International Journal of
Operations Research

International Journal of Operations Research Vol. 8, No. 2, 18–26 (2011)

A New Option Pricing Model for Stocks in Uncertainty Markets

Jin Peng[1,*] **and Kai Yao**[2,]

[1]Institute of Uncertain Systems, College of Mathematical and Information Sciences,

Huanggang Normal University, Hubei, China - 438000

[2]Uncertainty Theory Laboratory, Department of Mathematical Sciences

Tsinghua University, Beijing, China - 100084

Received August 2010; *Revised October 2010*; *Accepted November 2010*

Abstract—This paper presents a type of stock models for uncertain markets by using uncertainty theory. Firstly, a brief history of stock models and some methodologies are reviewed. Next, some useful concepts and properties about uncertain process are recalled. Then, a stock model for uncertain markets is formulated by the tool of uncertain differential equation. Some option pricing formulas on the proposed uncertain stock model are investigated and some numerical calculations are illustrated. Finally, some remarks are made in the concluding section.
Keywords—Uncertainty theory, uncertain process, finance, stock model, option price.

1. INTRODUCTION

In 1973, Black and Scholes (1973) proposed the famous Black-Scholes model in stock market based on the geometric Brownian motion, and gave an option pricing formula. After that, the formula was used to evaluate other financial derivative. Nowadays, it has become an indispensable tool in financial market. In 2004, Liu (2004) founded a credibility theory to study the behavior of fuzzy phenomena. Soon afterwards Liu (2008) proposed a stock model based on a fuzzy process. Then Qin and Li (2008) derived the corresponding European option pricing formulas. Following that, Gao and Gao (2008) presented a mean-reverting stock model for fuzzy markets, which was generalized by Peng (2008) later.

However, a lot of surveys showed that some imprecise quantities, such as information and knowledge represented by human language, behave neither like randomness nor like fuzziness. In order to model these imprecise quantities, uncertainty theory was founded by Liu (2007) in 2007 and refined by Liu (2010a) in 2010 based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. Lots of researchers have contributed in this area. Gao (2009) showed some properties of continuous uncertain measure. You (2009) proved some convergence theorems of uncertain sequences. Liu and Ha (2010) gave a formula of the expected value of function of uncertain variables. With the development of uncertainty theory, Liu (2009) proposed uncertain programming which is essentially a type of mathematical programming involving uncertain variables. Besides, Liu (2010c) introduced uncertain risk analysis and reliability analysis to deal with system risk and reliability. Moreover, Li and Liu (2009) proposed uncertain logic to deal with uncertain knowledge via uncertainty theory, and Liu (2009c) developed uncertain entailment in the framework of uncertain logic. In order to derive consequences from uncertain knowledge or evidence, Liu (2010b) introduced uncertain inference and proposed the first inference rule, and Gao, Gao and Ralescu (2010) extended the inference rule to the case with multiple antecedents and with multiple if-then rules. In addition, Zhu (2010) presented uncertain optional control and applied it to a portfolio selection model. In order to collect and interpret expert's experimental data by uncertainty theory, Liu (2010a) introduced a questionnaire survey and proposed uncertain statistics for determining uncertainty distributions.

In order to study the evolution of uncertain phenomena with time, Liu (2008) proposed a concept of uncertain process in 2008, and Liu (2009b) designed a canonical process in 2009. In addition, Liu (2009b) invented uncertain calculus to deal with differentiation and integration of function of uncertain processes, and Liu (2008) defined uncertain differential equation. After that, Chen and Liu (2010) gave an existence and uniqueness theorem for uncertain differential equations. By means of

* Corresponding author's email: pengjin01@tsinghua.org.cn

uncertain differential equation, Liu (2009b) proposed a stock model for uncertain markets that are essentially a kind of markets consistent with uncertain measure. Following that, Chen (2010) derived an American option pricing formula.

While Liu's stock model describes stock prices in short-run properly, it can not describe stock prices in long-run, as the stock prices fluctuate around some average price in long-run. In this paper, we will present a new uncertain stock model to describe the stock prices in long-run and prove its corresponding option pricing formulas. The remainder of this paper is structured as follows. The next section is intended to introduce some concepts of uncertain process. A new type of stock model for uncertain markets and its extensions are formulated in Section 3. After that, European option pricing formulas and American option pricing formulas for the proposed uncertain stock model are investigated in Section 4 and Section 5, respectively. Finally, some remarks are made in Section 6.

