the second workstation and its input buffer. Expressions for the average and the variance of the number in system can then be obtained from equations 3-7.

Letting $q = \pi_{01}^N$, *and* $s = \pi_{10}^N$ to simplify the notation, and letting *C* = the number in system gives,

$$\textit{Average Number in System} = E[C] = \frac{K}{\pi_{10}^1} + 2K\left(\frac{1}{q}\right) + 3K\left(\frac{s}{q^2}\right) + 4K\left(\frac{s^2}{q^3}\right) + 5K\left(\frac{s^3}{q^4}\right) + \cdots,.$$

Simplifying we get,

$$\textit{Average Number in System} = K\left(\frac{1}{\pi_{10}^1} + \frac{2q - s}{(q-s)^2}\right) \tag{8}$$

The average number in system can be expressed as a function of f_i *and* r_i,

$$\textit{Average Number in System} = \frac{f_2 r_1 (f_1 r_2 + r_1 r_2 + b) - f_1 r_1 r_2^2}{(f_2 r_1 - f_1 r_2)(f_1 r_2 + r_1 r_2)} \tag{9}$$

The variance of the number in system is found by first finding $E[C^2]$ and then subtracting $E[C]^2$,

$$E[C^2] = \frac{K}{\pi_{10}^1} + 2^2 K\left(\frac{1}{q}\right) + 3^3 K\left(\frac{s}{q^2}\right) + 4^4 K\left(\frac{s^2}{q^3}\right) + 5^5 K\left(\frac{s^3}{q^4}\right) + \cdots$$

$$= K\left(\frac{1}{\pi_{10}^1} - \frac{1}{s} + \frac{1}{s}\left(\frac{q^2(q+s)}{(q-s)^3}\right)\right).$$

Subtracting $E[C]^2$ gives,

$$\textit{Variance of the Number in System} = K\left(\frac{1}{\pi_{10}^1} - K\left(\frac{1}{\pi_{10}^1} + \frac{2q-s}{(q-s)^2}\right)^2 + \frac{4q^2 - 3qs + s^2}{(q-s)^3}\right) \tag{10}$$

4. COMPARISON WITH TWO-MOMENT G/G/1 APPROXIMATIONS AND SIMULATIONS

As mentioned in section 1, one approach for analyzing automated workstations receiving input from automated workstations is the application of general two-moment G/G/1 approximations. Another more time consuming approach is to use simulation. In this section we show how the current model performs with respect to predicting the average number in system when compared against simulation results and two popular G/G/1 approximations.

The system used in these comparisons can be thought of as two automated workstations in series, where the first workstation always has jobs to process. Under this system description the mean and variance of the job interdeparture times from the first workstation may be found using equations (1) and (2). Similarly the mean and variance of the processing times (actual work time plus downtime) at the second workstation can be calculated. With these values computed a number of two-moment G/G/1 approximations may be applied to estimate the number of jobs in the second workstation and its queue. The two G/G/1 approximations used here are approximations for the average number in queue from Sakasegawa (1977) and Yu (1977), and Kramer and Lagenbach-Belz (1976). The approximations in Sakasegawa (1977) and Yu (1977) are the same as that used in Hopp and Spearman (2001). Whitt (1983) uses an approximation that is the approximation in Kramer and Lagenbach-Belz (1976) when the squared coefficient of variation of the interarrival times is less than one, and is the approximation in Sakasegawa (1977) and Yu (1977) when the square coefficient of variation of the interarrival times is greater than one. By adding the workstation utilization to these approximations we get an estimate for the average number in system.

The current model is compared to simulation results of a continuous time system with exponentially distributed operating times between failures and repair times, to estimate the impact of the discrete model approximation. The simulation results will also serve as the best estimate of the average number in system. A variety of systems were simulated to represent different interarrival and service time coefficients of variation, and different workstation-two utilizations. Within each service time coefficient of variation and utilization range, 20 different systems were simulated. Each simulation started with a 100,000 time unit warm-up (the fixed processing time for a single job is one time unit). After the warm-up period, the simulations were run until 1,000,000 jobs were completed on the second workstation. 30 replications were conducted and the best estimate for a systems average number in system was taken as the average of the 30 replications.

A summary of the simulation results is presented in Table 1. In Table 1, the simulation results are separated into test sets. Within each set the range for the workstation processing time (work time plus downtime) and interarrival time coefficients of variation, and utilizations for the second workstation are shown. The results presented are the average, maximum, and minimum (over the number of systems within a test set) absolute percent difference from simulation.

The average percent differences in Table 1 are plotted over the test sets in Figure 4. As can be seen, the performance of the model is very consistent over a range of workstation coefficients of variation and utilizations. Also, the average errors when using G/G/1 approximations can be very large. In general the approximation of Kramer and Lagenbach-Belz (1976) outperforms the approximation in Sakasegawa (1977) and Yu (1977) although they are very similar in performance. This is not surprising since they are very similar in functional form. Additionally the results confirm that the G/G/1 approximations do perform better for higher utilizations. Both approximations can be viewed as modifications of the upper bound in Kingman (1962) for a G/G/1 queue, which becomes tighter as utilization approaches one.

