

# Tail Approximation of Value-at-Risk under Multivariate Regular Variation

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**Abstract**—This paper presents a general tail approximation method for evaluating the Value-at-Risk of any norm of random vectors with multivariate regularly varying distributions. The main result is derived using the relation between the intensity measure of multivariate regular variation and tail dependence function of the underlying copula, and in particular extends the tail approximation discussed in Embrechts et al. (2009) for Archimedean copulas. The explicit tail approximations for random vectors with Archimedean copulas and multivariate Pareto distributions are also presented to illustrate the results.

**Keywords**—Value-at-risk, tail dependence function, multivariate regular variation, archimedean copulas, risk management.

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## 1. INTRODUCTION

Value-at-Risk (VaR) is one of the most widely used risk measures in financial risk management (McNeil et al., 2005). Given a non-negative random variable  $X$  representing loss, the VaR at confidence level  $p, 0 < p < 1$ , is defined as the  $p$ -th quantile of the loss distribution:

$$\text{VaR}_p(X) := \inf\{t \in \mathfrak{R} : \Pr\{X > t\} \leq 1 - p\}.$$

In risk analysis for multivariate portfolios, we are often more interested in calculating the VaR for aggregated data than for a single loss variable. For a non-negative random vector  $X = (X_1, \dots, X_d)$  representing various losses in a multivariate portfolio, we need to calculate  $\text{VaR}_p(\|X\|)$  where the norm  $\|\cdot\|$  determines the manner in which the data are aggregated. Calculating  $\text{VaR}_p(\|X\|)$  is in general a difficult problem, but its tail estimates as  $p \rightarrow 1$  are often tractable for various multivariate loss distributions. Good tail approximations of  $\text{VaR}_p(\|X\|)$  as  $p \rightarrow 1$  can be used to accurately estimate the risk, as measured by the VaR, for extreme losses.

Tail asymptotics of  $\text{VaR}_p(\sum_{i=1}^d X_i)$ , as  $p \rightarrow 1$ , for loss vectors with Archimedean copulas and regularly varying margins are obtained in Albrecher et al. (2006), Alink et al. (2004, 2007), Embrechts et al. (2009), and Kortschak et al. (2009). These tail estimates for the VaR of sums are asymptotically linear functions of the VaR of the univariate margin, with a proportionality constant that depends on the tail dependence of the underlying Archimedean copula and marginal heavy-tail index. Such asymptotic relations can be also used to analyze the structural properties of the VaR of aggregated

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sums for multivariate portfolios, such as the subadditivity property of the VaR as well as the property on how the dependence and marginal parameters would affect estimates of extreme risks. In this paper, we develop a general and more unified approach to derive tail asymptotics of  $\text{VaR}_p(\|X\|)$ , as  $p \rightarrow 1$ , for loss vectors that have multivariate regularly varying distributions. Our method is based on the link between multivariate regular variation and tail dependence functions of copulas, and the tail estimates obtained previously in the literature can be obtained from our tail asymptotics as special cases.

The paper is organized as follows. In Section 2, we define copulas, tail dependence functions and multivariate regular variation. The relationship between tail dependence functions and multivariate regular variation obtained in Li and Sun (2009) is highlighted. In Section 3, we give a tail approximation of VaRs for loss vectors that follow multivariate regularly varying distributions. Explicit approximations for Archimedean copulas, Archimedean survival copulas and multivariate Pareto distributions are also presented. For notational convenience, we denote hereafter by  $[a, b]$  the Cartesian product  $\prod_{i=1}^d [a_i, b_i]$  where  $a, b \in \overline{\mathfrak{R}}^d := [-\infty, \infty]^d$  and  $a_i \leq b_i$  for each  $i$ . Also denote vector  $(w_1^p, \dots, w_d^p)$  by  $w^p$  where  $p > 0, w_i \geq 0$  for each  $i$ , and  $I := \{1, \dots, d\}$  denotes the index set. The maximum and minimum of  $a$  and  $b$  are denoted as  $a \vee b$  and  $a \wedge b$  respectively.

## 2. TAIL DEPENDENCE OF MULTIVARIATE REGULAR VARIATION

Let  $X = (X_1, \dots, X_d)$  be a non-negative random vector with distribution  $F$ . For any norm  $\|\cdot\|$  on  $\mathfrak{R}^d$ , tail estimates for  $\text{VaR}_p(\|X\|)$  as  $p \rightarrow 1$  boil down to finding the limit of  $\Pr\{\|X\| > t\} / \Pr\{X_i > t\}$  as  $t \rightarrow \infty$ ,  $1 \leq i \leq d$ , for which the multivariate regular variation suits well. The following definition can be found in Finkenstädt et al. (2004) pp. 185-286 (also see Basrak (2000))

**Definition 1** A random vector  $X$  or its distribution  $F$  is said to be *multivariate regularly varying* (MRV) if there exists a Radon measure  $\mu$  (i.e., the measure is finite on compact sets), called the *intensity measure*, on  $\mathfrak{R}^d \setminus \{\mathbf{0}\}$  such that

$$\lim_{t \rightarrow \infty} \frac{\Pr\{X \in tB\}}{\Pr\{\|X\| > t\}} = \mu(B), \quad (1)$$

for any relatively compact set  $B \subset \overline{\mathfrak{R}}^d \setminus \{\mathbf{0}\}$  that satisfies  $\mu(\partial B) = 0$ .