2. PRELIMINARIES

Uncertain process was defined by Liu (2008) as a sequence of uncertain variables indexed by time or space. In this section, we will introduce some useful definitions and properties about uncertain process.

Definition 1. (Liu (2008))

Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set

$$\{X_t \in B\} = \{\gamma \mid X_t(\gamma) \in B\}$$

is an event.

An uncertain process X_t is said to have independent increments if

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where t_0 is the initial time and $t_1, t_2, \cdots, t_k$ are any times with $t_0 < t_1 < \cdots < t_k$. For this case, X_t is said to be an independent increment process.

Theorem 1. (Liu (2010d), Extreme Value Theorem)

Let X_t be an independent increment process with a continuous uncertainty distribution $\Phi_t(x)$ at each time t. Then the supremum

$$\sup_{0 \le t \le s} X_t$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0 \le t \le s} \Phi_t(x).$$

Moreover, if f is an increasing function, then

$$\sup_{0 \le t \le s} f(X_t)$$

has an uncertainty distribution

$$\Upsilon(x) = \inf_{0 \le t \le s} \Phi_t(f^{-1}(x)).$$

Definition 2. (Liu (2009b))

An uncertain process C_t is said to be a canonical process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_t has stationary and independent increments,

(iii) every increment $C_{s+t} - C_s$ is a normally distributed uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

If C_t is a canonical process, then the uncertain process $G_t = \exp(et + \sigma C_t)$ is called a geometric canonical process, where e is called the log-drift and σ is called the log-diffusion.

Definition 3. (Liu (2009b))

Let X_t be an uncertain process and let C_t be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \mathbf{L} < t_k = b$, the mesh is written as $\mathrm{D} = \max_{1 \le i \le k} | t_{i+1} - t_i |$. Then the uncertain integral of X_t with respect to C_t is

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\mathrm{D} \to 0} \sum_{i=1}^{k} X_{t_i} (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is an uncertain variable.

Definition 4. (Liu (2008))

Suppose C_t is a canonical process, and f and g are two given functions. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

is called an uncertain differential equation.

Let X_t be the bond price, and Y_t the stock price. Assume that the stock price Y_t follows a geometric canonical process. Then Liu's stock model (2009b) is written as follows,

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + s\, Y_t\mathrm{d}C_t \end{cases}$$

where r is the riskless interest rate, e is the stock drift, and s is the stock diffusion, and C_t is a canonical process. Liu [13] gave European option pricing formulas for Liu's stock model, and Chen [4] gave American option pricing formulas.

3. A NEW STOCK MODEL

In this section, we give some new uncertain stock models for financial markets as a counterpart of Black-Karasinski's model (1991).

Let X_t be the bond price, and Y_t the stock price. Then a new uncertain stock model with mean-reverting process may be represented as follows,

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = (m - a\, Y_t)\mathrm{d}t + s\, Y_t\mathrm{d}C_t \end{cases} \tag{1}$$

where r, m, a, s are some given positive constants, and C_t is a canonical process.

This model incorporates a general economic behavior: mean reversion. That is, when the stock price $Y_t > m/a$, it has a negative drift, thus the price tends to fall more likely; when the stock price $Y_t < m/a$, it has a positive drift, thus the price tends to rise more likely. Thus the stock prices appear to be pulled back to a long-run average level m/a over time.

A natural generalization for the above stock model is

$$\begin{cases} \mathrm{d}X_t = r_t X_t\mathrm{d}t \\ \mathrm{d}Y_t = (m_t - a_t Y_t)\mathrm{d}t + s_t Y_t\mathrm{d}C_t \end{cases}$$

where r_t, m_t, a_t, s_t are deterministic functions of the time t, and C_t is a canonical process.

Now we assume that there are multiple stocks whose prices are determined by multiple canonical processes. For this case, we have a multi-factor stock model in which the bond price X_t and the stock prices Y_{it} are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_{it} = (m_i - a_i Y_{it})\mathrm{d}t + \sum_{j=1}^{n} s_{ij}\mathrm{d}C_{jt}, \quad i = 1, 2, \mathbf{L}, k \end{cases}$$

where r, m_i, a_i, s_{ij} are deterministic real numbers and C_{jt} are canonical processes for $i = 1, 2, \mathbf{L}, k$ and $j = 1, 2, \mathbf{L}, n$, respectively.