Table 1. Summary of comparisons the model with simulation and two-moment G/G/1 approximations

				Absolute Percent Difference From Simulation								
Test Set	Workstation	Workstation	No. of	New Model			Sakasegawa-Yu			KLB		
Number	CV Range	Util. Range	Systems	Avg.	Max	Min	Avg.	Max	Min	Avg.	Max	Min
1	0 to 1	80%-90%	20	3.14%	9.81%	0.06%	26.06%	59.04%	2.71%	20.28%	58.97%	0.77%
2	1 to 2	80%-90%	20	1.49%	5.35%	0.03%	18.10%	61.76%	0.60%	16.14%	60.99%	0.65%
3	2 to 3	80%-90%	20	2.12%	6.01%	0.03%	32.01%	91.44%	5.93%	28.31%	83.76%	3.00%
4	3 to 4	80%-90%	20	2.37%	4.88%	0.08%	23.27%	78.73%	4.75%	20.16%	69.99%	3.49%
5	4 to 5	80%-90%	20	1.86%	5.84%	0.06%	20.08%	56.49%	2.21%	16.95%	50.96%	0.54%
6	0 to 1	90%-95%	20	1.83%	5.88%	0.01%	16.74%	60.95%	1.43%	12.67%	51.13%	0.87%
7	1 to 2	90%-95%	20	1.03%	2.26%	0.02%	16.25%	69.88%	0.30%	15.32%	67.41%	0.46%
8	2 to 3	90%-95%	20	1.77%	4.04%	0.08%	12.22%	69.39%	0.44%	10.98%	66.42%	0.08%
9	3 to 4	90%-95%	20	1.18%	3.89%	0.10%	12.56%	76.66%	0.15%	11.29%	72.36%	0.07%
10	4 to 5	90%-95%	20	2.44%	5.85%	0.20%	13.18%	35.52%	1.97%	11.66%	33.06%	3.34%
11	0 to 5	80%-95%	20	1.70%	6.12%	0.04%	22.59%	129.74%	0.29%	18.04%	93.79%	0.40%

5. SUMMARY

In this research we have developed an analytical model for the average number in queue, the variance of the number in queue, and queue size distribution for an automated server, which receives inputs from an automated workstation. The analytical solution is derived from a discrete time Markov chain model of two automated workstations in series with infinite buffer capacity. We show that the analytical model performs much better on average than two-moment G/G/1 approximations applied to such systems. In such automated systems it is very possible to have large coefficients of variation for the time jobs spend in the workstation. The analytical expressions derived are more complicated than existing two-moment G/G/1 approximations, yet they are very easily implemented in spreadsheets.

The next step in this research is to examine more than two workstations in series. This has been conducted in Nagarajan and Kim (2006), where two-moment linking equations are developed to estimate the mean and variance of workstation interdeparture times. Additional extensions are to workstations that can process multiple jobs in parallel, and to series of workstations that have different fixed job processing times.

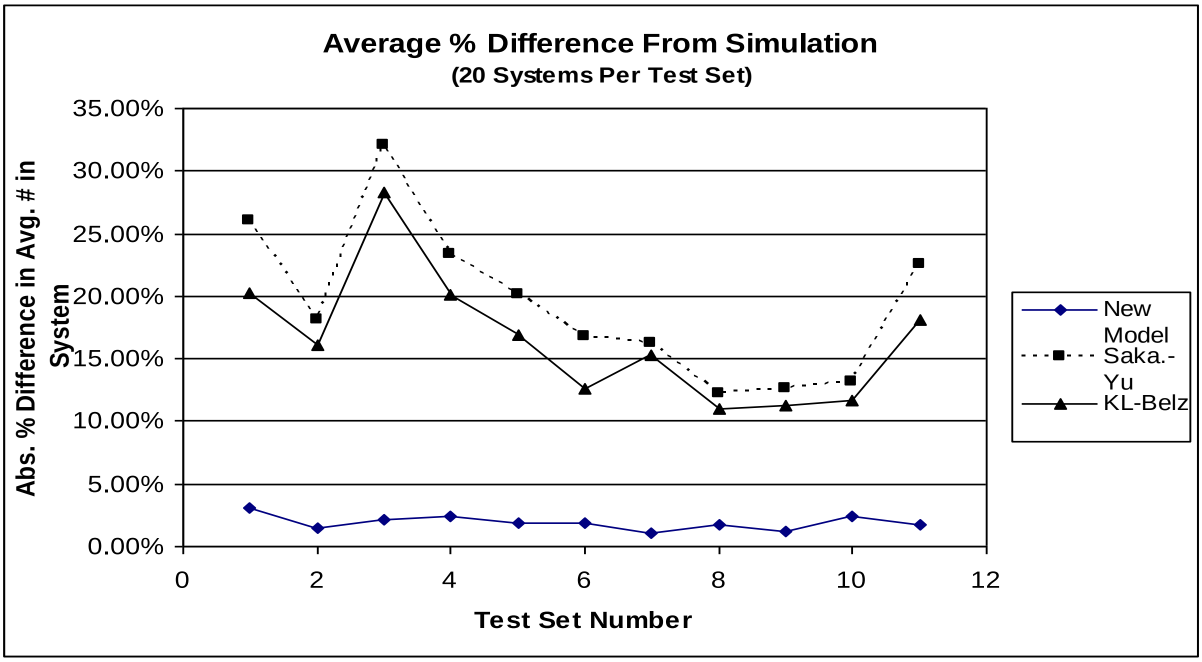

Figure 4. Average percent difference in the time average number in system. The average percent difference is taken over 20 systems within a test set.

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