Note that here  $\overline{\mathfrak{R}}^d$  is compact and the punctured version  $\mathfrak{R}^d \setminus \{\mathbf{0}\}$  is modified via one-point uncompactification. It must be emphasized that the intensity measure  $\mu$  in (1) depends on the choice of norm  $\|\cdot\|$ , but intensity measures for any two different norms are proportional and thus equivalent. Observe also that for any non-negative MRV random vector  $X$ , its non-degenerate univariate margins  $X_i$  have regularly varying right tails, that is,

$$\overline{F}_i(t) := \Pr\{X_i > t\} = t^{-\beta} L(t), \quad t \geq 0, \quad (2)$$

where  $\beta > 0$  is the marginal heavy-tail index and  $L(t)$  is a slowly varying function with  $L(xt)/L(t) \rightarrow 1$  as  $t \rightarrow \infty$  for any  $x > 0$ . The multivariate regular variation defined in Definition 1 describes multivariate heavy tail phenomena. The detailed discussions on multivariate regular variation and its various applications can be found in Resnick (1987, 2007).

All the extremal dependence information of an MRV vector  $X$  is encoded in the intensity measure  $\mu$ , which can be further decomposed, using the copula approach, into the rank-invariant tail dependence and marginal heavy-tail index. A copula  $C$  is a distribution function, defined on the unit cube  $[0, 1]^d$ , with uniform one-dimensional margins. Given a copula  $C$ , if one defines

$$F(t_1, \dots, t_d) := C(F_1(t_1), \dots, F_d(t_d)), \quad (t_1, \dots, t_d) \in \mathfrak{R}^d, \quad (3)$$

then  $F$  is a multivariate distribution with univariate margins  $F_1, \dots, F_d$ . Given a distribution  $F$  with margins  $F_1, \dots, F_d$ , there exists a copula  $C$  such that (3) holds. If  $F_1, \dots, F_d$  are all continuous, then the corresponding copula  $C$  is unique and can be written as

$$C(u_1, \dots, u_d) := F(F_1^{-1}(t_1), \dots, F_d^{-1}(t_d)), \quad (u_1, \dots, u_d) \in [0, 1]^d.$$

Thus, for multivariate distributions with continuous margins, the univariate margins and the multivariate rank-invariant dependence structure (as described by their copulas) can be separated (Nelsen, 2006).

Using copulas we can define tail dependence functions. Let  $X = (X_1, \dots, X_d)$  have continuous margins  $F_1, \dots, F_d$  and copula  $C$ . The lower and upper tail dependence functions, denoted by  $b(\cdot)$  and  $b^*(\cdot)$  respectively, are introduced in Joe et al. (2010), Klüppelberg et al. (2008), and Nikoloulopoulos et al. (2009) as follows,

$$b(w) := \lim_{u \rightarrow 0^+} \frac{C(u w_j, \forall j \in I)}{u}, \quad \forall w = (w_1, \dots, w_d) \in \mathfrak{R}_+^d;$$

$$b^*(w) := \lim_{u \rightarrow 0^+} \frac{\bar{C}(1-u w_j, \forall j \in I)}{u}, \quad \forall w = (w_1, \dots, w_d) \in \mathfrak{R}_+^d, \quad (4)$$

provided that the limits exist, where  $\bar{C}$  denotes the survival function of  $C$ . Since the lower tail dependence function of a copula is the upper tail dependence function of its survival copula, we only focus on upper tail dependence in this paper. Define the upper exponent function  $a^*(\cdot)$  of copula  $C$  as

$$a^*(w) := \sum_{s \in I, s \neq \emptyset} (-1)^{|s|-1} b_s^*(w_i, i \in S), \quad (5)$$

where  $b_s^*(w_i, i \in S)$  denotes the upper tail dependence function of the margin of  $C$  with component indices in  $S$ . It is shown in Joe et al. (2010) that tail dependence functions  $\{b_s^*(w_i, i \in S)\}$  and the exponent function  $a^*(w)$  are uniquely determined from each other.

The link between tail dependence functions and multivariate regular variation is obtained in Li and Sun (2009) as follows.

**Theorem 1** Let  $X = (X_1, \dots, X_d)$  be a non-negative MRV random vector with intensity measure  $\mu$ , copula  $C$  and continuous margins  $F_1, \dots, F_d$ . If the margins are tail equivalent (i.e.  $\bar{F}_i(t)/\bar{F}_j(t) \rightarrow 1$  as  $t \rightarrow \infty$  for any  $i \neq j$  with heavy-tail index  $\beta > 0$  (see (2)), then the upper tail dependence function  $b^*(\cdot)$  exists and

$$1. \quad b^*(w) = \frac{\mu([w^{-1/\beta}, \infty])}{\mu([1, \infty] \times \mathfrak{R}_+^{d-1})} \quad \text{and} \quad a^*(w) = \frac{\mu([0, w^{-1/\beta}]^c)}{\mu([0, 1] \times \mathfrak{R}_+^{d-1})^c};$$

$$2. \quad \frac{\mu([w, \infty])}{\mu([0, 1]^c)} = \frac{b^*(w^{-\beta})}{a^*(1)} \quad \text{and} \quad \frac{\mu([0, w]^c)}{\mu([0, 1]^c)} = \frac{a^*(w^{-\beta})}{a^*(1)}.$$

Therefore, the tail dependence function and intensity measure are equivalent in the sense that the Radon measure generated by the tail dependence function is a rescaled version of the intensity measure with marginal scaling functions being of Pareto type. In contrast, the tail dependence function describes the rank-invariant extremal dependence extracted from the intensity measure.