4. EUROPEAN OPTIONS

A European option gives one the right, but not the obligation, to buy or sell a stock at a specified time for a specified price. Assume that a European call option has a strike price K and an expiration time T. If Y_T is the final price of the underlying stock, then the payoff from buying a European call option is $(Y_T - K)^+$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rT)(Y_T - K)^+$.

Definition 5.

Assume a European call option has a strike price K and an expiration time T. Then this option has price

$$f_c = \exp(-rT)E[(Y_T - K)^+].$$

Theorem 2. (European Call Option Pricing Formula)

Assume a European put option for the stock model (1) has a strike price K and an expiration time T. Then the European call option price is

$$f_c = \frac{\sqrt{3}s}{pa}\exp(-rT)(1-\exp(-aT))\ln\left(1+\exp\left(-\frac{p}{\sqrt{3}}b\right)\right) \tag{2}$$

where $b = \left(aK - m - \exp(-aT)(aY_0 - m)\right) / \left(s - s\ \exp(-aT)\right)$.

Proof.

It follows from the chain rule that

$$\begin{aligned} \mathrm{d}(\exp(at)Y_t) &= a\ \exp(at)Y_t\mathrm{d}t + \exp(at)\mathrm{d}Y_t \\ &= a\ \exp(at)Y_t\mathrm{d}t + \exp(at)(m - aY_t)\mathrm{d}t + \exp(at)s\ \mathrm{d}C_t \\ &= m\exp(at)\mathrm{d}t + s\ \exp(at)\ \mathrm{d}C_t. \end{aligned}$$

Integration on both sides of above equation yields

$$\exp(at)Y_t - Y_0 = m\int_0^t \exp(as)\mathrm{d}s + s\int_0^t \exp(as)\mathrm{d}C_s.$$

This means

$$Y_t = \frac{m}{a} + \exp(-at)(Y_0 - \frac{m}{a}) + s\ \exp(-at)\int_0^t \exp(as)\mathrm{d}C_s.$$

Then we have

$$\begin{aligned} f_c &= \exp(-rT)E[(Y_T - K)^+] \\ &= \exp(-rT)\int_0^{\infty} M\{Y_T - K \ge s\}\mathrm{d}s \\ &= \exp(-rT)\int_d^{\infty} M\left\{s\ \exp(-aT)\int_0^T \exp(au)\ \mathrm{d}C_u \ge s\right\}\mathrm{d}s \end{aligned}$$

where

$$d = \left(K - \frac{m}{a} - \exp(-aT)(Y_0 - \frac{m}{a})\right).$$

Since $\int_0^T \exp(au)\ \mathrm{d}C_u$ has a normal uncertainty distribution $N\left(0, \int_0^T \exp(au)\ \mathrm{d}u\right)$, we have

$$\begin{aligned} M\left\{s\ \exp(-aT)\int_0^T \exp(au)\mathrm{d}C_u \ge s\right\} &= M\left\{\frac{s}{a}\exp(-aT)(\exp(aT)-1)x \ge s\right\} \\ &= M\left\{x > as / \left(s\ (1-\exp(-aT))\right)\right\} \end{aligned}$$

where x is a normal uncertain variable. So

$$
\begin{aligned}
f_c &= \exp(-rT)\int_d^{\infty} M\left\{x > \frac{a s}{s - s\exp(-aT)}\right\}\mathrm{d}s \\
&= \frac{s}{a}\exp(-rT)(1-\exp(-aT))\int_b^{\infty} M\{x > s\}\mathrm{d}s \\
&= \frac{s}{a}\exp(-rT)(1-\exp(-aT))\int_b^{\infty}\left(1+\exp\left(\frac{p}{\sqrt{3}}s\right)\right)^{-1}\mathrm{d}s \\
&= \frac{\sqrt{3}s}{pa}\exp(-rT)(1-\exp(-aT))\ln\left(1+\exp\left(-\frac{p}{\sqrt{3}}b\right)\right)
\end{aligned}
$$

where

$$b = \left(aK - m - \exp(-aT)(aY_0 - m)\right)/\left(s - s\exp(-aT)\right).$$

The European call option pricing formula is verified.

Example 1.

Assume that a stock is presently selling for a price $Y_0 = 30$, the riskless interest rate $r = 8\%$ per annum, $m = 0.1$, $a = 0.06$ and $s = 7.5$. Consider a European call option with an expiration time $T = 0.25$ and a strike price $K = 31$. By the European call option pricing formula (2), we effectively calculate out that the price is 0.2245.

Assume that a European put option has a strike price K and an expiration time T. If Y_T is the final price of the underlying stock, then the payoff from buying a European put option is $(K - Y_T)^+$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rT)(K - Y_T)^+$.

Definition 6.

Assume a European put option has a strike price K and an expiration time T. Then this option has price

$$f_p = \exp(-rT)E[(K - Y_T)^+].$$

Theorem 3. (European Put Option Pricing Formula)

Assume a European put option for the stock model (1) has a strike price K and an expiration time T. Then the European put option price is

$$f_p = \frac{\sqrt{3}s}{pa}\exp(-rT)(1-\exp(-aT))\left(\ln\left(1+\exp\left(\frac{p}{\sqrt{3}}b\right)\right) - \ln\left(1+\exp\left(\frac{p}{\sqrt{3}}g\right)\right)\right). \tag{3}$$

where

$$b = \left(aK - m - \exp(-aT)(aY_0 - m)\right)/\left(s - s\exp(-aT)\right)$$

and

$$g = \left(-m - \exp(-aT)(aY_0 - m)\right)/\left(s - s\exp(-aT)\right).$$

Proof.

According to the definition of expected value of uncertain variable, we have

$$
\begin{aligned}
f_p &= \exp(-rT)E[(K-Y_T)^+] \\
&= \exp(-rT)\int_0^\infty M\{(K-Y_T)^+ \ge s\}\mathrm{d}s \\
&= \exp(-rT)\int_0^K M\{K-Y_T \ge s\}\mathrm{d}s \\
&= \exp(-rT)\int_0^K M\{Y_T \le s\}\mathrm{d}s \\
&= \exp(-rT)\int_0^K M\{Y_T \le s\}\mathrm{d}s \\
&= \exp(-rT)\int_{d-K}^d M\left\{s\ \exp(-aT)\int_0^T \exp(au)\mathrm{d}C_u \le s\right\}\mathrm{d}s \\
&= \frac{s}{a}\exp(-rT)(1-\exp(-aT))\int_g^b M\{x \le s\}\,\mathrm{d}s \\
&= \frac{s}{a}\exp(-rT)(1-\exp(-aT))\int_g^b \left(1+\exp\left(-\frac{p}{\sqrt{3}}s\right)\right)^{-1}\mathrm{d}s \\
&= \frac{\sqrt{3}s}{pa}\exp(-rT)(1-\exp(-aT))\left(\ln\left(1+\exp\left(\frac{p}{\sqrt{3}}b\right)\right)-\ln\left(1+\exp\left(\frac{p}{\sqrt{3}}g\right)\right)\right)
\end{aligned}
$$

where

$$d = \left(K - m/a - \exp(-aT)(Y_0 - m/a)\right),$$

$$b = \left(aK - m - \exp(-aT)(aY_0 - m)\right)/\left(s - s\ \exp(-aT)\right),$$

$$g = \left(-m - \exp(-aT)(aY_0 - m)\right)/\left(s - s\ \exp(-aT)\right)$$

and x is a normal uncertain variable. The European put option pricing formula is verified.

Example 2.

Assume that a stock is presently selling for a price $Y_0 = 30$, the riskless interest rate $r = 8\%$ per annum, $m = 0.1$, $a = 0.06$ and $s = 7.5$. Consider a European put option with an expiration time $T = 0.25$ and a strike price $K = 29$. By the European put option pricing formula (3), we effectively calculate out that the price is 0.4531.

5. AMERICAN OPTIONS

An American option gives one the right, but not the obligation, to buy or sell a stock before a specified time for a specified price. Assume that an American call option has a strike price K and an expiration time T. If Y_t is the price of the underlying stock at some time t, then the payoff from buying an American call option is

$$\sup_{0\le t\le T}(Y_t - K)^+.$$

Considering the time value of money resulted from the bond, the present value of this payoff is

$$\sup_{0\le t\le T}\exp(Y_t - K)^+.$$

Definition 7.