Since intensity measures for any two different norms are proportionally related, the rescaled intensity measures, such as  $\mu(\cdot) / \mu([1, \infty] \times \mathfrak{R}_+^{d-1})$  and  $\mu(\cdot) / \mu([0, 1]^c)$ , do not depend on the choice of norms.

3. TAIL APPROXIMATION

In this section, we derive the tail asymptotics for  $VaR_p(\|X\|)$  as  $p \rightarrow 1$ , for any fixed norm  $\|\cdot\|$  on  $\mathfrak{R}^d$ . The results discussed in Embrechts et al. (2009) can be obtained by taking the  $l_1$ -norm and the tail dependence function of Archimedean copulas.

Consider a non-negative MRV random vector  $X = (X_1, \dots, X_d)$  with intensity measure  $\mu$ , joint distribution  $F$  and margins  $F_1, \dots, F_d$  that are tail equivalent with heavy-tail index  $\beta > 0$ . Without loss of generality, we use  $F_1$  to define the following limit:

$$q_{\|\cdot\|}(\beta, b^*) := \lim_{t \rightarrow \infty} \frac{\Pr\{\|X\| > t\}}{\bar{F}_1(t)}, \tag{6}$$

where  $b^*$  denotes the upper tail dependence function of  $X$ . This limiting constant depends on the intensity measure  $\mu$ , which in turn depends on heavy-tail index  $\beta$ , tail dependence function  $b^*$  and norm  $\|\cdot\|$ .

**Theorem 2** If  $F$  is absolutely continuous and the partial derivative  $\partial^d b^*(v) / \partial v_1 \dots \partial v_d$  exists everywhere, then  $q_{\|\cdot\|}(\beta, b^*)$  has the following representation

$$q_{\|\cdot\|}(\beta, b^*) = (-1)^d \int_W \frac{\partial^d b^*(w^{-\beta})}{\partial w_1 \dots \partial w_d} dw = \int_{W^{-1/\beta}} \frac{\partial^d b^*(v)}{\partial v_1 \dots \partial v_d} dv \tag{7}$$

where  $W = \{w \geq 0 : \|w\| > 1\}$  and  $W^{-1/\beta} = \{v \geq 0 : \|v^{-1/\beta}\| > 1\}$ .

**Proof.** It follows from (1) that

$$\frac{\Pr\{\|X\| > t\}}{\Pr\{X_1 > t\}} = \frac{\Pr\{X \in tW\}}{\Pr\{X \in tW_1\}} \rightarrow \frac{\mu(W)}{\mu(W_1)}, \text{ as } t \rightarrow \infty,$$

where  $W = \{w : \|w\| > 1\}$ , and  $W_1 = \{w : w_1 > 1\}$ . Let  $\tilde{\mu}(\cdot) := \mu(\cdot) / \mu(W_1)$ , then by Theorem 1 (1),  $\tilde{\mu}(\cdot)$  is the measure generated by  $\tilde{\mu}([w, \infty]) = b^*(w^{-\beta})$ . Thus the partial derivative

$$\tilde{\mu}'(w) := (-1)^d \frac{\partial^d}{\partial w_1 \dots \partial w_d} \frac{\mu([w, \infty])}{\mu([1, \infty] \times \mathfrak{R}_+^{d-1})} = (-1)^d \frac{\partial^d b^*(w^{-\beta})}{\partial w_1 \dots \partial w_d}$$

exists everywhere for  $w > 0$ . Since  $F$  is absolutely continuous,  $\tilde{\mu}(\cdot)$  is absolutely continuous with respect to the Lebesgue measure, it follows from the Radon-Nikodym Theorem that

$$q_{\|\cdot\|}(\beta, b^*) = \frac{\mu(W)}{\mu(W_1)} = \int_W \tilde{\mu}'(w) dw,$$

and by the uniqueness of the Radon-Nikodym derivative,  $\tilde{\mu}'(w) \geq 0$  is the Radon-Nikodym derivative of the intensity measure  $\tilde{\mu}$  with respect to the Lebesgue measure. Therefore, using Theorem 1 (1) we obtain the first expression in (7). The second expression in (7) follows by taking variable substitutions.

**Remark 1**

1. It follows from the non-negativity of the Radon-Nikodym derivative  $\tilde{\mu}'(w)$  that

$$0 \leq \frac{1}{\beta^d} \prod_{i=1}^d w_i^{1+\beta} \tilde{\mu}'(w) = \frac{\partial^d b^*(v)}{\partial v_1 \dots \partial v_d}, \text{ with } v_i = w_i^{-\beta}, w_i > 0, 1 \leq i \leq d,$$

which shows that  $\partial^d b^*(\mathbf{v}) / \partial v_1 \dots \partial v_d \geq 0$  for all  $\mathbf{v} \geq \mathbf{0}$ . It is easy to see that  $W^{-1/\beta} \subseteq W^{-1/\beta'}$  for any  $\beta \leq \beta'$ , and thus by (7),  $q_{\|\cdot\|}(\beta, b^*)$  is non-decreasing in  $\beta$ . This extends Theorem 2.5 in Alink et al. (2004) to multivariate regular variation with respect to any norm.