Assume an American call option has a strike price K and an expiration time T. Then this option has price

$$f_c = E[\sup_{0\le t\le T}\exp(-rt)(Y_t - K)^+].$$

Theorem 4. (American Call Option Pricing Formula)

Assume an American call option for the stock model (1) has a strike price K and an expiration time T. Then the American call option price is

$$f_c = \int_0^\infty \sup_{0\le t\le T}\left(1+\exp\left(\frac{p\,\left(a\ \exp(rt)s + (aK-m) + (m-aY_0)\exp(-at)\right)}{\sqrt{3}s\ (1-\exp(-at))}\right)\right)^{-1}\mathrm{d}s. \qquad (4)$$

Proof.

We first calculate the uncertainty distribution Υ_t of $\exp(-rt)(Y_t - K)^+$. For each $t \in (0, T]$, it is obvious that $\Upsilon_t(x) = 0$ provided $x < 0$. When $x > 0$, we have

$$\begin{aligned}\Upsilon_t(x) &= M\left\{\exp(-rt)(Y_t - K)^+ \le x\right\} \\ &= M\left\{\exp(-rt)\left(\frac{m}{a} + \exp(-at)(Y_0 - \frac{m}{a}) + s\ \exp(-at)\int_0^t \exp(as)\mathrm{d}C_s - K\right)^+ \le x\right\} \\ &= M\left\{s\int_0^t \exp(as)\mathrm{d}C_s \le \exp(rt + at)x + \left(K - \frac{m}{a}\right)\exp(at) - \left(Y_0 - \frac{m}{a}\right)\right\} \\ &= M\left\{s\int_0^t \exp(as)\mathrm{d}s\ \ x \le \exp(rt + at)x + \left(K - \frac{m}{a}\right)\exp(at) - \left(Y_0 - \frac{m}{a}\right)\right\}\end{aligned}$$

where x is a normal uncertain variable. So we obtain

$$\begin{aligned}\Upsilon_t(x) &= M\left\{x \le \left(a\ \exp(rt + at)x + \left(aK - m\right)\exp(at) - \left(aY_0 - m\right)\right) / \left(s\ (\exp(at) - 1)\right)\right\} \\ &= \left(1 + \exp\left(-\frac{p\left(a\ \exp(rt)x + (aK - m) + (m - aY_0)\exp(-at)\right)}{\sqrt{3}s\ (1 - \exp(-at))}\right)\right)^{-1}.\end{aligned}$$

It follows from the extreme value theorem that

$$\sup_{0\le t\le T} \exp(-rt)(Y_t - K)^+$$

has an uncertainty distribution

$$\Upsilon(x) = \inf_{0\le t\le T}\left(1 + \exp\left(-\frac{p\left(a\ \exp(rt)x + (aK - m) + (m - aY_0)\exp(-at)\right)}{\sqrt{3}s\ (1 - \exp(-at))}\right)\right)^{-1}.$$

Thus we get

$$\begin{aligned}f_c &= E[\sup_{0\le t\le T} \exp(-rt)(Y_t - K)^+] \\ &= \int_0^{\infty} M\left\{\sup_{0\le t\le T} \exp(-rt)(Y_t - K)^+ \ge s\right\}\mathrm{d}s \\ &= \int_0^{\infty} 1 - \inf_{0\le t\le T}\left(1 + \exp\left(-\frac{p\left(a\ \exp(rt)s + (aK - m) + (m - aY_0)\exp(-at)\right)}{\sqrt{3}s\ (1 - \exp(-at))}\right)\right)^{-1}\mathrm{d}s \\ &= \int_0^{\infty} \sup_{0\le t\le T}\left(1 + \exp\left(\frac{p\left(a\ \exp(rt)s + (aK - m) + (m - aY_0)\exp(-at)\right)}{\sqrt{3}s\ (1 - \exp(-at))}\right)\right)^{-1}\mathrm{d}s.\end{aligned}$$

The American call option pricing formula is verified.

Example 3.

Assume that a stock is presently selling for a price $Y_0 = 30$, the riskless interest rate $r = 8\%$ per annum, $m = 0.1$, $a = 0.06$ and $s = 7.5$. Consider an American call option with an expiration time $T = 0.25$ and a strike price $K = 31$. By the American call option pricing formula (4), we effectively calculate out that the price is 0.2244.

Assume that an American put option has a strike price K and an expiration time T. If Y_t is the price of the underlying stock at some time t, then the payoff from buying an American put option is

$$\sup_{0\le t\le T}(K - Y_t)^+.$$

Considering the time value of money resulted from the bond, the present value of this payoff is

$$\sup_{0\le t\le T} \exp(K - Y_t)^+.$$

Definition 8.

Assume an American put option has a strike price K and an expiration time T. Then this option has price

$$f_p = E[\sup_{0\le t\le T} \exp(-rt)(K - Y_t)^+].$$

Theorem 5. (American Put Option Pricing Formula)

Assume an American put option for the stock model (1) has a strike price K and an expiration time T. Then the American put option price is

$$f_p = \int_0^{\infty} \sup_{0\le t\le T}\left(1+\exp\left(-\frac{p\left(-a\exp(rt)s+(aK-m)+(m-aY_0)\exp(-at)\right)}{\sqrt{3}s\,(1-\exp(-at))}\right)\right)^{-1} \mathrm{d}s. \tag{5}$$

Proof.

We first calculate the uncertainty distribution Υ_t of $\exp(-rt)(K-Y_t)^+$. For each $t\in(0,T]$, it is obvious that $\Upsilon_t(x)=0$ provided $x<0$. When $x>0$, we have

$$\begin{aligned}
\Upsilon_t(x) &= M\left\{\exp(-rt)(K-Y_t)^+ \le x\right\}\\
&= M\left\{\exp(-rt)\left(K-\frac{m}{a}-\exp(-at)(Y_0-\frac{m}{a})-s\,\exp(-at)\int_0^t \exp(as)\mathrm{d}C_s\right)^+ \le x\right\}\\
&= M\left\{s\int_0^t \exp(as)\mathrm{d}C_s \ge -\exp(rt+at)x+\left(K-\frac{m}{a}\right)\exp(at)-\left(Y_0-\frac{m}{a}\right)\right\}\\
&= M\left\{s\int_0^t \exp(as)\mathrm{d}s\ x \ge -\exp(rt+at)x+\left(K-\frac{m}{a}\right)\exp(at)-\left(Y_0-\frac{m}{a}\right)\right\}
\end{aligned}$$

where x is a normal uncertain variable. So we obtain

$$\begin{aligned}
\Upsilon_t(x) &= M\left\{x \ge \left(-a\exp(rt+at)x+\left(aK-m\right)\exp(at)-\left(aY_0-m\right)\right)/\left(s\,(\exp(at)-1)\right)\right\}\\
&= \left(1+\exp\left(\frac{p\left(-a\exp(rt)x+(aK-m)+(m-aY_0)\exp(-at)\right)}{\sqrt{3}s\,(1-\exp(-at))}\right)\right)^{-1}.
\end{aligned}$$

It follows from the extreme value theorem that

$$\sup_{0\le t\le T}\exp(-rt)(Y_t-K)^+$$

has an uncertainty distribution

$$\Upsilon(x) = \inf_{0\le t\le T}\left(1+\exp\left(\frac{p\left(-a\exp(rt)x+(aK-m)+(m-aY_0)\exp(-at)\right)}{\sqrt{3}s\,(1-\exp(-at))}\right)\right)^{-1}.$$

Thus we get

$$\begin{aligned}
f_p &= E[\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+]\\
&= \int_0^{\infty} M\left\{\sup_{0\le t\le T}\exp(-rt)(K-Y_t)^+ \ge s\right\}\mathrm{d}s\\
&= \int_0^{\infty} 1-\inf_{0\le t\le T}\left(1+\exp\left(\frac{p\left(-a\exp(rt)x+(aK-m)+(m-aY_0)\exp(-at)\right)}{\sqrt{3}s\,(1-\exp(-at))}\right)\right)^{-1}\mathrm{d}s\\
&= \int_0^{\infty} \sup_{0\le t\le T}\left(1+\exp\left(-\frac{p\left(-a\exp(rt)s+(aK-m)+(m-aY_0)\exp(-at)\right)}{\sqrt{3}s\,(1-\exp(-at))}\right)\right)^{-1}\mathrm{d}s.
\end{aligned}$$

The American put option pricing formula is verified.