2. If we choose  $\|\cdot\|$  to be the  $\ell_\infty$ -norm, i.e.,  $\|w\| = \max\{w_i, i = 1, \dots, d\}$ , then  $W^{-1/\beta} = \{w \geq \mathbf{0} : \wedge_{i=1}^d w_i < 1\}$ . In this case,  $q_{\|\cdot\|}(\beta, b^*)$  is independent of  $\beta$  and depends on the tail dependence function only.

The limiting proportionality constant  $q_{\|\cdot\|}(\beta, b^*)$  takes particularly tractable forms for Archimedean copulas as the following corollaries show.

**Corollary 1 (Archimedean Copula)** Consider a random vector  $(X_1, \dots, X_d)$  which satisfies the assumptions of Theorem 2. Assume that  $(X_1, \dots, X_d)$  has an Archimedean copula  $C(u_1, \dots, u_d) = \psi^{-1}(\sum_{i=1}^d \psi(u_i))$ , where the generator  $\psi$  is regularly varying at 1 with tail index  $\alpha > 1$ . Then

$$q_{\|\cdot\|}(\beta, b^*) = \int_{W^{-1/\beta}} \frac{\partial^d}{\partial v_1 \dots \partial v_d} \sum_{s \subseteq I, s \neq \emptyset} [(-1)^{|s|-1} (\sum_{j \in s} v_j^\alpha)^{1/\alpha}] dv_1 \dots dv_d, \quad (8)$$

where  $W^{-1/\beta} = \{v \geq \mathbf{0} : \|v^{-1/\beta}\| > 1\}$ .

**Proof.** From Joe et al. (2010), the upper tail dependence function  $b^*(v)$  can be expressed by the upper exponent functions of the margins:

$$b^*(v) := \sum_{s \subseteq I, s \neq \emptyset} (-1)^{|s|-1} a_s^*(v_i, i \in s),$$

where  $a_s^*(v_i, i \in s) = \lim_{v_i \rightarrow 0, i \in s} a^*(v)$  is the upper exponent function of the margin of  $C$  with component indices in  $s$ .

It is given in Genest et al. (1989) that the upper exponent function of Archimedean copula is  $a^*(v) = (\sum_{j=1}^d v_j^\alpha)^{1/\alpha}$ .

Therefore,  $a_s^*(v_i, i \in s) = (\sum_{j \in s} v_j^\alpha)^{1/\alpha}$  and the result follows.

**Corollary 2 (Archimedean Survival Copula)** Consider a random vector  $(X_1, \dots, X_d)$  which satisfies the assumptions of Theorem 2. Assume that  $(-X_1, \dots, -X_d)$  has an Archimedean copula  $C(u_1, \dots, u_d) = \phi^{-1}(\sum_{i=1}^d \phi(u_i))$ , where the generator  $\phi$  is regularly varying at 0 with tail index  $\alpha > 0$ . Then

$$q_{\|\cdot\|}(\beta, b^*) = \int_B \frac{\partial^d}{\partial x_1 \dots \partial x_d} \left( \sum_{i=1}^d x_i^{-\alpha} \right)^{-1/\alpha} dx_1 \dots dx_d, \quad (9)$$

where  $B = \{x : \|x^{-1}\| > 1\}$ .

**Proof.** It is given in Joe et al. (2010) that the upper tail dependence function of  $(-X_1, \dots, -X_d)$  is  $b^*(w) = (\sum_{j=1}^d w_j^{-\alpha})^{-1/\alpha}$ . Therefore,

$$q_{\|\cdot\|}(\beta, b^*) = (-1)^d \int_W \frac{\partial^d}{\partial w_1 \dots \partial w_d} b^*(w^{-\beta}) dw = (-1)^d \int_W \frac{\partial^d}{\partial w_1 \dots \partial w_d} \left( \sum_{j=1}^d w_j^{\alpha\beta} \right)^{-1/\alpha} dw,$$

where  $W = \{w : \|w\| > 1\}$ . If we substitute  $x_i^{-1}$  for  $w_i$ , then we obtain (9).

**Remark 2**

1. In Corollary 1, if we choose  $\|\cdot\|$  to be the  $l_\infty$ -norm, then the explicit expression of the limiting constant can be easily computed for Archimedean copulas. Take the bivariate case for example,

$$q_{l_\infty}(\beta, b^*) = \int_{W^{-1/\beta}} \frac{\partial^2}{\partial v_1 \partial v_2} [v_1 + v_2 - (v_1^\alpha + v_2^\alpha)^{1/\alpha}] dv_1 dv_2$$

where  $W^{-1/\beta} = \{v : v_1^{-1/\beta} \vee v_2^{-1/\beta} > 1\}$ . Hence,

$$\begin{aligned} q_{l_\infty}(\beta, b^*) &= \left( \int_0^\infty \int_0^1 + \int_0^1 \int_0^\infty - \int_0^1 \int_0^1 \right) \frac{\partial^2}{\partial v_1 \partial v_2} [v_1 + v_2 - (v_1^\alpha + v_2^\alpha)^{1/\alpha}] dv_1 dv_2 \\ &= 2[v_1 + v_2 - (v_1^\alpha + v_2^\alpha)^{1/\alpha}] \Big|_{v_1=0}^1 \Big|_{v_2=0}^\infty - [v_1 + v_2 - (v_1^\alpha + v_2^\alpha)^{1/\alpha}] \Big|_{v_1=0}^1 \Big|_{v_2=0}^1 \\ &= 2^{1/\alpha}, \end{aligned}$$

which only depends on the tail dependence of the Archimedean copula.

2. In Corollary 2, if we choose  $\|\cdot\|$  to be the  $l_1$ -norm, i.e.,  $\|x\| = \sum_{i=1}^d x_i$  for non-negative  $x_i, i=1, \dots, d$ , then the result is reduced to the limiting constant obtained in Proposition 2.2 of Embrechts et al. (2009). Using the  $l_1$ -norm is the most common way to aggregate data. The monotonicity property of  $q_{l_1}(\beta, b^*)$  with respect to  $\alpha$  for  $d \geq 2$  is also given in Embrechts et al. (2009):  $q_{l_1}(\beta, b^*)$  is increasing in  $\alpha$  when  $\beta > 1$ ;  $q_{l_1}(\beta, b^*)$  is decreasing in  $\alpha$  when  $\beta < 1$ ;  $q_{l_1}(\beta, b^*) = d$  when  $\beta = 1$ . It is further shown that for  $\alpha > 0$  and  $\beta > 1$  (or  $\beta < 1$ ), VaR is asymptotically subadditive (superadditive), i.e.,  $VaR_p(\sum_{i=1}^d X_i) < \sum_{i=1}^d VaR_p(X_i)$  (or  $VaR_p(\sum_{i=1}^d X_i) > \sum_{i=1}^d VaR_p(X_i)$ ) for  $p$  near 1.

Heavy-tailed scale mixtures of multivariate exponential distributions also have tractable tail dependence functions (Li, 2009, Li and Sun, 2009) and thus the limiting constant  $q_{\|\cdot\|}(\beta, b^*)$  can be calculated using Theorem 2 for these distributions. Consider the following multivariate Pareto distribution:

$$X = (X_1, \dots, X_d) = \left( \frac{T_1}{Z}, \dots, \frac{T_d}{Z} \right), \tag{10}$$

where  $Z$  is a positive random variable and the random vector  $(T_1, \dots, T_d)$ , independent of  $Z$ , has a min-stable multivariate exponential distribution with identical margins, that is, the distribution  $G$  of  $(T_1, \dots, T_d)$  satisfies that for all  $w \in \mathfrak{R}^d$ ,  $\min\{T_1/w_1, \dots, T_d/w_d\}$  has an exponential distribution (see Joe (1997) pp. 174-175 for details). The well-known Pickands representation shows that the survival function  $\bar{G}$  can be written as  $\bar{G}(t) = \exp\{-P(t)\}$ , where

$$P(t) := \int_{S^d} \bigvee_{i=1}^d s_i t_i U(ds),$$

is called the Pickands dependence function, and  $S^d := \{s \geq 0 : \sum_{i=1}^d s_i = 1\}$ ,  $U$  is a finite measure on  $S^d$  and  $\int_{S^d} s_i U(ds) = \lambda$  for  $1 \leq i \leq d$ . Observe that  $P(t)$  is homogeneous of order 1, that is,  $P(ct) = cP(t)$  for any  $c \geq 0$ . Let  $Z$  have the Laplace transform  $\varphi$ , and then the survival function  $\bar{F}$  of (10) can be written as

$$\bar{F}(x) = E(e^{-P(Zx)}) = E(e^{-ZP(x)}) = \phi(P(x)),$$

with the marginal survival function  $\bar{F}_i(x) = \phi(\lambda x)$ ,  $1 \leq i \leq d$ .

Assume that  $\phi(x)$ , and thus  $\bar{F}_i(x)$ , is regularly varying at  $\infty$  with heavy-tail index  $\beta > 0$  in the sense of (2). The upper tail dependence function of  $X$  in (10) can be calculated explicitly as follows,

$$b^*(w) = \lim_{t \rightarrow \infty} \frac{\bar{F}(tw)}{\bar{F}_i(t)} = \lim_{t \rightarrow \infty} \frac{\phi(tP(w))}{\phi(\lambda t)} = \left(\frac{P(w)}{\lambda}\right)^{-\beta} \quad (11)$$

For example, if  $Z$  in (10) has the gamma distribution with shape parameter (Pareto index)  $\beta > 0$  and scale parameter 1, then  $\phi(x) = (x+1)^{-\beta}$ . Note that in this case  $R = 1/Z$  has an inverse gamma distribution with shape parameter  $\beta > 0$  and scale parameter 1. It is known that  $R$  is regularly varying with heavy-tail index  $\beta$  and its survival function is given by

$$\Pr\{R > r\} = 1 - \frac{\Gamma(\beta, 1/r)}{\Gamma(\beta)}, \quad r > 0, \beta > 0,$$

where  $\Gamma(\beta, 1/r) = \int_{1/r}^{\infty} t^{\beta-1} e^{-t} dt$  is known as the upper incomplete gamma function and  $\Gamma(\beta) = \int_0^{\infty} t^{\beta-1} e^{-t} dt$  is the gamma function.