Example 4.

Assume that a stock is presently selling for a price $Y_0 = 30$, the riskless interest rate $r = 8\%$ per annum, $m = 0.1$, $a = 0.06$ and $s = 7.5$. Consider an American put option with an expiration time $T = 0.25$ and a strike price $K = 29$. By the American put option pricing formula (5), we effectively calculate out that the price is 0.4530.

6. CONCLUSION

The main contribution of this paper was to provide a mean-reverting stock model via uncertainty theory. Some option pricing formulas were proved for the stock model. In addition, some numerical experiments were given to illustrate the formulas.

ACKNOWLEDGEMENTS

This work is supported by the National Natural Science Foundation (Grant No.70671050 and No.60874067), the Hubei Provincial Natural Science Foundation, and the Innovation Team Project of Hubei Provincial Department of Education, China.

REFERENCES

1. Black, F. and Scholes, M.(1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81, 637-654.
2. Black, F. and Karasinski, P.(1991). Bond and option pricing when short-term rates are lognormal. *Financial Analysts Journal*, 47(4), 52-59.
3. Chen, X. and Liu, B. (2010). Existence and uniqueness theorem for uncertain differential equations. *Fuzzy Optimization and Decision Making*, 9(1), 69-81.
4. Chen, X. (2010). American option pricing formula for uncertain financial market. http://orsc.edu.cn/online/100701.pdf.
5. Gao, J. and Gao, X.(2008). Credibilistic option pricing: A new model. *Journal of Uncertain Systems*, 2(4), 243-247.
6. Gao, X.(2009). Some properties of continuous uncertain measure. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 17(3), 419-426.
7. Gao, X., Gao, Y. and Ralescu, D.A.(2010). On Liu's inference rule for uncertain systems. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 18(1), 1-11.
8. Li, X. and Liu, B. (2009). Hybrid logic and uncertain logic. *Journal of Uncertain Systems*, 3(2), 83-94.
9. Liu, B. (2004). *Uncertainty Theory: An Introduction to Its Axiomatic Foundations*, Springer-Verlag, Berlin.
10. Liu, B. (2007). *Uncertainty Theory*, 2nd ed., Springer-Verlag, Berlin.
11. Liu, B. (2008). Fuzzy process, hybrid process and uncertain process. *Journal of Uncertain Systems*, 2(1), 3-16.
12. Liu, B. (2009a). *Theory and Practice of Uncertain Programming*, 2nd ed., Springer-Verlag, Berlin.
13. Liu, B. (2009b). Some research problems in uncertainty theory. *Journal of Uncertain Systems*, 3(1), 3-10.
14. Liu, B. (2009c). Uncertain entailment and modus ponens in the framework of uncertain logic. *Journal of Uncertain Systems*, 3(4), 243-251.
15. Liu, B. (2010a). *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*, Springer-Verlag, Berlin.
16. Liu, B. (2010b). Uncertain set theory and uncertain inference rule with application to uncertain control. *Journal of Uncertain Systems*, 4 (In press).
17. Liu, B. (2010c). Uncertain risk analysis and uncertain reliability analysis. *Journal of Uncertain Systems*, 4(3), 163-170.
18. Liu, B. (2010d). Extreme value theorems of uncertain process with application to insurance risk models, http://orsc.edu.cn/online/100722.pdf.
19. Liu, Y. and Ha, M. (2010). Expected value of function of uncertain variables, *Journal of Uncertain Systems*, 4(3), 181-186.
20. Peng, J. (2008). A general stock model for fuzzy markets. *Journal of Uncertain Systems*, 2(4), 248-254.
21. Qin, Z. and Li, X. (2008). Option pricing formula for fuzzy financial market. *Journal of Uncertain Systems*, 2(1), 17-21.
22. You, C. (2009). Some convergence theorems of uncertain sequences. *Mathematical and Computer Modelling*, 49(3/4), 482-487.
23. Zhu, Y. (2010). Uncertain optimal control with application to a portfolio selection model. *Cybernetics and Systems*, 41 (In press).