If the distribution  $G$  of  $(T_1, \dots, T_d)$  in (10) is absolutely continuous, then the  $d$ -th order partial derivative of the Pickands dependence function  $P(w)$  exists, and thus the limit  $q_{||,||}(\beta, b^*)$  for such  $X$  can be obtained by using Theorem 2. Note that if  $(T_1, \dots, T_d)$  in (10) are independent, then the survival copula of  $X$  is Archimedean, and thus (10) includes Archimedean copulas as a special case. In contrast to Archimedean copulas, (10) models local dependence of  $(T_1, \dots, T_d)$  on top of the global dependence described by the mixture.

**Example 1** Let  $(T_1, \dots, T_d)$  have the following joint distribution

$$\bar{G}(t) := \prod_{1 \leq i < j \leq d} K_{ij}(\bar{H}_i(t_i), \bar{H}_j(t_j)) \prod_{i=1}^d \bar{H}_i(t_i),$$

where  $\bar{H}_i(t_i) = \exp\{-\alpha t_i\}$ ,  $1 \leq i \leq d$ , and

$$K_{ij}(u_1, u_2; \alpha_{ij}) = \exp\{-[(-\log u_1)^{\alpha_{ij}} + (-\log u_2)^{\alpha_{ij}}]^{1/\alpha_{ij}}\}, \quad 1 \leq u_1, u_2 \leq 1$$

is known as the extreme-value copula with dependence parameter  $\alpha_{ij}$  (Genest and Rivest, 1989). Marginally,  $T_i$  has the exponential distribution  $G_i(t) = 1 - \exp\{-\lambda t\}$  where  $\lambda = \alpha d$ . Jointly,  $T_i$  and  $T_j$  are coupled with copula  $K_{ij}$ . Clearly, the Pickands dependence function of  $G$  is given by

$$P(t) = -\log \bar{G}(t) = \alpha \left[ \sum_{1 \leq i < j \leq d} (t_i^{\alpha_{ij}} + t_j^{\alpha_{ij}})^{1/\alpha_{ij}} + \sum_{i=1}^d t_i \right]$$

It then follows from (11) that

$$b^*(w) = d^\beta \left[ \sum_{1 \leq i < j \leq d} (w_i^{\alpha_{ij}} + w_j^{\alpha_{ij}})^{1/\alpha_{ij}} + \sum_{i=1}^d w_i \right]^{-\beta}.$$

The limit  $q_{\|\cdot\|}(\beta, b^*)$  can be obtained by using Theorem 2.

Note that the absolute continuity of  $F$  in Theorem 2 is also necessary for (7). Here we give the bivariate case as an example.

**Example 2 (Bivariate Pareto Distribution of Marshall-Olkin Type)** The Marshall-Olkin distribution with rate parameters  $\{\lambda_1, \lambda_2, \lambda_{12}\}$  is the joint distribution of  $T_1 := E_1 \wedge E_{12}$  and  $T_2 := E_2 \wedge E_{12}$ , where  $\{E_S, S \subseteq \{1, 2\}\}$  is a sequence of independent exponentially distributed random variables, with  $E_S$  having mean  $1/\lambda_S$ . In the reliability context,  $T_1$  and  $T_2$  can be viewed as the lifetime of two components operating in a random shock environment where a fatal shock governed by the Poisson process with rate  $\lambda_S$  destroys all the components with indices in  $S \subseteq \{1, 2\}$  simultaneously (Marshall et al., 1967). In credit-risk modeling,  $T_1$  and  $T_2$  can be viewed as the times-to-default of various different counterparties or types of counterparty, for which the Poisson shocks might be a variety of underlying economic events (Embrechts et al., 2003).

Let  $(RT_1, RT_2)$  be defined as in (10), where  $R = 1/Z$  and  $Z$  follows a gamma distribution with shape parameter  $\beta > 0$  and scale parameter 1. From Li and Sun (2009), the bivariate distribution of  $(RT_1, RT_2)$  is regularly varying, and its upper tail dependence function is given by

$$b^*(w_1, w_2) = E\left(\frac{w_1 T_1^\beta}{E(T_1^\beta)} \wedge \frac{w_2 T_2^\beta}{E(T_2^\beta)}\right). \quad (12)$$

Since  $T_i$  is exponentially distributed with mean  $1/(\lambda_i + \lambda_{12})$  for  $i=1, 2$ , we have  $E(T_i^\beta) = \beta!(1/(\lambda_i + \lambda_{12}))^\beta$  for any positive integer  $\beta$ , thus

$$b^*(w_1, w_2) = \frac{1}{\beta!} E[w_1^{1/\beta} (\lambda_1 + \lambda_{12}) T_1 \wedge w_2^{1/\beta} (\lambda_2 + \lambda_{12}) T_2]^\beta.$$

Consider

$$\begin{aligned} & \Pr\{[w_1^{1/\beta} (\lambda_1 + \lambda_{12}) T_1 \wedge w_2^{1/\beta} (\lambda_2 + \lambda_{12}) T_2]^\beta > t\} \\ &= \Pr\{T_1 > (\lambda_1 + \lambda_{12})^{-1} (t/w_1)^{1/\beta}, T_2 > (\lambda_2 + \lambda_{12})^{-1} (t/w_2)^{1/\beta}\} \\ &= \exp\left\{-\left(\frac{\lambda_1}{(\lambda_1 + \lambda_{12}) w_1^{1/\beta}} + \frac{\lambda_2}{(\lambda_2 + \lambda_{12}) w_2^{1/\beta}} + \frac{\lambda_{12}}{(\lambda_1 + \lambda_{12}) w_1^{1/\beta} \wedge (\lambda_2 + \lambda_{12}) w_2^{1/\beta}}\right) t^{1/\beta}\right\}. \end{aligned}$$

Thus, for any positive integer  $\beta$ ,

$$\begin{aligned} b^*(w_1, w_2) &= \frac{1}{\beta!} \int_0^\infty \exp\left\{-\left(\frac{\lambda_1}{(\lambda_1 + \lambda_{12}) w_1^{1/\beta}} + \frac{\lambda_2}{(\lambda_2 + \lambda_{12}) w_2^{1/\beta}} + \frac{\lambda_{12}}{(\lambda_1 + \lambda_{12}) w_1^{1/\beta} \wedge (\lambda_2 + \lambda_{12}) w_2^{1/\beta}}\right) t^{1/\beta}\right\} dt \\ &= \frac{1}{\beta!} \int_0^\infty \beta s^{\beta-1} \exp\left\{-\left(\frac{\lambda_1}{(\lambda_1 + \lambda_{12}) w_1^{1/\beta}} + \frac{\lambda_2}{(\lambda_2 + \lambda_{12}) w_2^{1/\beta}} + \frac{\lambda_{12}}{(\lambda_1 + \lambda_{12}) w_1^{1/\beta} \wedge (\lambda_2 + \lambda_{12}) w_2^{1/\beta}}\right) s\right\} ds \\ &= \left(\frac{\lambda_1}{(\lambda_1 + \lambda_{12}) w_1^{1/\beta}} + \frac{\lambda_2}{(\lambda_2 + \lambda_{12}) w_2^{1/\beta}} + \frac{\lambda_{12}}{(\lambda_1 + \lambda_{12}) w_1^{1/\beta} \wedge (\lambda_2 + \lambda_{12}) w_2^{1/\beta}}\right)^{-\beta}. \end{aligned} \quad (13)$$

Observing the fact that  $b^*$  is not differentiable at  $(w_1, w_2)$  with  $(\lambda_1 + \lambda_{12}) w_1^{1/\beta} = (\lambda_2 + \lambda_{12}) w_2^{1/\beta}$ .

Consider a simple case where  $\lambda_1 = \lambda_2 = 0$  and  $\lambda_{12}, \beta > 0$ . In this case  $T_1 = T_2$  and  $b^*(w_1, w_2) = w_1 \wedge w_2$ . Observe that  $\partial^2 b^*(w_1, w_2) / \partial w_1 \partial w_2 = 0$  for any  $w_1 \neq w_2$ , and thus it follows from (7) that  $q_{\|\cdot\|}(\beta, b^*) = 0$  for any norm  $\|\cdot\|$ . On the other hand, however, (7) does not hold because it follows from (6) that



$$q_{\|\cdot\|}(\beta, b^*) = \lim_{t \rightarrow \infty} \frac{\Pr\{2RT_1 > t\}}{\Pr\{RT_1 > t\}} = 2^\beta \neq 0.$$

**Remark 3** In Example 2, if we set  $\lambda_{12} = 0$ , then  $T_i = E_i, i = 1, 2$ . In this case, it is known that the survival copula of  $(RT_1, RT_2)$  is a bivariate Clayton copula. On the other hand, (13) becomes  $b^*(w_1, w_2) = (w_1^{-1/\beta} + w_2^{-1/\beta})^{-\beta}$ . This is the upper tail dependence function of  $(-X_1, -X_2)$  from the bivariate case of Corollary 2, where the dependence parameter is  $1/\beta$ .

The next theorem, an application of Theorem 2, provides a method of obtaining VaR approximations in higher dimensions from one-dimensional VaR.

**Theorem 3** Consider a non-negative random vector  $X = (X_1, \dots, X_d)$  with MRV distribution function  $F$  and continuous margins  $F_1, \dots, F_d, d \geq 2$ . Assume that the margins are tail equivalent with heavy-tail index  $\beta > 0$ , and the tail dependence function  $b^* > 0$ . Then

$$\lim_{p \rightarrow 1} \frac{\text{VaR}_p(\|X\|)}{\text{VaR}_p(X_1)} = q_{\|\cdot\|}(\beta, b^*)^{1/\beta}. \quad (14)$$

**Proof.** Let  $G$  be the distribution function of  $\|X\|$ . From (6),  $\frac{\bar{G}(t)}{\bar{F}_1(t)} \rightarrow q_{\|\cdot\|}(\beta, b^*)$  as  $t \rightarrow \infty$ , i.e.,  $\bar{F}_1(t) \approx \frac{\bar{G}(t)}{q_{\|\cdot\|}(\beta, b^*)}$ , where ‘ $\approx$ ’ denotes tail equivalence. Hence we have  $t \approx \bar{F}_1^{-1}(q_{\|\cdot\|}(\beta, b^*)^{-1} \bar{G}(t))$ .

Define  $u := \bar{G}(t)$ . Then  $\bar{G}^{-1}(u) \approx \bar{F}_1^{-1}(q_{\|\cdot\|}(\beta, b^*)^{-1} u)$ . Since  $F_1$  is regularly varying at  $\infty$  with heavy-tail index  $\beta > 0$ , we have from Proposition 2.6 of Resnick (2007) that  $\bar{F}_1^{-1}(t)$  is regularly varying at 0, or more precisely,  $\bar{F}_1^{-1}(cu) / \bar{F}_1^{-1}(u) \rightarrow c^{-1/\beta}$  as  $u \rightarrow 0^+$  for any  $c > 0$ . Thus  $\bar{F}_1^{-1}(q_{\|\cdot\|}(\beta, b^*)^{-1} u) / \bar{F}_1^{-1}(u) \rightarrow q_{\|\cdot\|}(\beta, b^*)^{1/\beta}$ . Therefore,  $\bar{G}^{-1}(u) \approx q_{\|\cdot\|}(\beta, b^*)^{1/\beta} \bar{F}_1^{-1}(u)$ , i.e.,  $\lim_{u \rightarrow 0^+} \bar{G}^{-1}(u) / \bar{F}_1^{-1}(u) = q_{\|\cdot\|}(\beta, b^*)^{1/\beta}$ . Replace  $u$  by  $1-p$  and we get (14).

This result gives an asymptotic estimate of  $\text{VaR}_p(\|X\|)$  which does not generally have a closed form expression. The closed forms of tail dependence functions occur more often than for explicit copula expressions. This is because we can obtain tail dependence functions from either explicit copula expressions or the closure properties of the related conditional distributions whose parent distributions do not have explicit copulas. Using Theorems 2 and 3, when the confidence level is close to 1 and the tail dependence functions exist, we can take  $q_{\|\cdot\|}(\beta, b^*)^{1/\beta} \text{VaR}_p(X_1)$  as an approximation to the VaR of the normed (aggregated) loss. We illustrate our results using an insurance premium pricing model with risk-adjusted loading.

**Example 3** The calculation of premium is usually affected by various elements: pure risk premium, risk loading, administrative expenses, and loading for investment risk, credit risk, operational risk, etc. We consider here a simplified premium pricing model that consists of the pure risk premium and risk loading only.

Let  $X$  denote the total claim incurred for one insurance portfolio. The pure risk premium is the mean  $E(X)$  of loss  $X$ , and the risk loading depends on the excess of a risk measurement over the mean loss. If the risk measure we use here is the Value-at-Risk, then the risk loading depends on the excess  $\text{VaR}_p(X) - E(X)$  of the  $p$ -th quantile over the pure risk premium. That is, if we hold the risk capital  $\text{VaR}_p(X)$ , we know that we would not have ruin with probability  $p$ , which is usually close to 1. If an insurance company receives the capital  $\text{VaR}_p(X) - E(X)$  from investors and invests the capital at the risk-free rate  $r_0$  (e.g., in government bonds), the company needs to pay the investors at a higher rate  $r > r_0$ , because their investment is exposed to risk. Thus, the premium paid by the policyholder in this simple pricing model is the sum of the pure risk premium and risk loading:

$$E(X) + (r - r_0)(\text{VaR}_p(X) - E(X)) = (1 - r + r_0)E(X) + (r - r_0)\text{VaR}_p(X).$$

In the univariate case, the loss distribution and model parameters can be fitted from loss data, and the calculation of the premium is straightforward.

If we consider losses  $X_1, X_2$  of two insurance portfolios, then the premium pricing involves  $\text{VaR}_p(X_1 + X_2)$ , whose evaluation becomes non-trivial because  $X_1$  and  $X_2$  are joint distributed with bivariate non-normal distribution. If a bivariate distribution with Archimedean copula and heavy-tailed, identically distributed margins fits to the bivariate loss data for  $X_1, X_2$ , then the tail estimate to the VaR as  $p$  goes to 1 can be obtained using our results. More precisely, the premium for the aggregation of these two portfolios is estimated as, when  $p$  is close to 1,

$$2(1 - r + r_0)E(X_1) + (r - r_0)q_{\|\cdot\|}(\beta, b^*)^{1/\beta} \text{VaR}_p(X_1),$$

where  $\beta$  is the heavy-tail index of the margin, and  $b^*$  is the upper tail dependence function of the Archimedean copula. Using Corollaries 1 and 2,  $q_{\|\cdot\|}(\beta, b^*)$  can be calculated numerically with estimated model parameters. The premium pricing for the aggregated portfolio then proceeds as that in the univariate case. Note that the extremal dependence of  $X_1$  and  $X_2$  is encoded in the constant  $q_{\|\cdot\|}(\beta, b^*)$  in the premium pricing.

#### 4. CONCLUDING REMARKS

In this paper, we developed a general tail approximation method for evaluating the VaR of any norm of random vectors with multivariate regularly varying distribution. Our method is based on the relation between the intensity measure of multivariate regular variation and tail dependence function of the underlying copula, and extends various results in the literature. Our results are illustrated by an insurance premium pricing model.

Explicit forms of VaR approximations for Archimedean copulas and multivariate Pareto distributions can be derived directly from our results. As mentioned in Remark 2, the additivity properties of VaR under Archimedean copulas depend on the heavy-tail index of the margins and the copula generators. Whether and how similar structural properties hold for multivariate regular variation are worth further investigation.

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